Equivariant orbit preserving diffeomorphisms

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Joint with Gerald W. Schwarz

Following: André Haefliger, Eliane Salem, G. W. Schwarz (1991)

Theorem (torus version)

$T \cong (S^1)^k$ torus; $T \circlearrowright M$ faithfully; M connected.

$\psi \colon M \to M$ diffeomorphism

- T-equivariant
- orbit-preserving

Then $\exists\, \eta \colon M \to T$

- T-invariant
- smooth

such that $\psi(x) = \eta(x) \cdot x \quad \forall x \in M$

$$\psi(a \cdot x) = a \cdot \psi(x) = a \cdot \eta(x) \cdot x = \eta(x) \cdot (a \cdot x)$$

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Actions of Tori on Orbifolds

André Haefliger
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Introduction

In this paper, we study smooth effective actions of a torus T^n of dimension n on an orbifold of dimension m. Such actions occur naturally in the study of Riemannian foliations on simply connected manifolds (see [9]).

The basic techniques used by several authors, Seifert, Orlik-Raymond ([11]), Fintushel ([4] and [5]), etc... for the study of actions of circles or tori on manifolds are easily generalized and hopefully clarified to apply to the more general case of orbifolds. In [2] F. Bonahon and L. Siebenmann have made a careful study of locally free actions of the circle on 3-orbifolds.

After giving in Section 1 the basic definitions, we study in Section 2 the general structure of invariant tubular neighborhoods of orbits by passing to their universal coverings. They are described in terms of three invariants: 1) a subgroup Γ_0 of the lattice $\Gamma = \mathbf{Z}^n$ in $\mathbf{R}^n$, 2) a central extension

$$0 \to \Gamma/\Gamma_0 \to \Lambda \to D \to 1,$$

where D is a finite group, 3) a faithful representation ρ (the slice representation) of K into the group of isometries $\mathrm{O}(B)$, where K is the maximal compact subgroup of a Lie group G constructed from 2), and B is a Euclidean ball of dimension $m - n + \dim K$.

The way tubular neighborhoods are glued together above the orbit space is studied in Section 4 where we use a basic result whose proof was given to us by G. Schwarz (cf. Section 3). There is an obstruction for the gluing which is an element of $H^3(W, \mathbf{Z}^n)$ (in the sense of Čech cohomology), where W is the orbit space. The different gluings are parameterized by elements of $H^2(W, \mathbf{Z}^n)$.

Compact Lie groups of type HS: motivation; definition

$T \circlearrowright$ an orbifold $\implies$ local slices to T-orbits: $H \circlearrowright \mathbb{R}^n/\Gamma$

(Γ finite, $H \subseteq T$ closed),

given by compact $K \circlearrowright \mathbb{R}^n$ for $1 \to \Gamma \to K \to H \xrightarrow{\pi} 1$

$$\stackrel{(\star)}{\implies} \quad \underbrace{K_0}_{\text{identity component}} \subseteq \underbrace{Z(K)}_{\text{centre}} \quad \Longleftrightarrow: \quad K \textbf{ of type HS}$$

$^{(\star)}$Proof: H abelian $\implies$ $\pi(kak^{-1}a^{-1}) = 1, \quad \forall k, a \in K$

$\implies \quad \forall a \in K, \quad k \mapsto kak^{-1}a^{-1}$ maps $\underbrace{K_0}_{\text{connected}}$ to $\underbrace{\Gamma}_{\text{discrete}}$ and $1 \mapsto 1$,

$\implies \quad \forall a \in K \; \forall k \in K_0, \; kak^{-1}a^{-1} = 1, \quad \implies \quad K_0 \subseteq Z(K)$

Theorem (non-abelian version)

K compact Lie of type HS; $K \circlearrowright M$ faithfully; M connected

$\psi\colon M \to M$ diffeomorphism

- K-equivariant
- orbit-preserving

Then $\exists\ \eta\colon M \to K$

- K-equivariant (conjugation on target)
- smooth

such that $\psi(x) = \eta(x) \cdot x \quad \forall x \in M$

type HS $:\Longleftrightarrow K_0 \subseteq Z(K)$

From Haefliger–Salem:

The aim of this paragraph is to prove the following theorem which is an easy application of the basic Lemma 3.2 whose proof was communicated to us by G. Schwarz.

3.1. Theorem. *A diffeomorphism h of an orbifold X commuting with an action of T^n and preserving the orbits is of the form $h(x) = f(\pi(x)) \cdot x$, where f is a smooth map of the space of orbits in T^n.*

3.2. Lemma. (G. Schwarz). *Let B be an open Euclidean ball centered at the origin of $\mathbf{R}^q$. Let K be a compact subgroup of the orthogonal group $\mathrm{O}(q)$, whose component K_0 of the identity is a torus in the center of K. Let H be a diffeomorphism of B such that $H(v) \in K \cdot v$ for all $v \in B$. Then there is a smooth map $F\colon B \to K$ such that $H(v) = F(v) \cdot v$.*

Counterexample: $S^1 \circlearrowright \mathbb{C}$

$\psi(z) := \bar{z}$ orbit-preserving diffeomorphism

$\nexists$ smooth $\eta\colon \mathbb{C} \to S^1$ such that $\psi(z) = \eta(z) \cdot z \ \forall z \in \mathbb{C}$

$\eta(re^{i\theta}) = e^{-2i\theta}$

From Haefliger–Salem:

Proof of the Lemma. By hypothesis, H preserves the spheres centered at the origin O of $\mathbf{R}^q$. Let H_t be the diffeomorphism of B defined by $H_t = \frac{1}{t}H(tv)$ for $0 < t \leq 1$ and by $H_0 =$ the derivative of H at O for $t = 0$. Note that $H_0 \in \mathrm{O}(q)$. The family H_t is smooth and $H_t(v) \in K \cdot v$ for each $t \in [0,1]$ and $v \in B$. After replacing H by $H_0^{-1} \circ H$, we can assume that H_0 is the identity.

Let E be the vector space $\mathbf{R}^q \times \mathbf{R}$ on which K acts as usual on the first factor

"**All-linear HS lemma**":

K compact Lie of type HS; $K \circlearrowright W$ linear action;

$\psi\colon W \to W$ orbit-preserving K-equivariant$^{(*)}$ linear isomorphism.

Then $\exists\, \gamma \in K$ such that $\psi(x) = \gamma \cdot x \;\; \forall x \in W$.

type HS $:\Longleftrightarrow K_0 \subseteq Z(K)$

$^{(*)}$ or: equivariant with respect to an automorphism of K that is trivial on K_0.

Reminder of theorem:

K compact Lie group of type HS; $K \circlearrowright M$ faithfully; M connected;
$\psi\colon M \to M$ orbit-preserving K-equivariant diffeomorphism. Then
$\exists$ smooth K-equivariant $\eta\colon M \to K$ such that $\psi(x) = \eta(x) \cdot x \;\; \forall x \in K$.

Outline of proof of theorem

abelian:

- "all-linear" version
- infinitesimal version
- linear action; non-linear ψ
- action by automorphisms on vector bundle
- slice theorem $\Longrightarrow$ general (abelian) case

non-abelian:

- K finite
- finite + abelian $\Longrightarrow$ general (type HS) case

Compact Lie groups of type HS: examples

K compact: type HS $:\Longleftrightarrow$ $K_0 \subseteq Z(K)$ $\Longleftrightarrow$ $K/Z(K)$ finite

Examples: K finite; K compact abelian Lie; their products

Non-product example: $(S^1 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$

with $\mathbb{Z}_2 = \{1, -1\}$, where $\mathbb{Z}_2 \circlearrowright S^1 \times \mathbb{Z}_2$ by $(a, \epsilon) \mapsto (\epsilon a, \epsilon)$.

Compact Lie groups of type HS act generically freely

K compact of type HS; $K \circlearrowright M$ faithful; $M/Z(K)$ connected .

Then K acts freely on the principal orbit type stratum M_{princ}.

Proof: on M_{princ}, all stabilizers are conjugate to some $H \subseteq K$.

$$N(H) \supseteq Z(K) \text{ and } K \text{ is of type HS} \implies K/N(H) \text{ is finite.}$$

For representatives $k_1, \ldots, k_r$ of the distinct cosets in $K/N(H)$,

$$M_{H_i} := \{\text{points with stabilizer } H_i\} \quad \text{for } H_i := k_i H k_i^{-1}.$$

$$M = \text{closure}(M_{H_1}) \sqcup \ldots \sqcup \text{closure}(M_{H_r}).$$

$M/Z(K)$ is connected and closure(M_{H_i}) are $Z(K)$-invariant $\implies$ $r = 1$;

action is faithful $\implies$ H is trivial.

Orbit-preserving equivariant smooth maps are diffeomorphisms

K compact $\circlearrowright M$;

$\psi\colon M \to M$ orbit-preserving K-equivariant smooth map.

Then ψ is a diffeomorphism.

Proof: *Step 1*: for $M = K/H$ homogeneous. *Step 2*: for $H \circlearrowright W$ linear.
Step 3: slice theorem + homogeneous case + linear case $\Longrightarrow$ general case

Theorem (stronger non-abelian version)

K compact of type HS; $K_0 \subseteq A \subseteq Z(K)$.

$K \circlearrowright M$ faithfully; M/A connected.

$\psi\colon M \to M$ orbit-preserving smooth map, equivariant w.r.t. an automorphism of K that is trivial on A.

Then $\exists$ smooth K-equivariant$^{(*)}$ $\eta\colon M \to K$ such that $\psi(x) = \eta(x) \cdot x \quad \forall x \in M$.

$^{(*)}$ with respect to twisted-conjugation on the target

Special case — Locally standard actions

$(S^1)^n \circlearrowright \mathbb{C}^n \times \mathbb{R}^l$

Orbit-preserving $(S^1)^n$-equivariant diffeomorphism:

$$\psi(z,t) \;=\; (\psi_1(z,t),\,\ldots,\,\psi_n(z,t);\; t_1,\,\ldots,\,t_l)$$

$\psi_i(z,t)$: $(S^1)_{i^{\text{th}}}$-equivariant; $(S^1)_{j^{\text{th}}}$-invariant $\forall j \neq i$
$\psi_i(x,t)$, x real: anti-symmetric in x_i; symmetric in x_j $\forall j \neq i$
Whitney (1943) $\Longrightarrow \exists$ smooth g_i such that

$$\begin{aligned} &\psi_i(x,t) = x_i g_i({x_1}^2,\ldots,{x_n}^2;t_1,\ldots,t_l) && \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}^l \\ \Longrightarrow\;\; &\psi_i(z,t) = z_i g_i(|z_1|^2,\ldots,|z_n|^2;t_1,\ldots,t_l) && \forall (z,t) \in \mathbb{C}^n \times \mathbb{R}^l \end{aligned}$$

orbit-preserving $\Longrightarrow |g_i| = 1$.

$$\eta_i(z,t) := g_i(|z_1|^2,\ldots,|z_n|^2;t_1,\ldots,t_l)$$

$\eta := (\eta_1,\ldots,\eta_n)$. Then $\psi(m) = \eta(m)\cdot m \quad \forall m = (z,t)$.

(e.g. Delzant, 1988)

Recall – Outline of proof of theorem

abelian:

(1) "all-linear" version

(2) infinitesimal version

(3) linear action; non-linear ψ

(4) action by automorphisms on vector bundle

(5) slice theorem $\Longrightarrow$ general (abelian) case

non-abelian:

(6) K finite

(7) finite $+$ abelian $\Longrightarrow$ general (type HS) case

(1) Abelian "all-linear" version of theorem

A compact abelian Lie; $A \circlearrowright W$ linear action;

$\psi\colon W \to W$ orbit-preserving A-equivariant linear isomorphism.

Then $\exists\, \gamma \in A$ such that $\psi(x) = \gamma \cdot x \;\; \forall x \in W$.

Sketch of proof:

$K_W :=$ kernel of the action

$W_{\text{princ}} :=$ principal orbit type stratum $=$ points with stabilizer K_W

- *Trivial action*: take $\gamma = 1$.
- *Irreducible action:* take γ s.t. $\psi(x) = \gamma \cdot x$ for some $x \in W_{\text{princ}}$.
- *Inductive step:* suppose $W = W_1 \oplus W_2$; assume the theorem holds for W_1 and for W_2; take γ such that $\psi(x+y) = \gamma \cdot (x+y)$ for some $x \in (W_1)_{\text{princ}}$ and $y \in (W_2)_{\text{princ}}$.

(2) Infinitesimal version of theorem

T torus. $T \circlearrowright W$ linearly. ξ_t smooth family of vector fields on W, everywhere tangent to orbits. Then
$\exists$ smooth $\alpha_t \colon W \to \mathrm{Lie}(T)$ such that $\xi_t(x) = \alpha_t(x) \cdot x \quad \forall x \in W$.

(3) Linear action; non-linear ψ (abelian group)

A compact abelian, $A \circlearrowright W$ linearly.
$\psi \colon W \to W$ orbit-preserving A-equivariant diffeomorphism.
"All linear" version $\Longrightarrow$ WLOG $d\psi|_0(x) = \mathrm{Id}$

$$\Longrightarrow \quad \psi_t(x) := \begin{cases} x & \text{if } t = 0 \\ \frac{1}{t}\psi(tx) & \text{if } t \in (0,1] \end{cases} \quad \text{is smooth.}$$

$\xi_t(\psi_t(x)) = \frac{d}{dt}\psi_t(x)$ defines time-dependent vector field ξ_t, tangent to orbits. Infinitesimal version $\Longrightarrow$ $\xi_t(y) = \alpha_t(y) \cdot y$ for some $\alpha_t \colon W \to \mathrm{Lie}(T)$. $\eta_t(x) := \exp \int_0^t \alpha_\tau(x) d\tau$ satisfies
$\psi_t(x) = \eta_t(x) \cdot x \quad \forall x \in W$.

(4) Version for vector bundles (abelian group)

A compact abelian, $H \subseteq A$ closed, $H \circlearrowright W$ linearly

$A \circlearrowright \Omega := A \times_H W \xrightarrow{\pi} A/H$

$\psi\colon \Omega \to \Omega$ orbit-preserving A-equivariant diffeomorphism.

Step 1: $\exists$ A-invariant smooth map $\hat{\eta}\colon \Omega \to A$ such that
$\pi(\psi(x)) = \hat{\eta}(x) \cdot \pi(x) \;\; \forall x \in \Omega$

Step 2: By Step 1, WLOG $\pi(\psi(x)) = \pi(x) \;\; \forall x \in \Omega$.
By the deformation argument, $\exists\ \eta_H\colon W \to H$ such that
$\psi([1, w]) = \eta_H(w) \cdot [1, w] \;\; \forall w \in W$. Then $\eta([a, w]) := a \cdot \eta_H(w)$
satisfies $\psi(x) = \eta(x) \cdot x \;\; \forall x \in \Omega$.

(5) General abelian case: follows by Koszul's slice theorem.

(6) Version for finite group

K finite $\circlearrowright M$ faithfully; $M/Z(K)$ connected.

$\psi \colon M \to M$ orbit-preserving smooth map.

Then $M = \cup_k C_k$ where $C_k := \{x \in M \mid \psi(x) = k \cdot x\}$

Baire category theorem $\implies$ $\cup_k$interior(C_k) is dense
$\implies$ $M = \cup_k$closure(interior(C_k)).

This union is disjoint. (On the intersection of the kth and k'th sets, the differentials of $x \mapsto k \cdot x$ and $x \mapsto k' \cdot x$ coincide with those of ψ, hence with each other; this implies $k = k'$.)

C_k are $Z(K)$-invariant and $M/Z(K)$ is connected $\implies \exists! k$ such that $C_k \neq \emptyset$. For it, $\psi(x) = k \cdot x \quad \forall x \in M$.

(7) General case

$K_0 \subseteq A \subseteq Z(K)$; $\quad M/A$ connected.

$M' :=$ principal orbit type stratum for $A \circlearrowright M$. Then $M' \subseteq M$ is open dense; $A \circlearrowright M'$ is free; M'/A is connected.

$\psi \colon M \to M$, K-orbit preserving, equivariant with respect to an automorphism of K that fixes A.

$\overline{\psi} \colon M'/A \to M'/A$, (K/A)-orbit preserving. Version for finite group $\Longrightarrow \exists\, \overline{\gamma} \in K/A$ such that $\overline{\psi}([x]) = \overline{\gamma} \cdot [x] \quad \forall [x] \in M'/A$.
$\gamma \in K$ representative for $\overline{\gamma} \in K/A$.

$\gamma^{-1}\psi \colon M \to M$ is A-equivariant and preserves A-orbits. Abelian case $\Longrightarrow \eta' \colon M \to A$ such that $\gamma^{-1}\psi(x) = \eta'(x) \cdot x \quad \forall x \in M$.
Take $\eta(x) = \gamma\eta'(x)$.

Fails for homeomorphisms

$S^1 \circlearrowright \mathbb{C}$.

$\psi(z) := e^{i/|z|} \cdot z$

$\nexists$ continuous $\eta\colon \mathbb{C} \to S^1$ such that $\psi(x) = \eta(x) \cdot x \ \ \forall x$

Fails without compactness

$\mathbb{R} \circlearrowright \mathbb{R}$ flow of $e^{-1/x^2} \frac{\partial}{\partial x}$.

$$\psi = \begin{cases} \text{time 1 map} & \text{on } [0, \infty) \\ \text{time } -1 \text{ map} & \text{on } (-\infty, 0] \end{cases}$$

$\nexists$ smooth $\eta\colon \mathbb{R} \to \mathbb{R}$ such that $\psi(x) = \eta(x) \cdot x \ \ \forall x$.

Fails without faithfulness

$S^1 \circlearrowright S^1 \times S^1$ by $a\colon (b,c) \mapsto (a^2 b, c)$.

$\psi(b,c) := (cb, c)$.

$\nexists$ smooth $\eta\colon S^1 \times S^1 \to S^1$ such that $\psi(x) = \eta(x) \cdot x \ \ \forall x$.

Fails for general K

$SO(3) \circlearrowright S^2$.

$\psi(x) := -x$, the antipode.

$\nexists$ smooth $\eta\colon S^2 \to \mathrm{SO}(3)$ such that $\psi(x) = \eta(x) \cdot x \ \ \forall x$.

Fails if only assume $\mathbf{K_0}$ =torus

$O(2) \circlearrowright \mathbb{R}^2 \times \mathbb{R}$ by $g \cdot (u, \xi) = (gu, (\det g)\xi)$.

$\psi(u,\xi) = (u, -\xi)$.

$\nexists$ smooth $\eta\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ such that $\psi(x) = \eta(x) \cdot x \ \ \forall x$.

Thank you!