Peterson Schubert calculus

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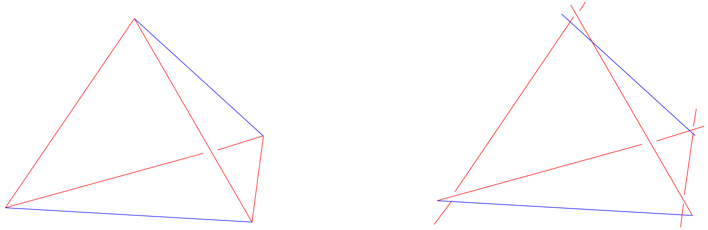
Schubert Calculus: some history

"Kalkül der Abzählenden Geometrie" (Calculus of Enumerative Geometry), Hermann Schubert, 1874.

Questions in algebraic geometry with finite # of solutions

- How many lines in 3-space, in general, intersect four given lines?
- Principle of conservation of number:



Mathematical Essays and Recreations, 1898.

- The Magic Square
- The Fourth Dimension
- The Squaring of the Circle

Schubert calculus of the Grassmannian $Gr(k, n)$

Definition

The Grassmannian of k planes on $\mathbb{C}^n$ is denoted $Gr(k, n)$ and is given as a set by $\{V_k \subset \mathbb{C}^n\}$, where V_k is a k-dimensional subspace of $\mathbb{C}^n$.

The space of (all possible) blue lines through R_1 forms a three-dimensional subvariety X_1 of $Gr(2, 4)$. Similarly, the space of lines through each of R_2, R_3 and R_4 form three-dimensional subvarieties X_2, X_3, X_4, respectively.
The only blue lines that intersect R_1, R_2, R_3 and R_4 are in the intersection

$$X_1 \cap X_2 \cap X_3 \cap X_4.$$

If the intersections are transverse, we count the $\#$ of points.

Flag manifolds

Let G be a complex reductive Lie group, B a Borel subgroup. The coset space G/B is the set of flags. When $G = Gl(n, \mathbb{C})$, we get the flag manifold for type A_{n-1}.

Definition

The flag manifold $Fl(n, \mathbb{C})$ *is the collection* $\{V_\bullet\}$ *of all complete flags in* $\mathbb{C}^n$

$$V_\bullet := \{0\} = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = \mathbb{C}^n,$$

with $\dim(V_i) = i$.

- Identify $V_\bullet$ with any $n \times n$ invertible matrix $A \in Gl(n, \mathbb{C})$, where the subspace V_i is identified with the span of the leftmost i columns of A.
- This is unique up to right multiplication by an upper triangular matrix. Thus we identify $Fl(n, \mathbb{C})$ with $Gl(n, \mathbb{C})/B$, where B is the set of upper triangular matrices.

Schubert varieties

Definition

For any $w \in W$, the Schubert variety X_w is the closure of B orbits, $\overline{B \cdot wB} \subset G/B$.

- $\dim_{\mathbb{C}} X_w = \ell(w)$, where $\ell(w)$ is the length of the word w
- Each fundamental homology class $[X_w] \in H_{2\ell(w)}(G/B)$
- Fundamental classes $\{[X_w] : w \in W\}$ form a basis for $H_*(G/B)$
- Dual basis $\{\sigma_w : w \in W\}$ forms a basis for $H^*(G/B)$
- The class σ_w has support on $X^w := \overline{B_- \cdot wB}$
- $\{\sigma_w\}$ are called *Schubert classes*[1]

[1] also called opposite Schubert classes

What can be said of intersecting Schubert classes and opposite Schubert classes, in general position?

Formalization of Schubert's "Principle of Conservation of Number"

Intersections of Schubert varieties can be measured by the product structure in the cohomology ring of G/B.

We have a basis $\{\sigma_w\}$ of $H^*(G/B)$, as a module over $\mathbb{C}$. Therefore,

$$\sigma_u \cdot \sigma_v = \sum_w c_{u,v}^w \sigma_w$$

for some $c_{u,v}^w \in \mathbb{C}$

Theorem

The coefficients $c_{u,v}^w \geq 0$ and $c_{u,v}^w = 0$ unless $\ell(u) + \ell(v) = \ell(w)$.

Pseudo-Proof:

$$c_{u,v}^w \text{``} = \text{''} \#(X^u \cap X^v \cap X_w)$$

T action on G/B

T be a maximal torus of G (for $G = Gl(n, \mathbb{C})$, choose T diagonal matrices)

Properties of G/B:

- T acts on G/B on the left.
- X_w is invariant under T.
- The T fixed points G/B are isolated, given by

$$(G/B)^T = \{\tilde{w}B \in G/B : \tilde{w} \in N(T)\}$$

where $\tilde{w}B = \tilde{v}B$ if $\tilde{w} = \tilde{v}$ in $N(T)/T$. Thus $(G/B)^T$ is in 1-1 correspondence with $W := N(T)/T$. When $G = Gl(n, \mathbb{C})$, $W \cong S_n$.

$\Longrightarrow$ $H_T^*(G/B)$ natural ring to consider.

Properties of equivariant cohomology H_T^*:

- T-equivariant map $f : Y \to X$ induces $f^* : H_T^*(X) \to H_T^*(Y)$
- $Y \subset X$ a T-equivariant inclusion of manifolds, or of varieties when X is smooth. $[Y]$ represents an equivariant cohomology class in $H_T^*(Y)$
- $H_T^*(X)$ is a module over $H_T^*(pt)$ induced by $X \to pt$
- $H_T^*(pt) \cong S(\mathfrak{t}^*)$, a polynomial ring with $\dim(T)$ variables

Equivariant Cohomology for Flag Manifolds

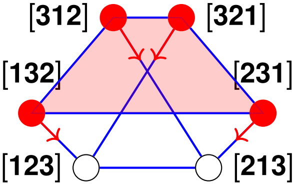
Properties of equivariant cohomology ring $H_T^*(G/B)$:

Theorem (Chang-Skjelbred '74, Hsiang '75)

The inclusion of the fixed point set $G/B^T \hookrightarrow G/B$ induces an injection

$$H_T^*(G/B) \longrightarrow H_T^*((G/B)^T) = \bigoplus_{v \in W} H_T^*(pt)$$

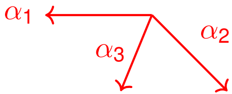
- Since X_w is invariant under T, $\{\sigma_w\}_{w \in W}$ are a basis for $H_T^*(G/B)$ as a module over $H_T^*(pt)$.
- We obtained σ_w by duality but we it has has a geometric interpretation
- σ_w is supported on $X^w := \overline{B_- wB/B}$



Basis statement:

$$\sigma_u \cdot \sigma_v = \sum_w c_{u,v}^w \sigma_w, \quad c_{u,v}^w \in H_T^*(pt) \cong \mathbb{C}[\mathfrak{t}].$$

The choice of Borel results in a choice of positive roots $\alpha_1, \ldots, \alpha_{n-1}$, the weights of the T action on $Lie(B)$.



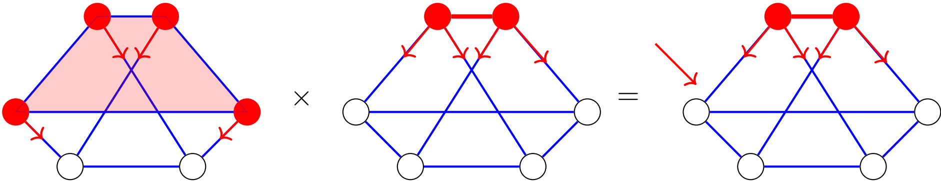
Theorem (Gr, 1999)

The polynomial $c_{u,v}^w$ is a polynomial in $\alpha_1, \ldots, \alpha_{n-1}$ with nonnegative coefficients, for all $u, v, w \in W$.

This theorem is the equivariant analog of saying that the intersections of Schubert varieties occur in positively oriented points, which can then be added up to get an intersection number.

Example (Multiplying two Schubert classes in $Gl(3,\mathbb{C})/B$)

- Basis $\{\sigma_u\}_{u\in W}$. So $\sigma_u\sigma_v = \sum_w c_{u,v}^w \sigma_w$.
- $H_T^*(G/B) \to \oplus_{u\in W} H_T^*(uB/B)$: $\sigma_w \mapsto \oplus_{v\in W}\sigma_w|_v$
- Image of the moment map $\mu(G/B) \subseteq \mathfrak{t}^*$ is convex hull of $\mu((G/B)^T)$
- H_T^*(fixed point) is identified with polynomials on $\mathfrak{t}^*$
- Each Schubert class has an associated 6-tuple of polynomials.
- $Gl(3,\mathbb{C})$: a Schubert class σ_w restricted to a fixed point is a product of arrows, or 0
- The product structure: multiply the weights at each fixed point



$$\sigma_{[132]} \cdot \sigma_{[312]} = \alpha_2\ \sigma_{[312]}$$

Peterson variety

- B Borel subgroup, $T \subseteq B \subseteq G$ maximal torus
- $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ simple roots
- W Weyl group
- $\mathfrak{t} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ Lie algebras
- $\mathfrak{g}_\alpha$ root space for α, with basis element $E_\alpha \in \mathfrak{g}_\alpha$
- $N_0 = \sum_{\alpha \in \Delta} E_\alpha$ regular nilpotent operator

The Peterson variety $Pet \subseteq G/B$

$$Pet := \{gB \in G/B | \ Ad(g^{-1})(N_0) \in \mathfrak{b} \oplus \bigoplus_{\alpha \in -\Delta} \mathfrak{g}_\alpha\}$$

Features

Related to integrable systems and quantum cohomology
Interesting stratification(s) that exhibit mirror symmetry phenomena
Rich combinatorial and algebraic structure
Less symmetry and "good properties" compared to G/B

Peterson variety in type A_{n-1}

Let N be the regular nilpotent operator, or matrix whose Jordan canonical form consists of one block with 1's on the superdiagonal and 0's elsewhere.

For $n = 3$,

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Definition

The Peterson variety $Y \subseteq Fl(n; \mathbb{C})$ *is the collection of flags over* $\mathbb{C}^n$ *satisfying* $NV_i \subseteq V_{i+1}$.

Example ($n = 3$)

Unique representation of the points in the Peterson Y given by:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 1 \\ 0 & 1 & 0 \end{pmatrix} \cup \begin{pmatrix} c & d & 1 \\ d & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where $a, b, c, d \in \mathbb{C}$.

Geometric considerations (type A_{n-1})

$$Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 1 \\ 0 & 1 & 0 \end{pmatrix} \cup \begin{pmatrix} c & d & 1 \\ d & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- $\dim_{\mathbb{C}}(Y) = n - 1$
- Y has (several) paving by affine varieties/cells.
- Y is not smooth for $n \geq 3$
- Y is a local complete intersection
- Y is normal if and only if $n \leq 3$

- T doesn't act on Y
- One-dimensional torus S acts on Y, where

$$S = \{diag(t^n, t^{n-1}, t^{n-2}, \dots, t)\}$$

for $t \in \mathbb{C}^*, ||t|| = 1$

We'd like to imitate Schubert calculus on Y.

Peterson Schubert basis

We'd like to imitate Schubert calculus on Y. Consider the composition j^*

$$H_T^*(G/B) \to H_S^*(G/B) \to H_S^*(Pet)$$

restrict action to S, then restrict to the Peterson.

Theorem (Harada-Tymoczko, Drellich)

(Type A_{n-1}:) Pick a subset $A = \{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, n-1\}$ with $i_1 < \cdots < i_k$, and let $v_A = s_{i_1} s_{i_2} \ldots s_{i_k}$. Define

$$p_A = j^*(\sigma_{v_A}),$$

where σ_{v_A} is the S-equivariant Schubert class obtained by restricting the T action to S. Then $H_S^*(Pet)$ is a free module over $H_S^*(pt)$ with basis $\{p_A\}$.
(All classical types:) More generally pick a subset $A = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\} \subset \{\alpha_1, \ldots, \alpha_n\}$ and let $v_A = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \ldots s_{\alpha_{i_k}}$. Then $p_A = j^*(\sigma_{v_A})$ is an S-equivariant basis for $H_S^*(Pet)$.

Structure constants in $H_S^*(Pet)$

$H_S^*(Pet)$ is a free module over $H_S^*(pt)$ with basis $\{p_A\}$ for $A \subseteq \{1, \ldots n-1\}$.

Definition

Let $A, B, C \subseteq \{\alpha_1, \ldots, \alpha_n\}$. Define $b_{A,B}^C$ to be the coefficient of p_C in

$$p_A \cdot p_B = \sum_{C \subset \{\alpha_1, \ldots, \alpha_n\}} b_{A,B}^C \, p_C, \quad b_{A,B}^C \in H_S^*(\{pt\}) \cong \mathbb{C}[t]$$

Is $b_{A,B}^C$ a polynomial in t with *nonnegative coefficients*?
Is there a combinatorially positive formula to count it?

Theorem (G.-Gorbutt)

In type A_{n-1}, Yes and Yes.

S-fixed points of the Peterson

There's a commutative diagram

$$\begin{array}{ccc} H_T^*(G/B) & \xrightarrow{i^*} & \bigoplus_{wB\in(G/B)^T} H_T^*(wB) \\ \Big\downarrow j^* & & \Big\downarrow \pi \\ H_S^*(Pet) & \xrightarrow{i^*} & \bigoplus_{wB\in(Pet)^S} H_S^*(wB) \end{array}$$

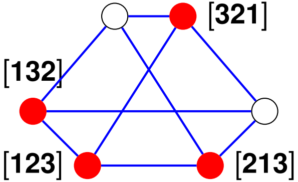
Fixed point set $(Pet)^S$

A subset of simple roots Δ

W_A, the Weyl group of the parabolic subgroup generated by A

w_A longest element of W_A

$$(Pet)^S = \{w_A | \ A \subseteq \Delta\}$$



Structure constants $b^C_{A,B}$ with $p_A p_B = \sum_C b^C_{A,B}\, p_C$

Observation

Let $A, B \subseteq \{1, \ldots, n-1\}$ be nonempty. Then $b^C_{A,B} \neq 0$ implies

- C contains $A \cup B$ due to support considerations of p_C, and
- $|C| \leq |A| + |B|$ due to degree considerations

Geometric picture:

Theorem (Tymoczko)

The Bruhat decomposition BwB/B of the flag variety intersects the Peterson variety in a paving by affine cells.

This paving is not *a priori* the way that the classes p_A come about.

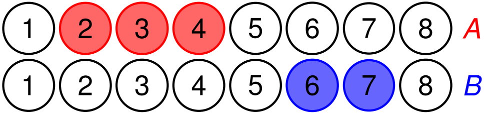
- $p_A = j^*(\sigma_{v_A})$ where σ_{v_A} are supported on B_- orbits
- *Pet* are singular and do not a priori have Poincaré duality in the sense of pairing homology and cohomology.
- The product $p_A p_B$ does not a priori have positive coefficients in the expansion.

Combinatorial Positivity for $b_{A,B}^{C}$

$p_A p_B = \sum_C b_{A,B}^{C}\, p_C$ in type A_{n-1}

Lemma (Harada-Tymoczko, 2011)

Suppose A and B are distinct nonempty, nonadjacent consecutive sequences of $\{1, \ldots, n-1\}$. Then $p_A \cdot p_B = p_{A \cup B}$.

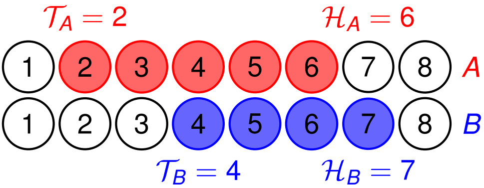


Recasting:

- $b_{A,B}^{A \cup B} = 1$ when A, B are consecutive, but $A \cup B$ not consecutive.
- If A is not consecutive, write $A = A_1 \cup A_2 \cup \cdots \cup A_k$ as a union of maximal consecutive sets A_k. Then $p_A = \prod_j p_{A_j}$. So in a moral sense, we only need consider when A is consecutive.

Structure constants $b_{A,B}^{C}$ with $p_A p_B = \sum_C b_{A,B}^{C}\, p_C$

$A \subseteq \{1, \ldots, n-1\}$ nonempty
$\mathcal{T}_A = min(A)$,
$\mathcal{H}_A = max(A)$



Theorem (G.-Gorbutt)

Let $A, B, C \subseteq \{1, \ldots, n-1\}$ be nonempty, with A, B consecutive.

1. *If $C = A \cup B$ is not consecutive, then $b_{A,B}^{C} = 1$.*
2. *If C is consecutive, $A \cup B \subseteq C$, and $|C| \leq |A| + |B|$,*

$$b_{A,B}^{C} = \frac{(\mathcal{H}_A - \mathcal{T}_B + 1)!(\mathcal{H}_B - \mathcal{T}_A + 1)!}{d!(\mathcal{T}_{A\cup B} - \mathcal{T}_C)!(\mathcal{H}_C - \mathcal{H}_{A\cup B})!(\max(\mathcal{T}_A, \mathcal{T}_B) - \mathcal{T}_C)!(\mathcal{H}_C - \min(\mathcal{H}_A, \mathcal{H}_B))!} t^d$$

where $d = |A| + |B| - |C|$.

This is manifestly positive! Integral!

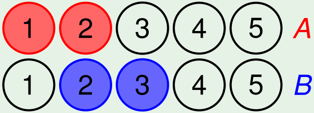
$$b_{A,B}^C = \frac{(\mathcal{H}_A - \mathcal{T}_B + 1)!(\mathcal{H}_B - \mathcal{T}_A + 1)!}{d!(\mathcal{T}_{A\cup B} - \mathcal{T}_C)!(\mathcal{H}_C - \mathcal{H}_{A\cup B})!(\max(\mathcal{T}_A, \mathcal{T}_B) - \mathcal{T}_C)!(\mathcal{H}_C - \min(\mathcal{H}_A, \mathcal{H}_B))!} t^d$$

where $d = |A| + |B| - |C|$.

Example

Let $A = \{1,2\}$, $B = \{2,3\}$ and $C = \{1,2,3\}$.

$\mathcal{T}_A = 1 \quad \mathcal{H}_A = 2$



$\mathcal{T}_C = 1, \mathcal{H}_C = 3$

$\max(\mathcal{T}_A, \mathcal{T}_B) = 2$

$\mathcal{T}_B = 2 \qquad \mathcal{H}_B = 3$

$\min(\mathcal{H}_A, \mathcal{H}_B) = 2$

$$b_{12,23}^{123} = \frac{(2-2+1)!(3-1+1)!}{1!(1-1)!(3-3)!(2-1)!(3-2)!} t^{2+2-3} = \frac{3!}{1!} t = 6t.$$

Similarly, $b_{12,23}^{1234} = 3$. All other $b_{12,23}^C = 0$. Thus $p_{12}p_{23} = (6t)p_{123} + 3p_{1234}$.

Theorem (G.-Gorbutt)

Let $A, B, C \subseteq \{1, \ldots, n-1\}$ be nonempty, with A, B consecutive.

1. *If $C = A \cup B$ is not consecutive, then $b_{A,B}^{C} = 1$.*
2. *If C is consecutive, $A \cup B \subseteq C$, and $|C| \leq |A| + |B|$,*

$$b_{A,B}^{C} = \frac{(\mathcal{H}_A - \mathcal{T}_B + 1)!(\mathcal{H}_B - \mathcal{T}_A + 1)!}{d!(\mathcal{T}_{A\cup B} - \mathcal{T}_C)!(\mathcal{H}_C - \mathcal{H}_{A\cup B})!(\max(\mathcal{T}_A, \mathcal{T}_B) - \mathcal{T}_C)!(\mathcal{H}_C - \min(\mathcal{H}_A, \mathcal{H}_B))!} t^d$$

   where $d = |A| + |B| - |C|$.
3. *Otherwise, $b_{A,B}^{C} = 0$.*

Theorem (G.-Gorbutt)

Let A, B, C be any nonempty subsets of $\{1, \ldots, n-1\}$. Then $b_{A,B}^{C}$ is obtained as an explicit (positive) sum of products of terms $b_{A_i,B_j}^{C_k}$ where A_i, B_j are consecutive subsets of $\{1, \ldots, n-1\}$, and C_k is either a consecutive subset, or $C_k = A_i \cup B_j$.
Details not provided here!

The proof in type A_{n-1} is a crazy induction using localization to fixed points.... and of course involves some combinatorics.

A few words on the combinatorial proof

$Pet^S = Fl(n;\mathbb{C})^T \cap Pet$
Flags represented by block diagonal permutation matrices whose blocks are antidiagonal. Such a permutation is the long word w_A in the the subgroup of W generated by $\{s_i : i \in A\}$.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad A = \{1, 3, 4\}.$$

- Pet^S is indexed by $A \subseteq [n-1]$.
- $p_A = j^*(\sigma_{v_A})$ and for each fixed point w_A.
- p_A is determined by the set of restrictions $p_A|_{w_C}$ for $C \subset [n-1]$.

- Restricting to fixed points provides a way to compute products inductively.

$$(p_A \cdot p_B)\,|_{w_C} = \sum_D b^D_{A,B} p_D|_{w_C}$$

- Solve for $b^C_{A,B}$:

$$b^C_{A,B} p_C|_{w_C} = p_A|_{w_C} p_B|_{w_C} - \sum_{\substack{D \\ A\cup B\subset D\subset C}} b^D_{A,B} p_D|_{w_C}$$

- Calculate $p_D|_{w_C}$
- With some algebra:

$$b^C_{A,B}\frac{(w+m+n)!}{w!x!} = \binom{w+m}{w}\binom{y+m}{x}\binom{w+n}{y}\binom{z+n}{x} - \sum_{\substack{0\le i\le m \\ 0\le j\le n \\ i+j<m+n}} \binom{w+m+j}{i,\ j,\ m-i,\ x-i-j,\ z-x+j,\ y-x+i}\binom{w+i+n}{w+i+j}$$

Not positive!

Key combinatorial step: a positive formula for the right hand side.

A combinatorial identity

$$b^C_{A,B}\frac{(w+m+n)!}{w!x!} = \binom{w+m}{w}\binom{y+m}{x}\binom{w+n}{y}\binom{z+n}{x}$$
$$-\sum_{\substack{0\le i\le m\\0\le j\le n\\i+j<m+n}}\binom{w+m+j}{i,\ j,\ m-i,\ x-i-j,\ z-x+j,\ y-x+i}\binom{w+i+n}{w+i+j}$$

Theorem (G.-Gorbutt)

Let $m,n,w,x,y,z\in\mathbb{Z}$ with $w+x=y+z$. Then

$$\binom{w+m}{w}\binom{y+m}{x}\binom{w+n}{y}\binom{z+n}{z}$$
$$=\sum_{\substack{0\le i\le m\\0\le j\le n}}\binom{w+m+j}{i,\ j,\ m-i,\ x-i-j,\ z-x+j,\ y-x+i}\binom{w+i+n}{w+i+j}.$$

One way to count it

Question (Count strings of $m+n+w$ beads of 7 colors with:)

- m blue and green beads: i green and $m-i$ blue
- n red and orange beads: j orange and $n-j$ red
- # of green beads and yellow beads differ by a fixed constant α
- # of orange beads and purple beads differ by a fixed constant β
- No red bead lies immediately to the right of a blue bead
- w yellow, purple, and white beads: $\alpha+i$ yellow, $\beta+j$ purple, $w-(\alpha-i+\beta-j)$ white beads

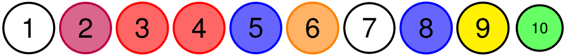
Multinomial counts number of all sequences without the red beads. Multiply by the number of ways to insert red beads without putting any of them immediately to the right of a blue bead.

$$\sum_{\substack{0\leq i\leq m\\ 0\leq j\leq n}} \binom{w+m+j}{i,\ j,\ m-i,\ w-\alpha-\beta-i-j,\ \beta+j,\ \alpha+i}\binom{w+i+n}{n-j}$$

The role of combinatorics

$$b_{A,B}^{C}\frac{(w+m+n)!}{w!x!} = \binom{w+m}{w}\binom{y+m}{x}\binom{w+n}{y}\binom{z+n}{x}$$
$$- \sum_{\substack{0\le i\le m\\ 0\le j\le n\\ i+j<m+n}} \binom{w+m+j}{i,\ j,\ m-i,\ x-i-j,\ z-x+j,\ y-x+i}\binom{w+i+n}{w+i+j}$$
$$= \binom{w+m+n}{m,\ n, x-m-n, z-x+n,\ y-x+m}$$

What does Peterson Schubert calculus have to do with 7-colored-beaded strings?



On positivity

Conjecture (G.-Mihalcea-Singh)

Let $\alpha_1, \ldots, \alpha_d$ be the weights of the action of S on $Lie(N)$, the Lie algebra of the unipotent radical. Then $b^C_{A,B} \in H^*_S(pt)$ is a polynomial in $\alpha_1, \ldots, \alpha_d$ with non-negative coefficients.

- Schubert calculus on G/B is positive, even in $H^*_S(G/B)$.

$$\sigma_u \cdot \sigma_v = \sum_w c^w_{u,v} \sigma_w, \quad c^w_{u,v} \in H^*_S(pt)$$

  $c^w_{u,v}$ are polys in one variable with nonnegative coefficients.
- Therefore

$$p_A \cdot p_B = j^*(\sigma_{v_A}) \cdot j^*(\sigma_{v_B}) = j^*(\sigma_{v_A} \cdot \sigma_{v_B}) = \sum_w c^w_{v_A, v_B} j^* \sigma_w$$

  Then

$$j^*(\sigma_w) = \sum_A (\text{presumably positive}) p_A$$

  implies the desired positivity.

$$j^*(\sigma_w) = \sum_A (\text{presumably positive}) p_A$$

- Recall σ_w is the Poincaré dual of the opposite Schubert class $[X^w]$.
- Pet has a paving by affine cells $Pet^0(A) = BwB/B \cap Pet$.

What is needed:

- $j_*(\overline{Pet^0(A)}) = \sum_{u \in W} c_{A,u}[X_u]$ (in homology) is a positive sum (true by a theorem due to W. Graham)
- Pairing between homology classes $\overline{Pet^0(A)}$ and cohomology classes p_A (requires some geometry)
- $j^*(\sigma_w) = \sum_D (c_{D,w}) p_D$

Then the product has a positive expansion:

$$p_A \cdot p_B = j^*(\sigma_{v_A} \cdot \sigma_{v_B}) \sum_w c^w_{v_A, v_B} j^* \sigma_w = \sum_{w \in W, D \subset \Delta} c^w_{v_A, v_B} c_{D,w} p_D$$

Questions:

- What about positivity for other nilpotent Hessenberg varieties?
- Do combinatorial formulas for nilpotent Hessenberg Schubert calculus result in formulas for other calculi?
- Can understanding this geometry build on our understanding of Hessenberg varieties of Springer fibers?
- How can other pavings by affine (cells or varieties) be useful?

Thank you!