BALANCE AND RIGIDITY CONDITIONS OF SUBVARIETIES IN THE COMPLEX TORUS

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Introduction.

According to good compactification theorem for any algebraic variety X in the complex torus there is a toric compactification $M \supset (\mathbb{C}^*)^n$ such that the closure $\overline{X}$ ox X in M does not intersect orbits in M whose dimension is smaller then codimension of X in M. If M_1 is a good compactification for X and $\pi : M_1 \to M$ is a proper equivariant map then M_1 also is a good compactification for X.

Problem. Fix a positive 0-divisor D at the union M^{n-k} of $n-k$ dimensional orbits of a complete toric variety $M \supset (\mathbb{C}^*)^n$. Under what condition on D there is an algebraic variety $Y \subset (\mathbb{C}^*)^n$ with $\dim Y = k$ which does not intersect orbits of dimension smaller then k whose intersection with M^{n-k} equal to D? If there are such varieties Y than describe all of them.

Let M be a complete complex toric variety.

About 25 year ago I completely solved this problem for the case $n = 2$, $k = 1$ dealing with algebraic curves in toric surface (see [1]). This solution provides an elementary proof of the classical Weil reciprocity law and to its converse statement (previously unknown). It also was one of the first results in tropical geometry.

The next step was made only very recently: I completely solved the this problem for the case when $n > 1$ is an arbitrary number and $k = n-1$ dealing with algebraic hypersurfaces in toric variety.

Conditions on zero divisor D necessary for solvability of the problem in these cases split into three groups: additive and multiplicative balance conditions and rigidity conditions.

In the talk I will present analogous necessary condition for general case and will explain why these conditions are also sufficient if $k = n-1$. This result could be considered as a natural continuation of [1].

Statements of results

For an orbit O of M we denote by $M(O)$ the complete toric variety equal to the closure of O in M. By $D(O)$ we denote the 0-cycle in $M(O)$ induced by the 0 cycle $A \in M \supset M(O)$.

Lemma. If $Y \subset M$ is a solution for Problem with 0-divisor D then $Y \cap M(O)$ is a solution for the induced Problem in toric variety $M(O)$ and the divisor $D(O)$.

For the case when $n > 1$ is an arbitrary number and $k = 1$ dealing with a curve in n-dimensional toric variety one can prove the following theorem.

Theorem on necessary conditions for curves For the solvability of the Problem for $k = 1$ the zero divisor D located at the union of $n - 1$ dimensional orbits has to satisfy the additive and multiplicative balance conditions and rigidity condition.

Corollary (on necessary conditions for general case). If the problem in the case $n > k \geq 1$ has at least one solution for zero divisor D then for any $n-k+1$ dimensional orbit O the induced zero divisor $D(O)$ has to satisfy the additive and multiplicative balance conditions and rigidity condition.

Theorem (on sufficient conditions for the case $k = n - 1$).

1) The problem has at least one solution Y if and only if for each two-dimensional orbit O the problem for the toric surface $M(O)$ and the 0-cycles $D(O)$ has at least one solution.

2) Moreover the intersection of any solution Y with the torus $(\mathbb{C}^*)^n$ can be defined by equation $Q = 0$ where Q is a Laurent polynomial whose Newton polyhedron Δ is defined up to a ship and can be found explicitly.

3) The coefficients of Q at monomials belonging to vertices and edges of Δ can be found explicitly up to a common factor.

4) All other coefficients of Q are arbitrary complex numbers.

Classical reciprocity laws.

The origin for balance conditions are the following classical reciprocity laws.

Theorem (Additive reciprocity law) For any meromorphic function f on a compact curve Γ the following condition holds:

$$\sum_{a\in\Gamma} ord_a f = 0.$$

Proof. Let $O\subset\Gamma$ be the set of poles and zeros of f. For each point $a\in O$ let γ_a be a small circle running around a in the contr clockwise direction. The the cycle $\gamma=\sum\gamma_a$ is equal to zero in $H_1(\Gamma\setminus O,\mathbb{Z})$. Thus

$$\frac{1}{2\pi}\int_\gamma \frac{df}{f}=0$$

Let f,g be a couple of meromorphic functions on Γ. Let u be a parameter about a point $a\in\tilde{\Gamma}$ such that $u(a)=0$. Assume that about the point a

$$f=bu^k+\dots,\ \ g=cu^m+\dots.$$

To the couple (f,g) and a point $p\in\tilde{\Gamma}$ one associate:

the **Weil symbol** $[f,g]_a=(-1)^{km}b^mc^{-k}\in\mathbb{C}^*$.

The following reciprocity law was discovered by A.Weil.

Theorem (Multiplicative reciprocity law) For any meromorphic functions f,g on a compact curve Γ the following condition holds:

$$\prod_{a\in\Gamma}[f,g]_a=1.$$

One can prove this reciprocity law in very different ways. One of its proof follows from the Abel.s type theorem which I will discuss later.

Another proof is topological: one can defined an element in $H^1(\Gamma\setminus O,\mathbb{Z})$ with values in $\mathbb{C}^*$ whose value on the cycle γ_a equals to $[f,g]_a$. The law follows now from the equality $\sum\gamma_a=0$ in $H_1(\Gamma\setminus O,\mathbb{Z})$. I use this topological approach in the talk.

Additive balance condition.

Consider the problem for general case $n > k \geq 1$ for the 0 -cycle D whose support belongs to the union of $(n-k)$ dimensional orbits. For any $(n-k)$ dimensional orbit O_α denote by $\mu_\alpha(D)$ the degree of the induced divisor $D(O_\alpha)$.

The closure $M(O)$ of any $(n-k+1)$ dimensional orbit O is $(n-k+1)$ dimensional variety. Any $(n-k)$ dimensional orbit O_α in $M(O)$ (which is automatically an orbit in M corresponds an irreducible integral vector v_α belonging to the lattice of one parameter groups in O.

Theorem (on additive balance condition) If the problem has at least one solution for a zero divisor D then it satisfies the following additive balance condition: for any $(n-k+1)$ dimensional orbit O the relation

$$\sum \mu_\alpha(D) v_\alpha = 0$$

holds, where the sum is taken over all $(n-k)$ dimensional orbits O_α in $M(O)$.

Graph with marked complex edges.

Let $M^1 = \{(O_i, A_i, B_i, T_i)\}$ be a finite collection of Riemann spheres O_i named **edges** with fixed three points:

two points A_i, B_i named **vertices** and a **marked point** T_i.

Let us choose some couples of spheres O_i,O_j and some identification of one vertex of O_i with one vertex of O_j or both vertices of O_i with both vertices of O_i.

The collection M^1 with the above identification we will consider as a singular complex curve Γ^1. It also could be considered as a graph with complex edges $\{O_i\}$ with the set of marked points T and the set of vertices $V \subset \Gamma^1$ obtained from the set $\cup\{A_i, B_i\}$ by the above identification.

Choose an order on the set of vertices on the sphere O_α at denote these vertices by v_α^1, v_α^2. The choice order allows to identify O_α with $\mathbb{C}P^1$ by the **natural isomorphism** $\rho_\alpha^{v_\alpha^1} : O_\alpha \to \mathbb{C}P^1$ such that

$$\rho_\alpha^{v_\alpha^1}(v_\alpha^1) = 0,\ \rho_\alpha^{v_\alpha^1}(v_\alpha^2) = 1,\ \rho_\alpha^{v_\alpha^1}(T_\alpha) = \infty.$$

The inverse map $u_\alpha^{v^1\alpha}$ to the map $\rho_\alpha^{v^1\alpha}$ we call a **natural parametrization** of O_α (there are two natural parametrization of the edge corresponding to two orderings of its vertices).

Abel's type theorem

A function $f : \Gamma^1 \to \mathbb{C}P^1$ is **allowed** if its restriction f_α on each edge $O_\alpha \sim \mathbb{C}P^1$ is a nonzero rational function and at each vertex v the following condition holds. Let u^v_α be the natural parametrization of an edge O_α containing v (such that $u^v_\alpha(0) = v$). Let C_α be the coefficient at the smallest degree of u^v_α in Laurent series of f_α at v. Then C_α **has to be independent of the choice of edge** O_α**, i.e.** $C_\alpha = C(v)$.

An allowed function is called **Laurent polynomial** if its restriction on any edge has no poles away from the vertices on the edge.

The only invertible Laurent polynomials are **monomials**, i.e. are functions $C \prod u_\alpha^{k_\alpha}$ where $C \neq 0$ is a constant, u_α are natural parameters and $k_\alpha \in \mathbb{Z}$.

The **principal divisor** (f) of an allowed function f is the divisor $\sum (ord_a f)a$ where the sum is taken over all points $a \in \Gamma \setminus V$. By a 0 divisor D on Γ^1 we mean a formal sum $\sum \mu(a)a$ where the function μ takes values in $\mathbb{Z}$ and has nonzero values at a finite set not intersecting V.

A **cycle** C in Γ^1 is a sequence of edges $O_1 \ , \dots, O_k$ and vertices $v_1, \dots, v_k$ such that $v_i \in O_i \cap O_{i+1}$ for $i = 1, \dots k-1$ and $v_k \in O_k \cap O_1$ With this cycle C one associate the set of identifications $\rho_i^{v_i} : O_i \to \mathbb{C}P^1$ of its edges with $\mathbb{C}P^1$.

Definition. A zero divisor $D = \sum \mu(a)a$ satisfies the multiplicative balance condition for the cycle $C = \{O_1, \dots, O_k; v_1, \dots, v_k\}$ if the following relation holds

$$\prod_{O_i \in C} \Big(\prod_{a \in O_i} (-\rho_i^{v_i}(a))^{\mu(a)} \Big) = 1.$$

Theorem (of Abel's type). A 0 divisor D in the graph Γ^1 is the principal divisor of an allowed function $f : \Gamma^1 \to \mathbb{C}$ if and only if D satisfies the multiplicative balance condition for every cycle in Γ^1.

If the condition is satisfied then f can be present explicitly and it is unique up to multiplication by a monomial $C \prod u_\alpha^{k_\alpha}$.

Multiplicative condition for hypersurfaces.

Consider the problem for the case $k = n - 1$ for the 0 -cycle D whose support belongs to the union of one dimensional orbits. For any two dimensional orbit O_α denote by $\mu_\alpha(D)$ the degree of the induced divisor $D(O_\alpha)$.

Each **one dimensional orbit** O_α has a marked point T_α which is equal to the image of $e \in (\mathbb{C}^*)^n$ under the factorization map. Its closure is isomorphic to $\mathbb{C}P^1$ and contains two null-orbits (A_α, B_α). Thus the closure of the union of one dimensional orbits has a natural structure of the graphΓ_1 with complex market edges.

Each **two dimensional orbit** O defines the cycle $C(O)$ in the graph Γ_1 containing all one-dimensional orbits belonging to the closure $M(O)$ of O: the set of one dimensional orbit in $M(O)$ has a natural cyclic order $Q_1, \dots, O_k$ Such cycles generate all cycle in Γ_1.

Theorem (on multiplicative balance condition for hypersurfaces) If the problem has at least one solution for a zero divisor D on the union of one dimensional orbits then for any two dimensional orbit O it satisfies the following multiplicative balance condition:

$$\prod_{O_i \in C(O)} \Big(\prod_{a \in O_i} (-\rho_i^{v_i}(a))^{\mu(a)} \Big) = 1.$$

Below we will state the multiplicative balance condition and the rigidity condition. To state the multiplicative balance condition some preparation is needed.

Consider a compact complex curve Γ and its meromorphic map $(f, g) : \Gamma \to (\mathbb{C}^*)^2$. The image of Γ is a curve in $(\mathbb{C}^*)^2$. This curve has to satisfy an equation $Q = 0$ where Q is a Laurent polynomial. Let $\Delta(Q)$ be it's Newton polygon, and let M_Δ be the toric surface associated with Δ.

Corollary. Applying the Abel's type theorem to M_Δ and to the image of the curve Γ one can obtain a proof of Weil reciprocity law for Γ and f, g (see [1]).

Interior product of vector with 2-form

Let ω be a 2-form and let w be a vector. The **interior product** of w with ω is the 1-form $i_w\omega$ whose value on a vector v is equal to

$$\omega(w \wedge v).$$

If $\omega = \sum p_{i,j} dx_i \wedge dx_j$ and $w = \sum w_i(\partial/\partial x_i)$ then $i_w\omega = \sum p_{i,j}(w_i dx_j - w_j dx_i)$. The definition implies that $i_w\omega(w) = 0$.

Let ω be a integral 2-form on the space N of one-parameter groups of the torus$(\mathbb{C}^*)^n$. i.e. let ω have integral values on a product $w \wedge v$ where $w, v \in \Lambda^*$. Then for any $w \in \Lambda^*$ the 1-form $i_w(\omega$ is a character, i.e.

$$i_w(\omega \in \Lambda.$$

The definition implies that the covector $i_w\omega$ vanish on any vector v proportional to w. Thus it is a character at a factor group $(\mathbb{C}^*)^n/H$ where H is a one parameter grout whose Lie algebra contains the vector w.

Integral 2-vector over $\mathbb{Z}/2\mathbb{Z}$ and a balanced set of integral vectors

Let $\{w_\alpha\}$ be a balanced set of integral vectors in N, i.e. such a set that that $\sum w_\alpha = 0$ and let ω be an integral 2-form.

Consider each vector w_α as a 1-cycle γ_α in $H^1(T^n, \mathbb{Z})$ where $T^n = \mathbb{R}^n/\Lambda$. Since $\sum w_\alpha = 0$ the cycle $\sum \gamma_\alpha$ is homological to zero in T^n, thus $\sum w_\alpha = \partial\sigma$ where σ is a 2-chain. The chain σ is defined up to addition of an integral 2-cycle σ. The value of ω on σ is an integral number. Thus we proved the following lemma.

Lemma $\int_\sigma \omega$ is a well defined element of $\mathbb{R}/\mathbb{Z}$.

Easy to check that this integral is a halfintegral number. O

Take any rational plane L in the space N of one parameter groups. Consider a projection $\pi : N \to L$ whose kernel is a complimentary to L rational space of dimension $(n-2)$. The set $\{\pi(w_\alpha\}$ is a balanced set of integral vectors in L. Since $\sum \pi(w_\alpha) = 0$ there a polygon $\pi(\sigma) \subset L$ whose sides are shifted vectors $\pi(w_\alpha$. By Pick formula

$$2\int_{\pi(\sigma)} \omega = 2\#(\sigma \cap \Lambda^*) - \#(\partial\sigma \cap \Lambda^*) + 2$$

where ω is the integral volume form and Λ^* is the integral lattice in L.

Corollary. $2\pi(\sigma)$ is an integral 2- vector on L. Modulo 2 it is equal to the sum of integral length of the vectors $\pi(w_\alpha$ multiplied by the standard integral 2-vector v_L on L. Since $\sum \pi(w_\alpha = 0$ it is also equal mod 2 to the sum of products of coordinats of vectors $\pi(w_\alpha$ multiplied by v_L.**Thus 2σ is an integral 2-vector explicitly computed modulo 2**.

Corollary. For any integral 2-form ω the following identity holds:

$$\exp(2\pi i \int_\sigma \omega) = (-1)^{\langle 2\sigma, \omega\rangle}.$$

Multiplicative balance condition

Let M be an n dimensional toric variety. Let D be a positive 0-cycle located at the union of $(n-1)$-dimensional orbits O_α of M. Denote by v_α the irreducible vector in the fan of M corresponding to O_α. Let μ_α be the degree of D at O_α. Let w_α be $\mu_\alpha v_\alpha$. Let C_α be the product of $A \cap O_\alpha$ in the factor group O_α.

Theorem (multiplicative balance conditions for $k = 1$). If the problem for D has at least one solution then for any integral 2-form ω on the space of one-parameter groups the following condition holds

$$\prod_{O_\alpha} i_{w_\alpha}\omega(C_\alpha) = \exp(2\pi i \int_\sigma \omega) = (-1)^{\langle 2\sigma, \omega \rangle}$$

where 2σ is an integral modulp 2 vector defined above.

Theorem (multiplicative balance conditions for $n > k \geq 1$). If the problem for D located on $(n-k)$ dimensional orbits of M has at least one solution then for any $(n-k+1$ dimensional orbit O the induced divisor $D(O)$ fohas to satisfy the above multiplicative balance conditions.

Rigidity condition

Let M be an n dimensional toric variety. Let D be a positive 0-cycle located at the union of $(n-1)$-dimensional orbits O_α of M. Denote by v_α the irreducible vector in the fan of M corresponding to O_α. Let μ_α be the degree of A at O_α.

Theorem (rigidity condition for the case $n > k = 1$. Assume that the multiplicities μ_α are not equal to zero exactly for two orbits, corresponding to the vectors v_1 and v_2.

Then $v_1 = -v_2$, or $v_i = \pm v$ for $i = 1, 2$ and the curve is a finite sum of a congruence class of curves $\Gamma_{\pm v}$ taken with integral coefficients, where $\Gamma_{\pm v}$ is the one-parameter group whose generating vector is proportional to v.

Under the factorization by $\Gamma_{\pm v}$ the curve became the 0-cycle on $D(O_{v_1})$ and on $D(O_{v_2})$ induced from the cycle D on corresponding orbits.

Theorem (rigidity conditions for $n > k \geq 1$). If the problem for D located on $(n-k)$ dimensional orbits of M has at least one solution then for any $(n-k+1$ dimensional orbit O the induced divisor $D(O)$ fohas to satisfy the above rigidity conditions.

Necessary conditions are sufficient for the case $k = n - 1$.

Let H be a positive Weil divisor in a complete toric variety $M \supset (\mathbb{C}^*)^n$. The intersection $H \cap (\mathbb{C}^*)^n$ can be defined by $Q = 0$. Newton polyhedron $\Delta(Q) = \Delta_Q$ of Q is defined up to a shift.

Lemma. If H not passing through null orbits of M, then the support function of Δ_Q is linear at each cone of the fan $\mathcal{F}_M$ of M.

Corollary. H is a Cartier divisor. Its equation in a toric affine chart is $\chi Q = 0$, where χ is an appropriate character (whose support function on the cone $\sigma \in \mathcal{F}_M$ corresponding to the affine chart is equal to the negative support function of Δ_Q.

Theorem. A divisor H having intersection numbers μ_α with orbits O_α exists if and only if μ_α satisfies the additive balance condition. If the condition is satisfied then there is a unique up to a shift polyhedron Δ such that H is defined in $\mathbb{C}^*)^n$ by an equation $Q = 0$ with $\Delta(Q) = \Delta$.

2) If D satisfies the multiplicative balance condition then one can construct a unique (up to a simple factor) allowed function on M^1 whose principal divisor is D.

3) If in addition D satisfies a rigidity condition then one can "push forward" the function f and defined (up to a constant factor) all coefficients of Laurent polynomial Q at the integral points belonging to edges and vertices of Δ.

4) All other coefficients of Q are arbitrary complex numbers.

REFERENCES.

1. A. Khovanskii. Newton polygons, curves on torus surfaces, and the converse Weil theorem, Russian Math. Surveys 52 (1997), no. 6, 1251–1279.