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Applications of the Chang Skjelbred Lemma to positive and non-negative Curvature

Michael Wiemeler (joint with O. Goertsches / L. Kennard & B. Wilking)

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living.knowledge

A question and a problem.

Question

What are the topological implications of positive sectional curvature?

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Problem

Classify manifolds admitting metrics of positive sectional curvature.

Examples of positively curved manifolds.

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Examples of positively curved manifolds.

- There are only very few examples of manifolds admitting metrics of positive sectional curvature.
- For $\dim M > 24$ all known examples are diffeomorphic to S^n, $\mathbb{C}P^n$, or $\mathbb{H}P^n$.
- Other examples are known in low dimensions.
- These are certain homogeneous spaces and biquotient spaces.

Topological implications of positive curvature.

For closed manifolds M the following is known:

Classical results

- Theorem of Gauß-Bonnet: $\sec(M^2) > 0 \Rightarrow M$ is diffeomorphic to S^2 or $\mathbb{R}P^2$.

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- Theorem of Gauß-Bonnet: $\sec(M^2) > 0 \Rightarrow M$ is diffeomorphic to S^2 or $\mathbb{R}P^2$.
- Theorem of Synge: $\sec(M^{2n}) > 0 \Rightarrow |\pi_1(M)| \leq 2$.
- Theorem of Bonnet-Myers: $\mathrm{Ric}(M^n) > 0 \Rightarrow |\pi_1(M)| < \infty$.

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- Theorem of Bonnet-Myers: $\mathrm{Ric}(M^n) > 0 \Rightarrow |\pi_1(M)| < \infty$.
- Gromov's Betti number Theorem: $\sec(M^n) \geq 0 \Rightarrow \sum_i b_i(M) < C(n)$.

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If M is a closed, even-dimensional positively curved manifold, then the Euler characteristic of M is positive.

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- But there are the following conjectures:

Hopf's Conjecture I

If M is a closed, even-dimensional positively curved manifold, then the Euler characteristic of M is positive.

Hopf's Conjecture II

$S^2 \times S^2$ does not admit a positively curved metric.

Remarks.

- The first conjecture would imply that $S^{2n+1} \times S^{2n'+1}$ does not admit a positively curved metric.

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- The first conjecture is true in dimensions two and four.

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Grove's Programme

Classify simply connected positively/non-negatively curved manifolds with large isometry group first.

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Classify simply connected positively/non-negatively curved manifolds with large isometry group first.

- Here "Classify" can also mean "compute topological invariants of..."

Previous Results for $\sec > 0$

- Some classification results for positively curved, simply connected manifolds with an isometric action of a torus T

Authors	$\dim M$	$\dim T$	Result
Grove, Searle (1994)	n	$\left[\frac{n+1}{2}\right]$	diffeo class
Wilking (2003)	$n \geq 10$	$\geq \frac{n}{4} + 1$	homotopy class
Fang, Rong (2005)	$n > 7$	$\left[\frac{n-1}{2}\right]$	homeo class
Amann, Kennard (2014)	$2n$	$\geq 2 \log_2 2n + 2$	$2 \leq x(M) \leq 2^{3(\log_2 2n)^2}$

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- There are also classification results specific to low dimensions.
- Dessai (2007)/Weisskopf (2017) have results on elliptic genera of two-connected positively curved manifolds, where $\dim T$ is independend of $\dim M$.

Previous Results for $\sec \geq 0$

- Some classification results for non-negatively curved simply connected manifolds M with isometric action of a torus T.

Authors	$\dim M$	$\dim T$	Result
W. (2015)	$2n$	n, $M^T \neq \emptyset$	equivar. diffeo. class.
Escher, Searle (2017)	n	isotropy maximal action	equivar. diffeo. class.

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- There are also classification results specific to low dimensions.
- It is conjectured that the dimension of a torus acting on a simply connected non-negatively curved manifold M is bounded from above by $[\frac{2}{3} \dim M]$.

Main results I.

Theorem (Goertsches, W. 2015)

Let M be a positively curved manifold with $H^{odd}(M;\mathbb{Q}) = 0$ *which admits an isometric action of a torus of type* GKM_3.

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Then there is a non-negatively curved torus manifold $\tilde{M}$ and a finite group G acting isometricly on $\tilde{M}$ such that

$$H^*(M;\mathbb{Q}) \cong H^*(\tilde{M}/G;\mathbb{Q}).$$

The GKM method I.

Let M be a manifold with $H^{\text{odd}}(M;\mathbb{Q}) = 0$ with an action of a torus T. Then:

- $H_T^*(M) \to H^*(M)$ is surjective with kernel generated by $H^{>0}(BT)$.

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- The natural map $\iota^*\colon H_T^*(M) \to H_T^*(M^T)$ is injective.

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- The natural map $\iota^*\colon H_T^*(M) \to H_T^*(M^T)$ is injective.
- Hence, to compute $H^*(M)$, it suffices to understand the image of ι^*

The GKM method II.

Lemma (Chang and Skjelbred 1974)

Let M be a rational cohomology manifold with an action of a torus such that $H_T^*(M)$ is a free module over $H^*(BT)$. Then:
The image of

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Lemma (Chang and Skjelbred 1974)

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$$M_1 = \{x \in M \ \dim Tx \leq 1\}.$$

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- By Allday-Franz-Puppe (2014), the conclusion of the above Lemma holds if and only if $H_T^*(M)$ is a reflexive $H^*(BT)$-module, i.e. the natural map

$$H_T^*(M) \to \mathrm{Hom}_R(\mathrm{Hom}_R(H_T^*(M),\ R)\ R),$$

is an isomorphism, where $R = H^*(BT)$.

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Why is equivariant formality natural in presence of positive/non-negative curvature?

- By the Bott conjecture, a simply connected non-negatively curved manifold would be rationally elliptic.
- By Hopf's conjecture a positively curved manifold of even dimension, should have positive Euler characteristic.
- Combining these two conjectures implies that a positively curved manifold M^{2n} has $H^{\mathrm{odd}}(M;\mathbb{Q}) = 0$.

The GKM condition.

Definition

Let $k \geq 2$ and M be a closed, orientable manifold with an action of a torus T such that

1. $\dim M = 2n$ is even.
2. $H^{\text{odd}}(M) = 0$
3. M^T consists of finitely many isolated points $p_1, \dots, p_m$

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4. At every fixed point p_i, any k weights of $T_{p_i}M$ are linearly independent.

Then the action is called of type GKM_k.

GKM Graphs.

- Conditions 3 + 4 imply that for a GKM manifold M, M_1 is a union of two-dimensional invariant spheres.
- We can associate to M a labeled graph (Γ, a) as follows:

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  - vertices of Γ are the points in M^T
  - If p_1, p_2 are contained in the same two-sphere $\subset M_1$, then connect p_1, p_2 by an edge.
  - For each edge e in Γ, let $a(e) \in H^2(BT)$ dual to the principal isotropy group of corresponding two-sphere.

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  - For each edge e in Γ, let $a(e) \in H^2(BT)$ dual to the principal isotropy group of corresponding two-sphere.
- If M is GKM$_k$, $k > 2$, then the graph Γ has higher dimensional faces.

Positively/non-negatively curved GKM manifolds.

Lemma

If M is GKM_3 and admits an invariant metric of positive (non-negative, respectively) curvature, then the two-dimensional faces of Γ have at most 3 (4, resp.) vertices.

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- The two-dimensional faces of Γ are GKM-graphs of four-dimensional invariant submanifolds of M fixed by codimension two tori of T.
- Now, the claim follows from classification results for four-dimensional positively/non-negatively curved manifolds with continuous symmetry by Hsiang–Kleiner and Searle–Yang.

The positively curved case I.

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If M is GKM_3 and admits an invariant metric of positive curvature, then the two-dimensional faces of Γ have at most 3 vertices.

- It follows from this lemma, that Γ is a complete graph with multiple edges.

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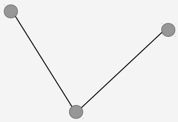

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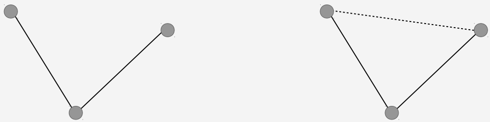

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- One can show that the weights are also the same as in one of these cases.

Main result I.

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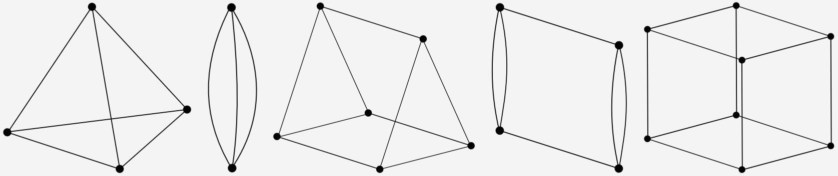
Non-negatively curved case I.

- In presence of of non-negative curvature and GKM_4 the 3-dimensional faces of Γ are of the following types: Δ^3, Σ^3, $\Delta^2 \times I$, $\Sigma^2 \times I$, I^3



GKM_4-manifolds with non-negative curvature.

- At each vertex p of Γ we have local product structure,

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Theorem

There is a normal covering

$$VEG(\prod_i \Delta^{n_i} \times \prod_j \Sigma^{m_j}) \to \Gamma$$

with finite deck transformation group G.

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- A GKM_n-graph labeling of $VEG(\prod_i \Delta^{n_i} \times \prod_j \Sigma^{m_j})$ defines torus manifold over $\prod_i \Delta^{n_i} \times \prod_j \Sigma^{m_j}$.
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  - Extend G-action on VEG to G-action on $\prod_i \Delta^{n_i} \times \prod_j \Sigma^{m_j}$, compatible with extended labeling.
- All this can be done.

Main result II.

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- Theorem also true for orientable orbifolds M, then $\tilde{M}$ is also an orbifold.
- Instead of M being non-negatively curved one can require M to be rationally elliptic to get the same conclusion.

Main result III.

Theorem (Kennard, W., Wilking 2019)

Assume $\sec(M^n) > 0$. *If M admits an isometric effective equivariantly formal action of* T^d, *with*

$$d \geq 8,$$

then M has the rational cohomology of S^n, $\mathbb{C}P^{\frac{n}{2}}$, *or* $\mathbb{H}P^{\frac{n}{4}}$.

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Tools for the proof I.

Theorem (Kennard, W., Wilking 2019)

Assume $\sec(M^n) > 0$ *and that there is an isometric effective action of a torus* T^d *of dimension*

$$d \geq 5.$$

Then every component F *of* M^T *has the rational cohomology of* S^m, $\mathbb{C}P^m$ *or* $\mathbb{H}P^m$.

Tools for the proof II.

Theorem (Kennard, W., Wilking 2019)

Let M^n be a equivariantly formal T-manifold such that for every subtorus $T' \subset T$ of codimensoin less than four and each component $F \subset M^{T'}$ we have

$$H^*(F;\mathbb{Q}) \cong H^*(S^m;\mathbb{Q}), H^*(\mathbb{C}P^m;\mathbb{Q}), H^*(\mathbb{H}P^m;\mathbb{Q}).$$

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Then

$$H^*(M;\mathbb{Q}) \cong H^*(S^n;\mathbb{Q}), H^*(\mathbb{C}P^{\frac{n}{2}};\mathbb{Q}), H^*(\mathbb{H}P^{\frac{n}{4}};\mathbb{Q}), H^*(\mathbb{O}P^2).$$

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Corollary

The isometry group of a potential positively curved metric on $S^{2n} \times S^{2n'}$ and $S^{2n} \times S^{2n'-1}$, $n' \leq n$ has rank at most seven.

Thank you!