Symplectic cohomological rigidity through toric degenerations

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Cohomological rigidity

Let $\mathcal{F}$ be a family of manifolds.
Fix $M, M' \in \mathcal{F}$.

If M, M' are diffeomorphic, then $H^*(M;\mathbb{Z}) \simeq H^*(M';\mathbb{Z})$ (as rings).

Question:
Does $H^*(M;\mathbb{Z}) \simeq H^*(M';\mathbb{Z})$ imply that M, M' are diffeomorphic?

If the answer is YES, then $\mathcal{F}$ is **cohomologically rigid**.

Examples:

- Surfaces are cohomologically rigid.
- Let M, M' be closed simply connected 4 manifolds.
  Then $H^*(M;\mathbb{Z}) \simeq H^*(M';\mathbb{Z})$ implies that M, M' are homeomorphic, but not diffeomorphic. [Freedman]

Toric Varieties

A **toric variety** is

- a irreducible closed variety X of complex dimension n, and
- a $(C^\times)^n$ action on X with a dense orbit.

Example: Hirzebruch surfaces

- $\Sigma_m := \mathbb{P}(\mathbb{C} \oplus \mathcal{O}(-m)) \to \mathbb{P}^1$ is a $\mathbb{P}^1$ bundles over $\mathbb{P}^1$.

Example: Bott manifolds

- Let $X_1 = \mathbb{P}^1$
- Given a holomorphic line bundle $L \to X_{n-1}$,
  $X_n := \mathbb{P}(L \oplus \mathbb{C})$ is $\mathbb{P}^1$ bundle over X_{n-1}.

Cohomological rigidity for toric varieties?

Smooth toric varieties are:

- easy to classify up to *equivariant* diffeomorphism, e.g. by **fans**;
- harder to classify up to diffeomorphism.

Theorem (Masuda, 2008)

Let X, X' be smooth algebraic varieties.
If there's an isomorphism $H^*_{(S^1)^n}(X;\mathbb{Z}) \to H^*_{(S^1)^n}(X';\mathbb{Z})$ with $c_1(X) \mapsto c_1(X')$, then X, X' are equivariantly diffeomorphic.

Question: (Masuda-Suh)
Are smooth toric varieties cohomologically rigid?

Some results

Theorem (Masuda-Panov, 2008)

If X is a Bott manifold and $H^*(X;\mathbb{Z}) \simeq H^*((\mathbb{P}^1)^n;\mathbb{Z})$, then X is diffeomorphic to $(\mathbb{P}^1)^n$.

Theorem (Choi-Masuda, 2012)

Let X, X' be Bott manifolds with $H^*(X;\mathbb{Q}) \simeq H^*(X';\mathbb{Q}) \simeq H^*((\mathbb{P}^1)^n;\mathbb{Q})$. If $H^*(X;\mathbb{Z}) \simeq H^*(X';\mathbb{Z})$, then X, X' diffeomorphic.

Cohomological rigidity holds in other special cases.
[Cho, Choi, Lee, Masuda, Panov, Park, Suh]

There are no known counterexamples.

Symplectic cohomological rigidity

Let $\mathcal{G}$ be a family of symplectic manifolds.
Fix $(M, \omega), (M', \omega') \in \mathcal{G}$.

If M, M' are symplectomorphic, there's an isomorphism $H^*(M; \mathbb{Z}) \to H^*(M'; \mathbb{Z})$ with $[\omega] \mapsto [\omega']$.

Question:
Does an isomorphism $H^*(M; \mathbb{Z}) \to H^*(M'; \mathbb{Z})$ with $[\omega] \mapsto [\omega']$ imply that M is symplectomorphic to M'?

If the answer is YES, then $\mathcal{G}$ is **symplectically cohomologically rigid**.

Example:

- Symplectic surfaces are symplectically cohomologically rigid.

Symplectic toric manifolds

A **symplectic toric manifold** is

- a $2n$-dimensional closed, connected manifold M;
- an integral symplectic form ω;
- a faithful $(S^1)^n$ action on M inducing $\xi_j \in \chi(M)$ for all $1 \leq j \leq n$;
- a moment map $\mu\colon M \to \mathbb{R}^n$, i.e., $\iota_{\xi_j}\omega = -d\mu_j$ for all j.

The **moment polytope** $\Delta := \mu(M)$ is a convex polytope.

Example:

- Any smooth projective toric variety $X \hookrightarrow \mathbb{P}^n$.
- In particular, every Bott manifold: **symplectic Bott manifold**.

Symplectic Bott manifolds

Let M be a symplectic Bott manifold with moment polytope Δ.

Then Δ is combinatorially equivalent to a hypercube; moreover, there's a strictly upper triangular integral matrix A and $\lambda \in \mathbb{Z}^n$ such that:

$$\Delta = \{p \in \mathbb{R}^n \mid \langle p, e_j \rangle \geq 0 \text{ and } \langle p, e_j + \sum_i A^i_j e_i \rangle \leq \lambda_j \ \forall j\}.$$

Additionally,

$$H^*(M;\mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_n] / \big(x_i^2 + \textstyle\sum_j A^i_j x_i x_j\big) \quad \text{and} \quad [\omega] = \textstyle\sum_i \lambda_i x_i,$$

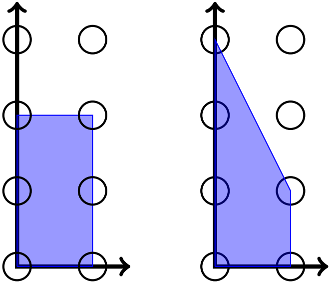
where x_j is dual to the preimage of the facet $\Delta \cap \{\langle p, e_j + \sum_i A^i_j e_i \rangle = \lambda_j\}$.

Symplectic Hirzebruch surfaces

The moment polytope of $\Sigma_m := \mathbb{P}(\mathbb{C} \oplus \mathcal{O}(-m))$ is a trapezoid;

$$H^*(M;\mathbb{Z}) = \mathbb{Z}[x_1, x_2]/\left(x_2^2, x_1^2 + m x_1 x_2\right).$$

Example: $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and Σ_2.



Symplectic rigidity for toric manifolds?

Symplectic toric manifolds are:

- easy to classify up to *equivariant* symplectomorphism;
- harder to classify up to symplectomorphism.

M, M' are equivariantly symplectomorphic exactly if $\Delta = \Delta' + c$. [Delzant]

Question:
Are symplectic toric manifolds symplectically cohomologically rigid?

Theorem (McDuff, 2011)

If M is a symplectic toric manifold and $H^*(M;\mathbb{Z}) \simeq H^*(\mathbb{P}^i \times \mathbb{P}^j;\mathbb{Z})$, then M is symplectomorphic to $\mathbb{P}^i \times \mathbb{P}^j$.

Other partial results. [Karshon, Kessler, Pinsonnault, McDuff]

Our main results

Theorem (Pabiniak-T)

Let M, M' be symplectic Bott manifolds with $H^*(M;\mathbb{Q}) \simeq H^*(M';\mathbb{Q}) \simeq H^*((\mathbb{P}^1)^n;\mathbb{Q})$. If there's an isomorphism $H^*(M;\mathbb{Z}) \to H^*(M';\mathbb{Z})$ with $[\omega] \mapsto [\omega']$, then M, M' are symplectomorphic.

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Corollary

If M is a symplectic toric manifold and $H^*(M;\mathbb{Z}) \simeq H^*((\mathbb{P}^1)^n;\mathbb{Z})$, then M is symplectomorphic to $(\mathbb{P}^1)^n$ with symplectic form $\omega_\lambda := \sum_i \lambda_i \pi_i^*(\omega_{\mathsf{FS}})$.

Proof of corollary:

By a result of Masuda and Panev, M is a symplectic Bott manifold.

Note: Strong rigidity also holds.

Proof of the main theorem

The key step is to construct new symplectomorphisms:

Otherwise, the proof is similar to the smooth case.

Proposition

Let M and M' be symplectic Bott manifolds.
Assume there exist $k < \ell$ and an isomorphism $H^*(M;\mathbb{Z}) \to H^*(M';\mathbb{Z})$ with $[\omega] \mapsto [\omega']$, $x_k \mapsto x'_k - \gamma x'_\ell$ for some $\gamma \in \mathbb{Z}$, and $x_i \mapsto x'_i$ for all $i \neq k$.
Then M, M' are symplectomorphic.

So we need to prove this proposition.

Toric degenerations

Let $X \subset \mathbb{P}^N$ be a smooth projective variety.

Fix a local coordinate system on X.

There's an associated semigroup $S = \cup_{m>0}\{m\} \times S_m \subset \mathbb{Z} \times \mathbb{Z}^n$.

The **Okounkov body** is $\Delta := \overline{\operatorname{conv} \cup_{m>0} \frac{1}{m} S_m}$.

Theorem (Harada-Kaveh, 2015)

Assume S is finitely generated.
$X_0 := \operatorname{Proj} \mathbb{C}[S]$ *is a projective toric variety with moment polytope* Δ.
There's a a continuous surjective map $\Phi\colon X \to X_0$ *that's a symplectomorphism on an open dense subset of* X.

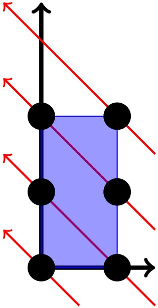
Key observation: If X_0 is smooth Φ is a symplectomorphism.

Idea of proof: Construct a toric degeneration of X,
i.e., a flat family $\pi\colon \mathcal{X} \to \mathbb{C}$ with generic fiber X and $\pi^{-1}(0) = X_0$.
Lift a radial vector field to construct flow.

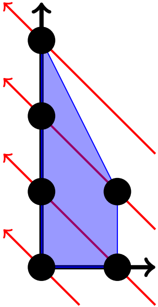
The "slide" operator

Fix $w \in \mathbb{Z}^n \setminus \mathbb{Z}^n_{\geq 0}$. Construct the **slide** of $Q \subseteq \mathbb{Z}^n_{\geq 0}$ along w by sliding each point as far as possible within $\mathbb{Z}^n_{\geq 0}$ in the direction w.

Example:



Slide in direction $-e_1 + e_2$.



Proposition

Let M, M' be symplectic Bott manifolds with moment polytopes Δ, Δ'. Assume there exists $k < \ell$ and an isomophism $H^*(M;\mathbb{Z}) \to H^*(M';\mathbb{Z})$ with $[\omega] \mapsto [\omega']$, $x_k \mapsto x'_k - \gamma x'_\ell$ for some $\gamma \in \mathbb{Z}$, and $x_i \mapsto x'_i$ for all $i \neq k$. If $(A')^k_\ell \geq |A^k_\ell|$, then there exists $c \geq 0$ so that

$$\mathcal{S}_{-e_k+ce_\ell}(m\Delta \cap \mathbb{Z}^n) = m\Delta' \cap \mathbb{Z}^n \quad \forall m \in \mathbb{Z}_{>0}.$$

Proposition

Let M, M' be symplectic toric manifolds with moment polytopes Δ, Δ' that are equal to $\mathbb{R}^n_{\geq 0}$ near 0. If there exists $k < \ell$ and $c \geq 0$ such that

$$\mathcal{S}_{-e_k+ce_\ell}(m\Delta \cap \mathbb{Z}^n) = m\Delta' \cap \mathbb{Z}^n \quad \forall m \in \mathbb{Z}_{>0},$$

then M is symplectomorphic to M'.