Realizability of $(\mathbb{Z}/p)^n$-equivariant chain complexes

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joint work with Henrik Rüping

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Workshop on Torus Actions in Topology

Motivation: Carlsson's conjecture

Let $G = \mathbb{Z}/p\mathbb{Z} \times \ldots \times \mathbb{Z}/p\mathbb{Z} = (\mathbb{Z}/p\mathbb{Z})^n$, $\mathbb{F}$ field of characteristic $p > 0$.

Conjecture (Carlsson, '80s)

Let $X \neq \varnothing$ be a finite G-CW complex. If the G-action is free, then

$$\sum_{i \geq 0} \dim_{\mathbb{F}} H_i(X; \mathbb{F}) \geq 2^n.$$

Example

G acts freely on the n-torus $S^1 \times \ldots \times S^1$ whose homology

$$H_*(S^1 \times \ldots \times S^1; \mathbb{F}) \cong H_*(S^1; \mathbb{F}) \otimes \ldots \otimes H_*(S^1; \mathbb{F})$$

has dimension 2^n.

Motivation: algebraic version

Let $G = \mathbb{Z}/p\mathbb{Z} \times \ldots \times \mathbb{Z}/p\mathbb{Z} = (\mathbb{Z}/p\mathbb{Z})^n$.

Conjecture (Counterexamples, Iyengar, Walker, '18)

Let C be a bounded chain complex of finitely generated free $\mathbb{F}G$-modules. If C is not acyclic, then

$$\sum_i \dim_{\mathbb{F}} H_i(C) \geq 2^n.$$

Remark

algebraic version $\implies$ Carlsson's conjecture

Motivation: Goal

Question (Iyengar, Walker)

Can the counterxamples be topologically realized in order to produce counterxamples to Carlsson's conjecture?

Answer (Rüping, S)

No!

Outline

- Counterexamples
- Tools for nonrealizability
- A spectral sequence for nonrealizability
- Apply spectral sequence to counterexamples

Counterexamples: Koszul complex

$G = (\mathbb{Z}/p\mathbb{Z})^n = \langle \tau_1 \rangle \times \ldots \times \langle \tau_n \rangle$

Note

$$\mathbb{F}G \cong \frac{\mathbb{F}[x_1, \ldots, x_n]}{(x_1^p, \ldots, x_n^p)}, \quad \tau_i - 1 \leftarrow\!\shortmid x_i.$$

Definition

The Koszul complex K_* of $\mathbb{F}G$ is the dga over $\mathbb{F}G$ with

- underlying graded algebra the exterior algebra of $\mathbb{F}G$ on n generators $z_1, \ldots, z_n$
- differential determined by $d(z_i) = x_i$.

Example

$n = 2, p = 2$:

$$\begin{array}{llllllll}
K_*: & 0 \longrightarrow \mathbb{F}Gz_1 \wedge z_2 & \xrightarrow{\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}} & \mathbb{F}Gz_1 \oplus \mathbb{F}Gz_2 & \xrightarrow{\begin{pmatrix} x_1 & x_2 \end{pmatrix}} & \mathbb{F}G & \longrightarrow 0 \\
H_*(K_*): & \mathbb{F}x_1x_2z_1 \wedge z_2 & & \mathbb{F}x_1z_1 \oplus \mathbb{F}x_2z_2 & & \mathbb{F} &
\end{array}$$

Counterexamples: Mapping cone

Fact

The homology of the Koszul complex of $\mathbb{F}G$ is an exterior algebra over $\mathbb{F}$ in n generators.

$$H_*(K_*) = \Lambda_{\mathbb{F}}(x_1^{p-1}z_1, \dots, x_n^{p-1}z_n)$$

For a cycle $w \in K_i$, let $\mathrm{Cone}_*(w)$ be the mapping cone of

$$w \wedge -\colon \Sigma^i K_* \to K_*.$$

Theorem (Iyengar, Walker)

If $n \geq 8$, p odd, then there exists a cycle $w \in K_2$ such that

$$\dim_{\mathbb{F}} H_*(\mathrm{Cone}_*(w)) < 2^n.$$

Counterexamples: Realizabililty result

Theorem (Rüping, S)

There is a topological space with a free G-action whose singular chain complex is quasi-isomorphic to $\mathrm{Cone}(w)_*$ as $\mathbb{F}G$-chain complexes if and only if

- *w is a boundary or*
- *$\deg(w) \leq 1$.*

Corollary

The counterexamples of Iyengar and Walker are not topologically realizable.

Example

$n = 2$, $p = 2$, $w = x_1 x_2 e_1 \wedge e_2 \in K_2$ yields nonrealizable $\mathrm{Cone}(w)_*$

$$\mathbb{F}G \xrightarrow{\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}} (\mathbb{F}G)^2 \xrightarrow{\begin{pmatrix} x_1 & x_2 \end{pmatrix}} \mathbb{F}G \xrightarrow{x_1 x_2} \mathbb{F}G \xrightarrow{\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}} (\mathbb{F}G)^2 \xrightarrow{\begin{pmatrix} x_1 & x_2 \end{pmatrix}} \mathbb{F}G$$

(Non)realizability: Tools

- Nonequivariantly: Every nonnegatively graded chain complex C_* over $\mathbb{F}$ with $H_0(C_*) \neq 0$ can be realized by a wedge of spheres.
- Smith: Obstruction theory for realizing free G-equivariant chain complexes via simply connected free G-spaces ($\mathrm{Cone}_*(w)$ has $H_1 \neq 0$)
- If $C_* \simeq C_*(X;\mathbb{F})$ for a free G-space X, then

$$H^*(C_*/G) \cong H^*(X/G;\mathbb{F}) \cong H^*(X \times_G EG;\mathbb{F})$$

  is a graded module over $H^*(BG;\mathbb{F})$
  - the $H^*(BG;\mathbb{F})$-action can be calculated algebraically from C_*
  - the annihilator ideal $\{a \in H^*(BG;\mathbb{F}) \mid x \cup a = 0 \text{ for all } x \in H^*(X/G;\mathbb{F})\}$ must be closed under Steenrod operations

  [Idea from Carlsson's counterexamples to Steenrod problem]
  (Doesn't provide obstructions for $\mathrm{Cone}(w)_*$)

Nonrealizability: Spectral sequence

$I = (x_1, \ldots, x_n) \subset \frac{\mathbb{F}[x_1,\ldots,x_n]}{(x_1^p,\ldots,x_n^p)} = \mathbb{F}G$ is nilpotent:
For $L = n(p-1)$: $I^L = (x_1^{p-1} \ldots x_n^{p-1}) \neq 0$, but $I^{L+1} = 0$.
For a cochain complex C^* of free right $\mathbb{F}G$-modules, consider spectral sequence from filtration $\{C^* I^{L-k}\}_k$:

$$0 \subset C^* I^L \subset \ldots \subset C^* I \subset C^*$$

Theorem (Rüping, S)

For the $\mathbb{F}$-cochains $C^*(X;\mathbb{F})$ of a free G-space X this yields a multiplicative spectral sequence converging to $H^*(X;\mathbb{F})$

- $E_1^{*,q} \cong H^q(X/G;\mathbb{F}) \otimes \frac{\mathbb{F}[y_1,\ldots,y_n]}{(y_1^p,\ldots,y_n^p)}$ *with* $\deg(y_j) = 1$
- *...*

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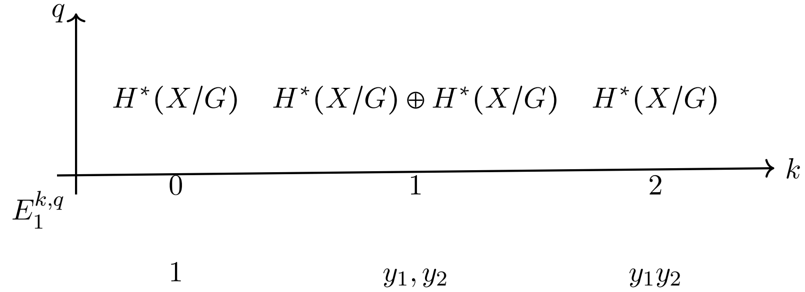
Example

$p = 2$, $n = 2$



Nonrealizability: Spectral sequence

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- $d_1(x \otimes y_j) = (-1)^{|x|} x \cup a_j$,
  where the a_j*'s form an explicit basis of* $H^1(BG;\mathbb{F})$

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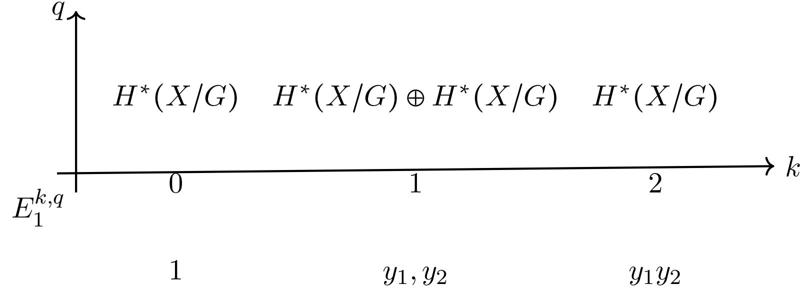
$p = 2$, $n = 2$



Nonrealizability: Spectral sequence

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For the $\mathbb{F}$-cochains $C^*(X;\mathbb{F})$ of a free G-space X this yields a multiplicative spectral sequence converging to $H^*(X;\mathbb{F})$

- $E_1^{*,q} \cong H^q(X/G;\mathbb{F}) \otimes \frac{\mathbb{F}[y_1,\ldots,y_n]}{(y_1^p,\ldots,y_n^p)}$ *with* $\deg(y_j) = 1$
- $d_1(x \otimes y_j) = (-1)^{|x|} x \cup a_j \quad , a_j \in H^1(BG;\mathbb{F})$ *explicit generator*

Strategy

To show that a free $\mathbb{F}G$-cochain complex C^* can not be realized topologically as $C^*(X;\mathbb{F})$:

- Calculate spectral sequence algebraically from definition.
- Calculate potential $H^1(BG;F)$-action on $H^*(C^*/C^*I) \cong H^*(X/G;\mathbb{F})$.
- Use this part of the multiplicative structure to find contradictions to the Leibniz rule.

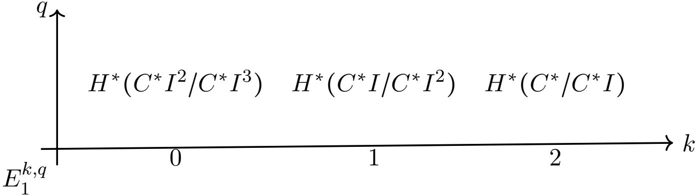
Nonrealizability: Application

Example

$n = 2$, $p = 2$, $w = x_1 x_2 e_1 \wedge e_2 \in K_2$ yields nonrealizable
$C^* = \mathrm{Cone}(w)^*$

$$\mathbb{F}G \xrightarrow{\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}} (\mathbb{F}G)^2 \xrightarrow{\begin{pmatrix} x_1 & x_2 \end{pmatrix}} \mathbb{F}G \xrightarrow{x_1 x_2} \mathbb{F}G \xrightarrow{\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}} (\mathbb{F}G)^2 \xrightarrow{\begin{pmatrix} x_1 & x_2 \end{pmatrix}} \mathbb{F}G$$

$\mathbb{F}G \cong \frac{\mathbb{F}[x_1, x_2]}{(x_1^2, x_2^2)}$, $I = (x_1, x_2)$, $I^2 = (x_1 x_2)$, $I^3 = 0$

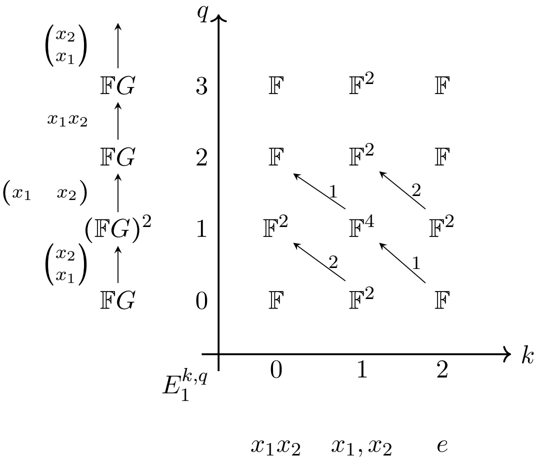


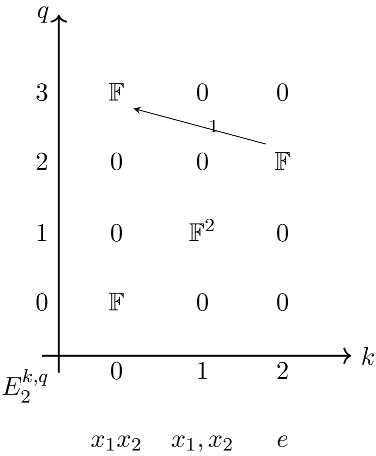
Nonrealizability: Application

$$\begin{tikzcd}
{} \\
\mathbb{F}G \arrow[u, "\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}"'] \\
\mathbb{F}G \arrow[u, "x_1 x_2"'] \\
(\mathbb{F}G)^2 \arrow[u, "\begin{pmatrix} x_1 & x_2 \end{pmatrix}"'] \\
\mathbb{F}G \arrow[u, "\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}"']
\end{tikzcd}$$

q	$H^*(C^*I^2)$	$H^*(C^*I/C^*I^2)$	$H^*(C^*/C^*I)$
3	$\mathbb{F}$	$\mathbb{F}^2$	$\mathbb{F}$
2	$\mathbb{F}$	$\mathbb{F}^2$	$\mathbb{F}$
1	$\mathbb{F}^2$	$\mathbb{F}^4$	$\mathbb{F}^2$
0	$\mathbb{F}$	$\mathbb{F}^2$	$\mathbb{F}$
$E_1^{k,q}$ $\quad k$	0	1	2
	x_1x_2	x_1, x_2	e

Nonrealizability: Application

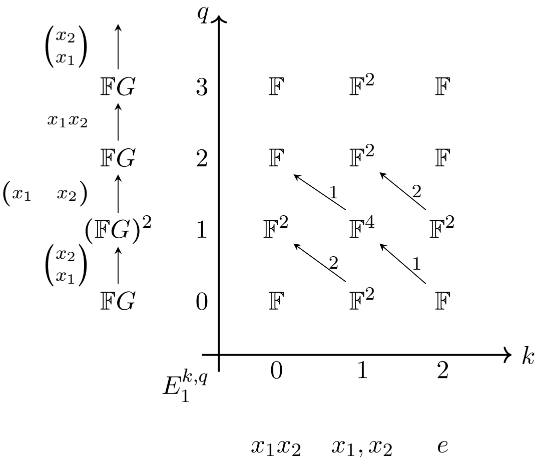


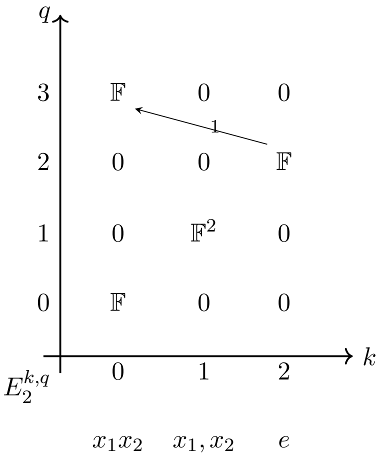


Claim: If C^* realizable, then the generator $c \in E_2^{2,2}$ is the product of two generators $b_1, b_2 \in E_2^{1,1}$, contradicting the Leibniz rule:

$$0 \neq d_2(c) = d_2(b_1 b_2) = d_2(b_1)b_2 \pm b_1 d_2(b_2) = 0$$

Nonrealizability: Application





Fact: Under
$H^q(C^*/C^*I) \otimes I^{L-k}/I^{L-k+1} \cong E_1^{k,q}, E_1^{*,q} \cong H^q(X/G) \otimes \frac{\mathbb{F}[y_1,\ldots,y_n]}{(y_1^p,\ldots,y_n^p)}$,
$x_1^{j_1} \ldots x_n^{j_n}$ corresponds to $y_1^{p-1-j_1} \ldots y_n^{p-1-j_n}$

Nonrealizability: Application

$$\begin{tikzcd}
{} \\
\mathbb{F}G \arrow[u, "\begin{pmatrix}x_2\\x_1\end{pmatrix}"'] \\
\mathbb{F}G \arrow[u, "x_1x_2"'] \\
(\mathbb{F}G)^2 \arrow[u, "\begin{pmatrix}x_1 & x_2\end{pmatrix}"'] \\
\mathbb{F}G \arrow[u, "\begin{pmatrix}x_2\\x_1\end{pmatrix}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
q & & & \\
3 & \mathbb{F} & \mathbb{F}^2 & \mathbb{F} \\
2 & \mathbb{F} & \mathbb{F}^2 & \mathbb{F} \\
1 & \color{red}{a_2, a_1} & \mathbb{F}^4 & \mathbb{F}^2 \\
0 & \mathbb{F} & \mathbb{F}^2 & \mathbb{F} \\
E_1^{k,q} & 0 & 1 & 2 & k \\
 & \overset{x_1x_2}{1} & \overset{x_1,x_2}{y_2, y_1} & \overset{e}{y_1y_2}
\end{tikzcd}
\qquad
\begin{tikzcd}
q & & & \\
3 & \mathbb{F} & 0 & 0 \\
2 & 0 & 0 & \mathbb{F} \arrow[ull, "1"'] \\
1 & 0 & \mathbb{F}^2 & 0 \\
0 & \mathbb{F} & 0 & 0 \\
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\end{tikzcd}$$

Set $a_2 = d_1(y_2) = d_1(x_1)$, $a_1 = d_1(y_1) = d_1(x_2)$.

Nonrealizability: Application

$$
\begin{array}{c}
\uparrow \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \\
\mathbb{F}G \\
\uparrow x_1 x_2 \\
\mathbb{F}G \\
\uparrow \begin{pmatrix} x_1 & x_2 \end{pmatrix} \\
(\mathbb{F}G)^2 \\
\uparrow \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \\
\mathbb{F}G
\end{array}
\qquad
\begin{array}{c|ccc}
q & & & \\
3 & \mathbb{F} & \mathbb{F}^2 & \mathbb{F} \\
2 & a_1 a_2 & \mathbb{F}^2 & a_1 a_2 y_1 y_2 \\
1 & a_2, a_1 & \mathbb{F}^4 & \mathbb{F}^2 \\
0 & \mathbb{F} & \mathbb{F}^2 & \mathbb{F} \\
\hline
E_1^{k,q} & 0 & 1 & 2 \quad k \\
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\begin{array}{c|ccc}
q & & & \\
3 & \mathbb{F} & 0 & 0 \\
2 & 0 & 0 & \mathbb{F} \\
1 & 0 & \mathbb{F}^2 & 0 \\
0 & \mathbb{F} & 0 & 0 \\
\hline
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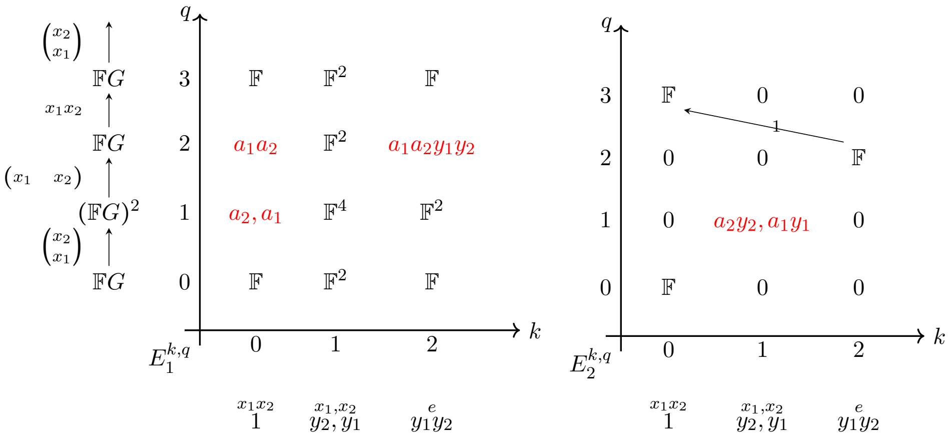
\end{array}
$$

Set $a_2 = d_1(y_2) = d_1(x_1)$, $a_1 = d_1(y_1) = d_1(x_2)$.
Note that $a_1 a_2 = d_1(a_2 y_1) = d_1(a_2 x_2) \neq 0$.

Nonrealizability: Application



Set $a_2 = d_1(y_2) = d_1(x_1)$, $a_1 = d_1(y_1) = d_1(x_2)$.
Thus the generator of $E_2^{2,2}$ is indeed a product of two generators in $E_2^{1,1}$.

Realizability: $\mathrm{Cone}(w)_*$

$G = (\mathbb{Z}/p\mathbb{Z})^n$, $\mathbb{F}$ field of characteristic $p > 0$, w cycle in Koszul complex K_*

The argument extends to establish the nonrealizability part of ...

Theorem (Rüping, S)

There is a topological space with a free G-action whose singular chain complex is quasi-isomorphic to $\mathrm{Cone}(w)_*$ as $\mathbb{F}G$-chain complexes if and only if w is a boundary or $\deg(w) \leq 1$.

Realizability:

- If w represents 0 in $H_i(K_*)$, then $\mathrm{Cone}(w)_*$ can be realized by $S^{i+1} \times T^n$.
- If w represents a nonzero homology class of degree 0, then $\mathrm{Cone}(w)_*$ is a contractible chain complex and realized by the empty space $X = \varnothing$.
- If w represents a nonzero homology class of degree 1, then $\mathrm{Cone}(w)_*$ can be realized by $S^3 \times T^{n-1}$.

Future directions

- Does $\dim_{\mathbb{F}} H_*(C_*) \geq 2^n$ hold asymptotically? [Hopkins, Seceleanu, Walker]
- Which weaker bound than 2^n holds?
- Does the algebraic version of Carlsson's conjecture hold for $p = 2$?
- The spectral sequence extends to finite p-groups. Is there a group theoretical interpretation of the d_1-differential?
- Action on Koszul homology

Thank you!