

Some modified toric constructions and their applications to cobordisms

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• Some Notation

M : free \mathbb{Z} -module of $\text{rk}(M) = n$, $M \cong \mathbb{Z}^n$

$M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$

$T_M := M_{\mathbb{R}} / M \cong \mathbb{T}^n$

For $K \leq M$: submodule of $\text{rk}(K) = k \leq n$

$\tilde{K} := K_{\mathbb{R}} \cap M \cong \mathbb{Z}^k$ ($\Rightarrow K \leq \tilde{K}$ with $[\tilde{K}:K] < \infty$)

$\iota_K: K \hookrightarrow \tilde{K}$ induces

$\xi_K: T_K \longrightarrow T_{\tilde{K}}$ surjective (covering) homom. with

$\text{Ker}(\xi_K) \cong \tilde{K}/K$

(Note that $T_K = K_{\mathbb{R}}/K \cong \mathbb{T}^k$, $T_{\tilde{K}}: \tilde{K}_{\mathbb{R}}/\tilde{K} \cong \mathbb{T}^k$)

$\iota_{\tilde{K}}: \tilde{K} \hookrightarrow M$ induces

$\xi_{\tilde{K}}: T_{\tilde{K}} \longrightarrow T_M$ an injective homom.

- Locally standard torus orbifold (l.s.t.o)

Def A. compact conn. effective $2n$ -dim orbifold X is a locally standard torus orbifold (l.s.t.o) if \exists effective action of $T_M (\cong T^n)$ on X s.t $\forall x \in X$, there exist

(1) T_M -stable neighborhood U

(2) $N \leq M$ with $\text{rk}(N) = n$ ($\varphi: N \hookrightarrow M$)

with $\zeta_N: T_N \rightarrow T_M$ the induced surj. homom

(3) (\tilde{U}, G, ψ) : an orbifold chart over U

with $G = \text{Ker } \zeta_N$,

U : weakly-eq. diffeom to C^n (the standard T^n -space)

$\psi: \tilde{U} \rightarrow U$: ζ_N -eq.map which induces

$\bar{\psi}: \tilde{U}/G \xrightarrow{\cong} U$: a homeomorphism.

Remarks

- (1) When $G = \{e\}$ for each orbifold chart,
 $\Rightarrow X$ is a locally standard torus manifold (l.s.t.m)
- (2) $X : l.s.t.o$
 $\Rightarrow P = X/T^n$ is a nice manifold with corners
In this case, X is called a l.s.t.o over P
- (3) l.s.t.o generalizes the notion of quasitoric orbifold studied by Poddar - Sarkar
- (4) l.s.t.o with boundary is defined in the usual manner.

Note 1 .

X : $2n$ -dim l.s.t.o over P (nice n -mfld with corners)

with $\tilde{\pi}_\omega: X \rightarrow P$ the orbit map.

$F_i \in \widetilde{\mathcal{F}}(P)$: a facet of P .

For $\forall x \in \tilde{\pi}_\omega^{-1}(F_i) \subset X$, its isotropy subgp T_{M_x} is a circle subgroup of T_M .

$\Rightarrow T_{M_x}$ determines $\pm \lambda_i \in M$ (not nec. primitive)

Def. A rational charac.fun (r.ch.fun) on P is

$$\lambda: \widetilde{\mathcal{F}}(P) \longrightarrow \mathbb{Z}^n \text{ s.t}$$

whenever $F_{i_1} \cap \dots \cap F_{i_k} \neq \emptyset$ for $F_j \in \widetilde{\mathcal{F}}(P)$

$\{\lambda(F_{i_1}), \dots, \lambda(F_{i_k})\}$ is lin. indep.

Note 2.

X : l.s.t.o with orbit map $\pi_U: X \rightarrow P$

$\dot{P} := P \setminus \text{small collar nbd of } \partial P$.

$\Rightarrow \pi_U|: \pi_U^{-1}(\dot{P}) \longrightarrow \dot{P}$ is a principal T^n -bundle

There exists a diffeom $\varphi: P \rightarrow \dot{P}$.

Let $\mu: E_X \rightarrow P$ be the following principal T^n -bundle

$$E_X := \varphi^*(\pi_U^{-1}(\dot{P})) \longrightarrow \pi_U^{-1}(\dot{P})$$

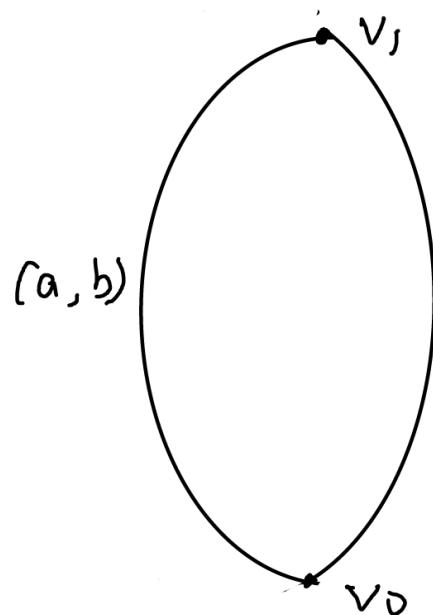
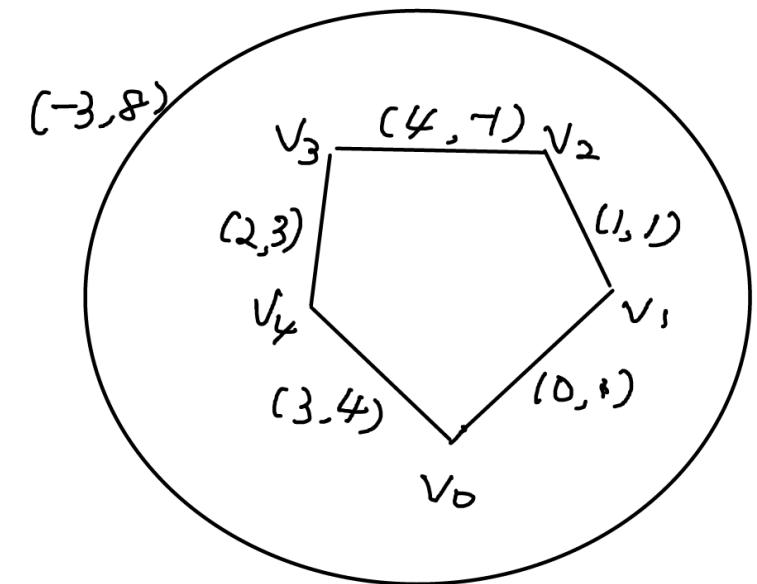
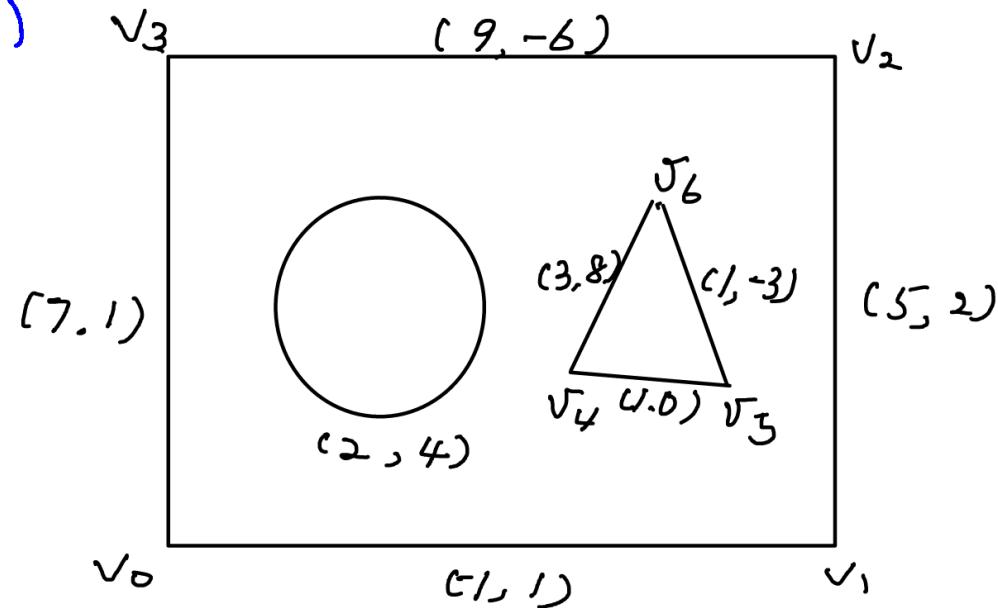
$$\begin{array}{ccc} & & \\ \mu \downarrow & & \downarrow \pi_U | \\ P & \xrightarrow{\varphi \cong} & \dot{P} \end{array}$$

Given a l.s.t.o X^n over P , we have

(1) $\lambda: F(P) \longrightarrow \mathbb{Z}^n$, an r-ch.fun.

(2) $\mu: E_X \longrightarrow P$, a principal T^n -bundle .

(E.g)



(c, d) with $ad - bc \neq 0$

(called an eye-shape)

[2-dim nice manifolds with corners & r-ch funs]

- Construction of l.s.t.o from r-ch.fun. and a principal T^n -bundle over P .

P : cpt. conn nice n-dim mfd with corners
 $\lambda : \tilde{F}(P) \longrightarrow \mathbb{Z}^n$: r-ch. fun
 $\mu : E \longrightarrow P$: principal T^n -bundle over P .
 F : codim k face of P .

$\Rightarrow \exists! \{F_{i_1}, \dots, F_{i_k}\} \subseteq \tilde{F}(P)$ s.t. $F = \text{a comp. of } F_{i_1} \cap \dots \cap F_{i_k}$

$$K(F) := \langle \lambda(F_{i_1}), \dots, \lambda(F_{i_k}) \rangle \leq M$$

$$\tilde{K}(F) := K(F)_R \cap M$$

$\gamma_{K(F)} : K(F) \xrightarrow{\quad} \tilde{K}(F)$ induces a surj. homom

$$\zeta_{K(F)} : T_{K(F)} \longrightarrow T_{\tilde{K}(F)}$$

$$G_F := \tilde{K}(F)/K(F) \cong \text{Ker}(\zeta_{K(F)})$$

$$\zeta_{\tilde{K}(F)} : T_{\tilde{K}(F)} \xrightarrow{\quad} T_M$$

$$T_F := \text{Im}(\zeta_{\tilde{K}(F)})$$

Define $X(P, \lambda, \mu) := E/\sim_\lambda$

where $x \sim_\lambda y$ in E

$$\Leftrightarrow \mu(x) = \mu(y)$$

& $x = ty$ for some $t \in T_F$

Define $\pi: X(P, \lambda, \mu) \longrightarrow P$, $[x] \longmapsto \mu(x)$.

Then $T^n \cap X(P, \lambda, \mu)$ & $P \cong X(P, \lambda, \mu)/T^n$.

Lemma 1 $X(P, \lambda, \mu)$ is a smooth l.s.t.o over P
with smooth orbit map $\pi: X(P, \lambda, \mu) \longrightarrow P$.

□

Notation. When μ is a trivial T^n -bundle, we
drop μ and write $X(P, \lambda)$.

Theorem 2. X : $2n$ -dim l.s.t.o. over P with the induced
 $\lambda: \widehat{F}(P) \rightarrow \mathbb{Z}$, r-ch. fun and
 $\mu: E_X \rightarrow P$, principal T^n -bundle.
 $\Rightarrow \exists T^n$ -equivariant orbifold diffeom from $X(P, \lambda, \mu)$
to X such that

$$\begin{array}{ccc} X(P, \lambda, \mu) & \xrightarrow{\cong} & X \\ \searrow & \curvearrowright & \swarrow \\ & P & \end{array}$$

□

Def Two data (P, λ, μ) and (P', λ', μ') are equivalent
 $\Leftrightarrow \exists \psi: P \rightarrow P'$ diffeom as mfds with corners
 $\exists \delta \in \text{Aut}(\mathbb{Z}^n)$ s.t
 $\lambda'(\psi(F)) = \pm \delta(\lambda(F))$ for $\forall F \in \widehat{F}(P)$
& $\mu \cong \psi^*(\mu')$

Theorem 3. $X(P, \lambda, \mu)$ and $X(P', \lambda', \mu')$ are
 \overline{T}^n -weakly equivariantly diffeomorphic
 $\Leftrightarrow (P, \lambda, \mu)$ and (P', λ', μ') are equivalent.

□

E.g (1) $S^4 = \{ (z_1, z_2, x) \in \mathbb{C}^2 \times \mathbb{R} \mid |z_1|^2 + |z_2|^2 + x^2 = 1 \}$
 $\overline{T}^2 \curvearrowright S^4 : (t_1, t_2) \cdot (z_1, z_2, x) = (t_1 z_1, t_2 z_2, x)$
 $\Rightarrow S^4 / \overline{T}^2 \cong P = \{ (x_1, x_2, x) \in \mathbb{R}^2 \times \mathbb{R} \mid x_1^2 + x_2^2 + x^2 = 1, x_i \geq 0 \text{ for } i=1, 2 \} : \text{eye-shape}$

$$\mathcal{F}(P) = \{F_1, F_2\}$$

where $F_i = \{ (x_1, x_2, x) \in P \mid x_i = 0 \}$

$\Rightarrow S^4 = X(P, \lambda)$ l.s.t.o over P where
 $\lambda(F_1) = (1, 0), \lambda(F_2) = (0, 1)$

(2) Let $\chi': \tilde{F}(P) \rightarrow \mathbb{Z}^2$: r-ch. fun defined by

$$\chi'(F_1) = (a, b)$$

$$\chi'(F_2) = (c, d) \text{ with } ad - bc \neq 0$$

Let $\phi: \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ be the homom defined by
the matrix

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$\Rightarrow \tilde{\psi}: T^2 \rightarrow T^2$ is the induced surj. homom
with $\text{Ker}(\tilde{\psi}) = \text{finite abelian group}$

$\Rightarrow \tilde{\psi} \times \text{Id}: T^2 \times P \xrightarrow{\quad} T^2 \times P$ induces a
surjective map $f_{\phi}: T^2 \times P / \sim_{\chi'} \xrightarrow{\quad} T^2 \times P / \sim_{\chi'}$

s.t

$$\begin{array}{ccc} S^4 & \xrightarrow{\quad} & \times(P, \chi', \mu) \\ \downarrow & & \cong \\ S^4 / \text{Ker } \phi & \xrightarrow{\quad} & \end{array}$$

• Construction of orbifolds with boundaries

Def A face-simple mfd with marked facets
= $(n+1)$ -dim oriented compact mfd with corners Y
with disjoint marked facets $\{P_1, \dots, P_m\} \subset F(Y)$.

st

- (1) Y is nice, i.e., \forall codim k face of Y is a conn. component of the intersection of a uniquely determined set of $\binom{k}{m}$ facets of Y , and
- (2) the vertex set $V(Y) = \coprod_{i=1}^m V(P_i)$

Let $Y[P_1, \dots, P_m]$ denote such mfd.

Let $\{F_1, \dots, F_m\} := F(Y) \setminus \{P_1, \dots, P_m\}$, and call it the remaining facets.

Def. A rational super ch. fun (rs-ch. fun) on

$\mathcal{Y}[P_1, \dots, P_m]$ is a function

$$\gamma : \{F_1, \dots, F_m\} \longrightarrow \mathbb{Z}^n \quad \text{s.t}$$

$\{\gamma(F_{i_1}), \dots, \gamma(F_{i_k})\}$ is lin. ind in \mathbb{Z}^n

whenever $F_{i_1} \cap \dots \cap F_{i_k} \neq \emptyset$.

$\gamma : \{F_1, \dots, F_m\} \longrightarrow \mathbb{Z}^n$: rs-ch. fun on $\mathcal{Y}[P_1, \dots, P_m]$

For each P_j , let $\mathcal{F}(P_j) := \{G_{j1}, \dots, G_{j, r_j}\}$

$\forall G_{ij} \quad \exists! F_{ji}$: remaining facet st

$$G_{ij} = P_j \cap F_{ji}$$

$\Rightarrow \lambda_j : \mathcal{F}(P_j) \longrightarrow \mathbb{Z}^n, \quad G_{ij} \longmapsto \gamma(F_{ji})$

is an r-ch. fun. on P_j .

Given

(1) $Y [P_1, \dots, P_m] : (n+1)\text{-dim face simple (orientable) mfd}$
with marked facets P_1, \dots, P_m

(2) $\varphi : \{F_1, \dots, F_{m'}\} = F(Y) - \{P_1, \dots, P_m\} \longrightarrow \mathbb{Z}^n$, rs-ch fun.

(3) $\zeta : E \longrightarrow Y$: smooth principal T^n -bundle over Y .

For F : face of Y of codim k , two cases:

Case 1. F is a face of P_i for some $i = 1, \dots, m$

$\Rightarrow \exists! \{F_{i1}, \dots, F_{ik}\}$: remaining facets s.t.
 $F = \text{a component of } F_{i1} \cap \dots \cap F_{ik} \cap P_i$

Case 2. F is not a face of any P_i

$\Rightarrow \exists! \{F_{i1}, \dots, F_{ik}\}$: remaining facets s.t.
 $F = \text{a component of } F_{i1} \cap \dots \cap F_{ik}$

Let

$$K(F) = \begin{cases} \langle \gamma(F_{i_1}), \dots, \gamma(F_{i_{k-1}}) \rangle & \text{in Case 1,} \\ \langle \gamma(F_{i_1}), \dots, \gamma(F_{i_k}) \rangle & \text{in Case 2.} \end{cases}$$

Define an eq. relation \sim_γ on E :

$$x \sim_\gamma y \iff \mu(x) = \mu(y) \text{ & } x = uy \text{ for some } u \in T_F$$

where T_F is defined similarly to the previous discussion.

Define $W(Y, \gamma, \tau) = E/\sim_\gamma$, and

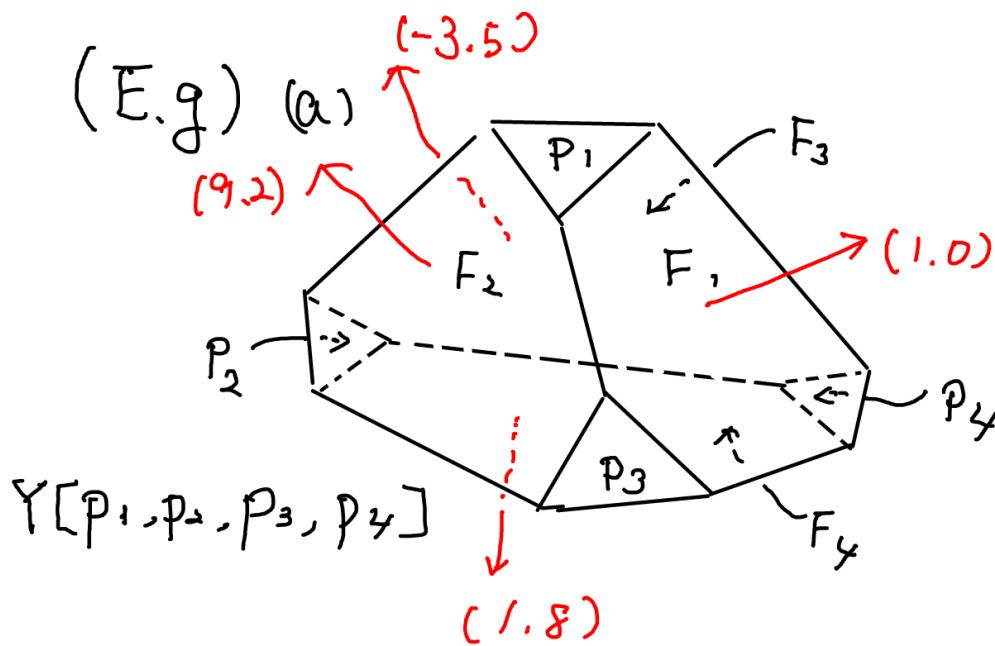
$$\pi: W(Y, \gamma, \tau) \longrightarrow Y, [x] \mapsto \mu(x).$$

Then $T^n \curvearrowright W(Y, \gamma, \tau)$ and π is the orbit map.

Theorem 4. Under the same notation,

$W(Y, \gamma, \tau)$ is a $(2n+1)$ -dim (orientable) effective T^n -orbifold with boundary s.t.

$\partial W(Y, \gamma, \tau)$ = disjoint union of (orientable) l.s.t.d.



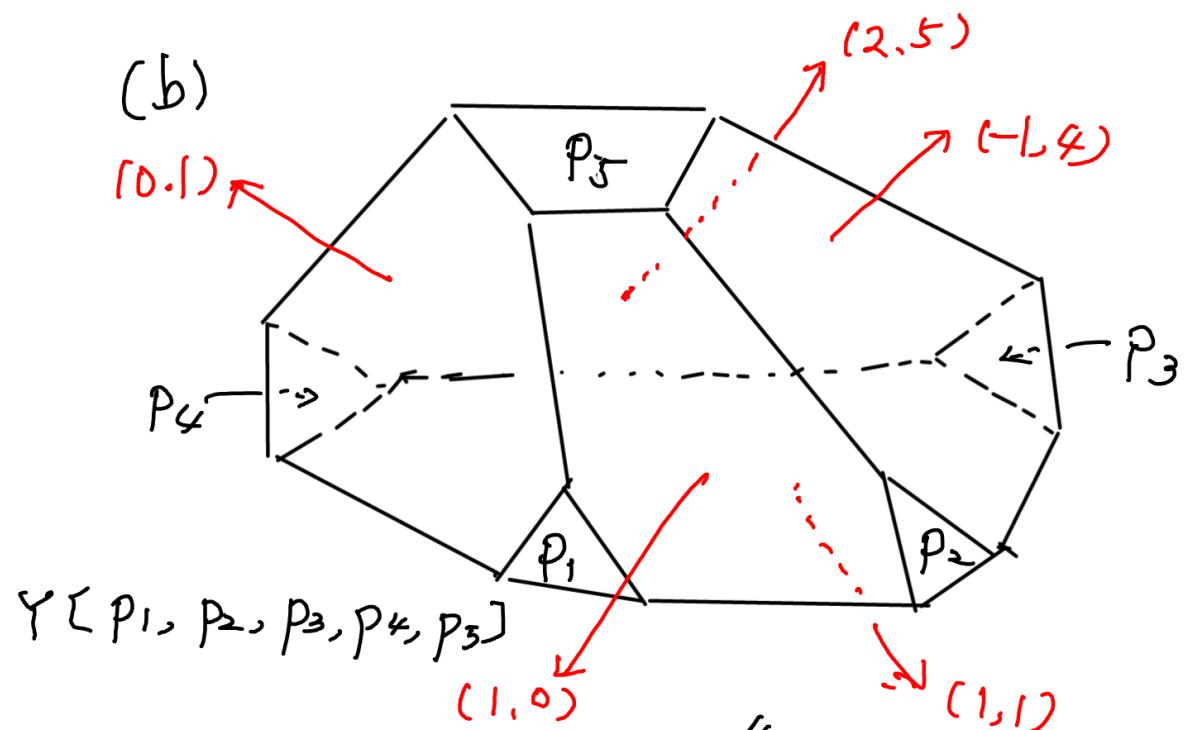
$$\tau: T^2 \times Y \longrightarrow Y$$

trivial T^2 -bundle over Y .

$\Rightarrow W(Y, \gamma, \tau)$ is a 5-dim T^2 -orbifold with

$$\partial W = \prod_{i=1}^4 \underbrace{\times (p_i, \lambda_i)}$$

called orbifold \mathbb{CP}^2 .



τ : trivial T^2 -bundle

$$\partial W(\gamma, \tau) = \underbrace{\bigcup_{i=1}^4 X(p_i, \lambda_i)}_{\parallel}$$

orbifold \mathbb{CP}^2

$$\underbrace{\perp X(p_5, \lambda_5)}_{\parallel}$$

orbifold Hirzebruch surface

- Application to equivariant cobordism of torus orbifolds

Def. Two $2n$ -dim (oriented) l.s.t.o X_1, X_2 are
equivariantly cobordant

$\Leftrightarrow \exists W : (2n+1)\text{-dim (oriented)} T^n\text{-orbifold s.t}$
 $\partial W \stackrel{\cong}{\underset{\xi}{\sim}} X_1 \sqcup (-X_2)$
 orient-pres. diffeom.

Thm 5. $X : 2n$ -dim (orient) l.s.t.o with $|X^{T^n}| = k$.
 $\Rightarrow X$ is eq. cobordant to $\coprod_{i=1}^k Z_i$
 where $Z_i : \text{orbifold } \mathbb{C}\mathbb{P}^n$

(Sketch of Proof)

$P := X/\tau^n$: n -dim nice mfd with corners

$\lambda: \mathcal{F}(P) = \{F_1, \dots, F_m\} \longrightarrow \mathbb{Z}^n$: assoc. r-ch. fun

$V(P) := \{v_1, \dots, v_k\}$: the vertices of P

$\tilde{Y} := P \times \Delta^1$: $(n+1)$ -dim. nice mfd with corners

$\mathcal{F}(\tilde{Y}) = \{F_i \times \Delta^1, \dots, F_m \times \Delta^1, Q \times \{\delta\}, Q \times \{1\}\}$

$V(\tilde{Y}) = \{v_i^j = v_i \times \{j\} \mid i=1, \dots, k, j=0, 1\}$

Y : vertex-cut of \tilde{Y} at the vertices

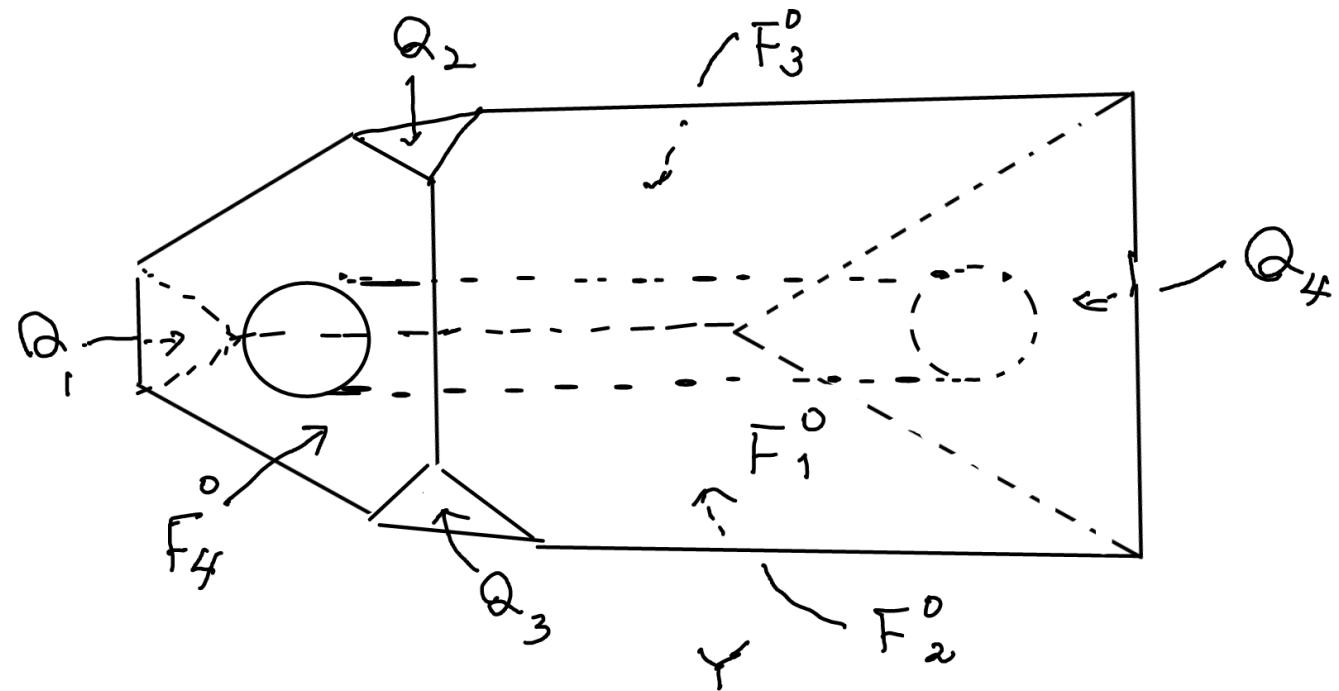
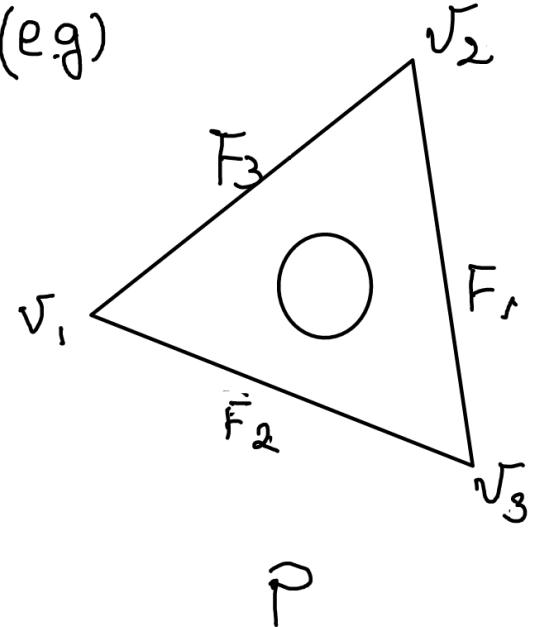
v_1^0, \dots, v_k^0

Y is an $(n+1)$ -dim. nice mfd with corners

Call the facets of Y by $F_i^0, i=1, \dots, m+1$

and $Q_i, i=1, \dots, k+1$ as in the following figure

(e.g)



$$\therefore \tilde{F}(Y) = \{F_1^0, \dots, F_{m+1}^0, Q_1, \dots, Q_{k+1}\}$$

$$V(Y) = \prod_{i=1}^{k+1} V(Q_i)$$

Define rs-ch. fun on $Y[Q_1, \dots, Q_{k+1}]$

$$\beta : F(Y) \longrightarrow \mathbb{Z}^n$$

by $\beta(F_j) = \begin{cases} \lambda(F_j), & 1 \leq j \leq m \\ \lambda_0 & j = m+1 \end{cases}$,

where λ_0 is any element in \mathbb{Z}^n which makes β a rs-ch. fun, and such λ_0 can be shown to exist.

Let $\tau' := \mu \times id : E \times \Delta^1 \longrightarrow P \times \Delta^1$

& $\tau := \iota^* \tau'$ where $\iota : Y \hookrightarrow \tilde{Y}$.

$\Rightarrow W(Y, \beta, \tau)$ is a T^n -orbifold with

$$\partial W = \coprod_{i=1}^k \underbrace{X_i}_{\text{orbifold}} \sqcup X$$

orbifold $\mathbb{C}\mathbb{P}^n$.



Corollary 6 X : $2n$ -dim (orient) l.s.t. \mathcal{O} with $X^{T^n} = \emptyset$.

$\Rightarrow X = \partial W$ where

W : $(2n+1)$ -dim (orient) effective T^n -orbifold.

• Some further results on 4-dim l.s.t.m.

Theorem 7. X : 4-dim ori. l.s.t. manifold with $X^{T^2} = \emptyset$
 $\Rightarrow X = \partial W$ where
 W : 5-dim ori. T^2 -manifold.

(Proof) Let $X = X(P, \lambda, \mu)$ where

P : 2-dim mfd with corners without vertices
 $\Rightarrow P$ is a bounded surface, say.
 $\partial P = \bigcup_{i=1}^k C_i$

By Corollary 6, $X = \partial W(Y, \eta, \tau)$ where $Y = P \times \Delta'$

$\text{Sing}(w(Y, \varphi, \tau))$ may occur only in $\coprod_{k=1}^k \pi^{-1}(C_i \times \{\partial\})$

Choose small nbd U_i of $C_i \times \{\partial\}$ in $Y = P \times \Delta'$
for each $i=1, \dots, k$ st

(i) $U_i \cong C_i \times \mathbb{R}_{\geq 0}^2$ as mfd with corners
(ii) $\overline{U_i} \cong C_i \times \Delta^2$ " "

(iii) $\overline{U_i} \cap \overline{U_j} = \emptyset$ if $i \neq j$ &

(iv) $\overline{U_i} \cap P \times \{\partial\} = \emptyset$

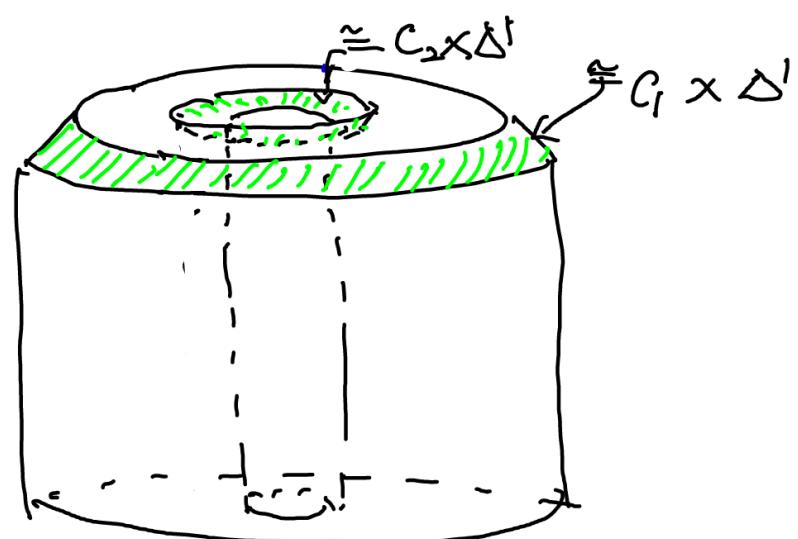
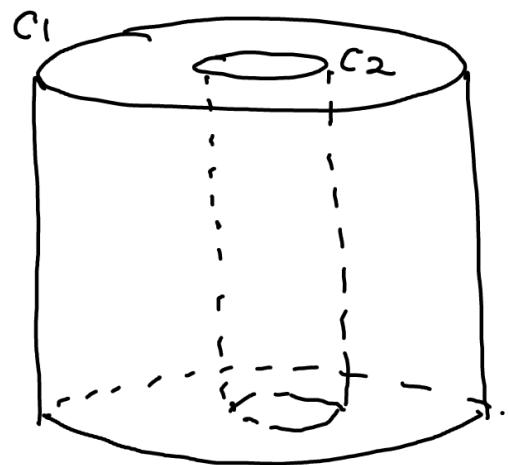
$$\Rightarrow \overline{U_i} \cap (Y \setminus \bigcup_{i=1}^k U_i) \cong C_i \times \Delta'$$

Let $w := w(Y, \varphi, \tau) - \bigcup_{i=1}^k \pi^{-1}(U_i)$.

$\Rightarrow w$ is T^2 -mfd with

$$\partial w = \coprod \left(\underbrace{T^2 \times C_i \times \Delta'}_{\uparrow} / \sim \right) \perp X$$

This is trivial T^2 -bundle because $H^2(C_i \times \Delta') = 0$.



Note (1) $(T^2 \times C_i \times \Delta')/\sim \cong C_i \times (T^2 \times \Delta'/\sim)$

(2) $T^2 \times \Delta'/\sim \cong L(p, q)$: lens space
weak reg.
diffeom.

On $F(\Delta')$ we have the function $\lambda: F(\Delta') \rightarrow \mathbb{Z}^2$

$$\begin{array}{ccc}
 \bullet & & \bullet \\
 \lambda(F_{i1}) & & \xrightarrow{\text{after autom of } \mathbb{Z}_2} & (1, 0) \\
 \parallel & & & (-g_i, p) \\
 (a, b) & & c, d & \text{st } p - g_i \text{ are rel. prime.} \\
 & & & \swarrow
 \end{array}$$

(3) $L(p, q)$ is T^2 -boundary. ■

Theorem 8. P : 2-dim nice mfd with corners at $\partial P \neq \emptyset$
 M : 4-dim l.s.E.m over P .
 $\Rightarrow M$ is eg-cobordant (by 5-dim T^2 -mfld) to
 some copies of $\mathbb{C}P^2$.

(Proof) P is obtained from a closed surface S_p by removing the interiors of polygons Q_1, \dots, Q_r , eye-shapes Q_{2r}, \dots, Q_{2s} , and disks Q_{3r}, \dots, Q_{3t} .

(For simplicity, assume $r = s = t$)

$$\therefore P = S_p \# Q_1 \# Q_2 \# Q_3$$

$M \cong X(P, \lambda, \tau)$, τ is trivial because $H^3(P; \mathbb{Z}) = 0$,

$$\cong (T^2 \times S_p) \# X(Q_1, \lambda|_{\mathcal{F}(Q_1)}) \# X(Q_2, \lambda|_{\mathcal{F}(Q_2)}) \# X(Q_3, \lambda|_{\mathcal{F}(Q_3)})$$

T^2 -eg. \sim cobordant $\underbrace{(T^2 \times S_p)}_{\uparrow T^2\text{-boundary}} \sqcup \bigcup_{i=1}^3 X(Q_i, \lambda|_{\mathcal{F}(Q_i)})$.

Now we can show the following :

(1) X : l.s.t.m over a disk D^2

$\Rightarrow X$ is T^2 -weakly equivariantly diffeom to $S^1 \times S^3$
where $T^2 \curvearrowright S^1 \times S^3$ is given by

$$(t_1, t_2) \cdot (z_1, z_2, z_3) = (t_1 z_1, t_2 z_2, z_3)$$

$S^1 \times S^3$ is T^2 -equivariantly a boundary.

(2) X : l.s.t.m over an eye-shape.

$\Rightarrow X$ is T^2 -weakly equivariantly diffeom to S^4

where $T^2 \curvearrowright S^4$ by

$$(t_1, t_2) \cdot (z_1, z_2, x) = (t_1 z_1, t_2 z_2, x)$$

(3) X : l.S.E.m over an n -gon

$\Rightarrow X$ is a quasi toric 4-manifold

By [Sarkar] X is T^2 -cobordant to some T^2 -cobordism
classes of \mathbb{CP}^2

Theorem 9.

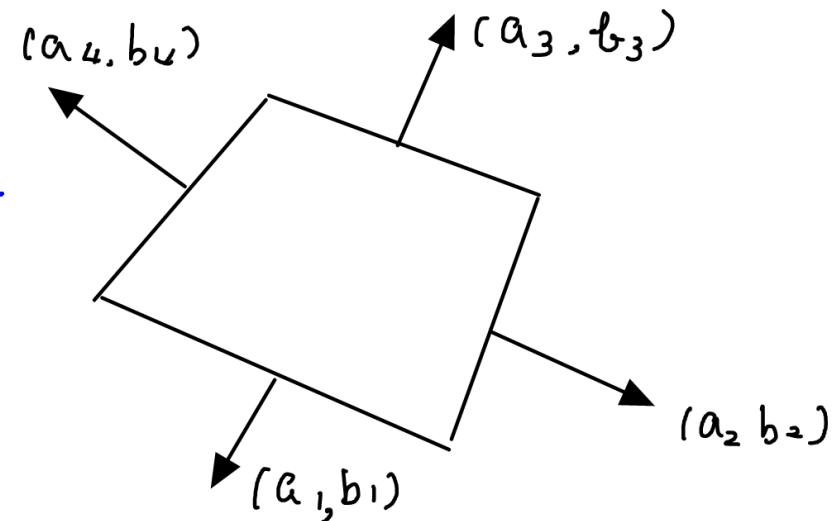
X : Orbifold Hirzebruch surface

corresp. to the figure.

If either $(a_1, b_1) = \pm(a_3, b_3)$
or $(a_2, b_2) = \pm(a_4, b_4)$,

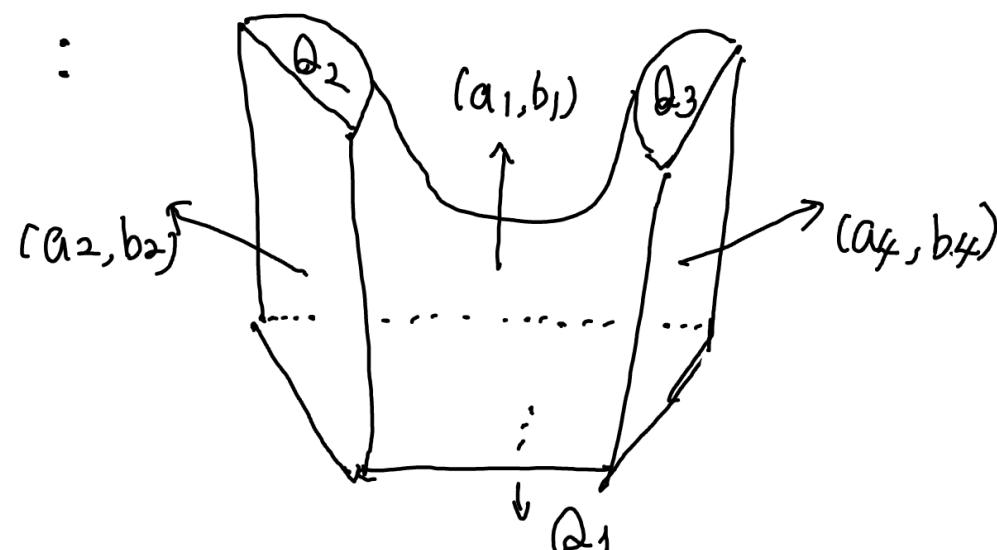
$$\Rightarrow X = 2W$$

where W = 5-dim orientable T^2 -orbifold.



(Proof) Assume $(a_1, b_1) = \pm(a_3, b_3)$

Y :



Consider

$Y[Q_1, Q_2, Q_3]$

with $\gamma: F(Y) \setminus \{Q_1, Q_2, Q_3\} \rightarrow \mathbb{Z}^2$

as in the figure

$\Rightarrow W(Y, \eta)$ is as an orient. effective orbifold at
 $\partial W = X(Q_1, \lambda_1) \sqcup X(Q_2, \lambda_2) \sqcup X(Q_3, \lambda_3)$
with $\pi_i = \exists |_{\mathcal{F}(Q_i)}$

Here $X(Q_1, \lambda_1) \equiv X$, and

$X(Q_2, \lambda_2), X(Q_3, \lambda_3)$: l.s.t.o over eye-shapes Q_2, Q_3 ,

These are T^2 -boundaries of 5-dim. orientable
 T^2 -orbifold.

$\therefore X$ is a boundary. ■

Corollary 10. $X :=$ Hirzebruch surface

$\Rightarrow X = \partial W$ where

W : 5-dim orient. T^2 -mfld.



Appendix: New construction of lens spaces.

$$P_i = \Delta^{n-i}$$

$$\mathcal{F}(P) = \{G_0, \dots, G_{n-1}\}$$

$\lambda: \mathcal{F}(P) \longrightarrow \mathbb{Z}^n$: hyper-char.fun (h -ch.fun)

i.e. whenever $G_{i_1} \cap \dots \cap G_{i_l} \neq \emptyset$, then

$\{\lambda(G_{i_1}), \dots, \lambda(G_{i_l})\}$ forms part of a basis of \mathbb{Z}^n .

$$\text{let } L(P, \lambda) := T^n \times \Delta^{n-i} / \sim_\lambda$$

where $(t, x) \sim (s, y) \Leftrightarrow x = y \text{ and } ts^{-1} \in T_F$,

and T_F is defined as before

$$\text{Let } H_\lambda := \mathbb{Z}^n / \langle \lambda(G_0), \dots, \lambda(G_{n-1}) \rangle$$

$$\Rightarrow \text{rk}(\langle \lambda(G_0), \dots, \lambda(G_{n-1}) \rangle) = n-1 \text{ or } n$$

Proposition 11. λ : h-ch fun on Δ^{n-1} .

$\Rightarrow L(\Delta^{n-1}, \lambda)$ is a $(2n-1)$ -dim T^n -orbifold
(actually a topological manifold), and

(1) if $\text{rk}(H_\lambda) = n-1 \Rightarrow L(\Delta^{n-1}, \lambda) \cong S^1 \times \mathbb{CP}^{n-1}$

(2) if $\text{rk}(H_\lambda) = n \Rightarrow L(\Delta^{n-1}, \lambda) \cong S^{2n-1}/H_\lambda$.

called generalized lens sp. \square

Now consider

Δ^n : n-dim. simplex with vertices v_0, \dots, v_n

Assume $\gamma: F(\Delta^n) \longrightarrow \mathbb{Z}^n$: rational function s.t.

$\{F_{i_1}, \dots, F_{i_l}\} \subset F(\Delta^n)$ s.t. $F_{i_1} \cap \dots \cap F_{i_l} \neq \emptyset$

$\{\gamma(F_{i_1}), \dots, \gamma(F_{i_l})\}$ forms a part of basis

whenever $n-l > 0$

Consider the vertex cut $\Upsilon := \text{VC}(\Delta^n)$ at every vertices of Δ^n .

$P_i \in F(\Upsilon)$: facet obtained from the vertex-cut of V_i

$$\Rightarrow P_i \cong \Delta^{n-1}$$

$$\forall G_{ij} \in F(P_i) \quad \exists! F_{ij} \in F(\Delta^n)$$

$$\text{s.t } G_{ij} = F_{ij} \cap P_i$$

Define $\lambda_i : F(P_i) \longrightarrow \mathbb{Z}^n$ by

$$G_{ij} \longmapsto \gamma(F_{ij}).$$

$\Rightarrow \lambda_i$ is a h-ch. fun on P_i

with $L(P_i, \lambda_i)$: generalized lens space.

(an orbifold.)

Now consider $Y[p_0, \dots, p_n]$, the vertex-cut $Y = VC(\Delta^n)$ with the marked facets p_1, \dots, p_n . With $\gamma : \{F_0, \dots, F_n\} \longrightarrow \mathbb{Z}^n$ as defined above,

Consider $W(Y, \gamma) = T^n \times Y / \sim_\gamma$
 $\Rightarrow W(Y, \gamma)$ is an $2n$ -dim T^n -orbifold s.t.
 $\partial W(Y, \gamma) = \coprod_{i=1}^n L(p_i, \lambda_i)$: gen. lens space

Proposition 12. Let

$\lambda_0 : \tilde{F}(\Delta^{n-1}) \longrightarrow \mathbb{Z}^n$ h-ch.fun. defined by

$$\begin{cases} F_0 \mapsto (-g_1, -g_2, \dots, -g_{n-1}, p), & (p-g_i) = 1 \forall i \\ F_i \mapsto p_i, & 1 \leq i \leq n-1. \end{cases}$$

$\Rightarrow L(\Delta^{n-1}, \lambda_0) = L(p : g_1, \dots, g_{n-1})$: usual lens sp.

• Special case when $n=2$

$$\Delta^{n+1} = \Delta' = [0, 1].$$

$$\lambda_0 : \mathcal{F}(\Delta') = \{\{0\}, \{1\}\} \longrightarrow \mathbb{Z}^2$$

$$\lambda_0(\{0\}) = (1, 0)$$

$$\lambda_0(\{1\}) = (-g, p) \quad \text{with} \quad \gcd\{p, g\} = 1.$$

$$\Rightarrow L(\Delta', \lambda_0) = L(p, g).$$

(We may assume $g < p$).

Lemma 13. Given $(a, b), (c, d) \in \mathbb{Z}^2$ s.t $|\det \begin{bmatrix} a & c \\ b & d \end{bmatrix}| = rs$

$$\text{with } \gcd\{a, b\} = t = \gcd\{c, d\}$$

$$\Rightarrow \exists (e, f) \in \mathbb{Z}^2 \text{ s.t } \gcd\{e, f\} = 1 \text{ and} \\ |\det \begin{bmatrix} a & e \\ b & f \end{bmatrix}| = 1 \text{ and } |\det \begin{bmatrix} e & c \\ f & d \end{bmatrix}| < r$$



Theorem 14. $L(p, q)$ is T^2 -equivariantly a boundary
i.e. $\exists W$: smooth, orient. 4-dim. T^2 -manifold
with $\partial W = L(p, q)$.

Moreover, we can find W s.t. $\pi_1(W) = 1$

(Proof) By Lemma 13, $\exists (g_1, p_1), \dots, (g_k, p_k) \in \mathbb{Z}^2$
such that

$$(g_1, p_1) = (1, 0)$$

$$(g_k, p_k) = (-g, p)$$

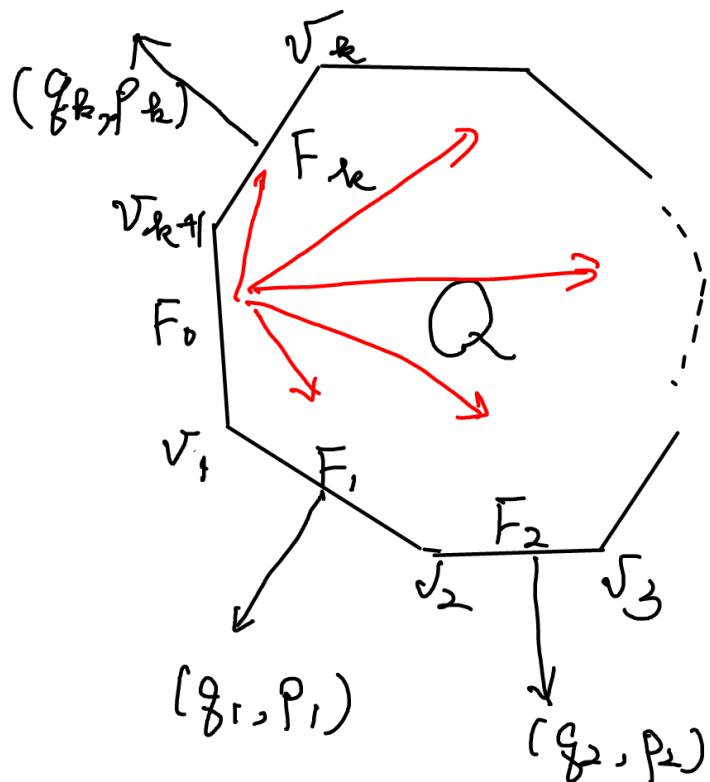
$$|\det \begin{bmatrix} g_i & g_{i+1} \\ p_i & p_{i+1} \end{bmatrix}| = 1 \quad \forall i = 1, \dots, k-1$$

Now consider $(k+1)$ -gon Q with

$$V(Q) = \{v_1, \dots, v_{k+1}\}$$

$$F(Q) = \{F_0 = (v_1, v_{k+1}), F_1 = (v_1, v_2), \dots, F_k = (v_k, v_{k+1})\}$$

$$\varphi : \{F_1, \dots, F_k\} \rightarrow \mathbb{Z}^2, \quad F_i \mapsto (g_i, p_i)$$



Consider $Q[F_0]$: Q with F_0 marked,
and construct

$$W(Q, \gamma) = T^*Q / \sim_\gamma$$

$$\Rightarrow \partial W = L(\Delta', \gamma_1)$$

$$= L(p \cdot q).$$

Since there is no singularity,
 W is a smooth mfd.

Let $E = \bigcup_{i=1}^n F_i \Rightarrow E$ is a def. retract of Q

$\Rightarrow \tilde{\pi}^{-1}(E)$ is a def. retract of $W(Q, \gamma)$

Each $\tilde{\pi}^{-1}(F_i) \cong S^2$ and $\tilde{\pi}^{-1}(F_i) \cap \tilde{\pi}^{-1}(F_{i+1}) = \text{pt}$ $\forall i = 1, \dots, k-1$

$$\therefore \tilde{\pi}^{-1}(E) = \bigvee^{k-1} S^2$$

$$\Rightarrow \pi_1(W(Q, \gamma)) \cong \pi_1(\bigvee^{k-1} S^2) = 1.$$

