Rigidity and flexibility of Hamiltonian 4-manifolds

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A combinatorial structure on Hamiltonian manifolds

We look at a symplectic manifold (M, ω) with a torus $T = (S^1)^k$-action that is *Hamiltonian*, with *moment map* $\Phi \colon M \to \mathfrak{t}^* \cong \mathbb{R}^k$:

$$\omega(\cdot, \xi_j) = d\Phi_j.$$

The Convexity Theorem (Guillemin-Sternberg, Atiyah 1982)

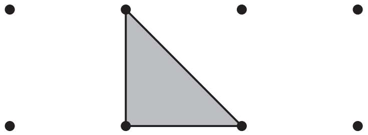
$\Phi(M)$ is a convex polytope: the convex hull of the images of the fixed points.

Special case: Delzant polytopes

If

$$k = \dim T = \frac{1}{2} \dim M$$

the action is *toric*, and (M, ω, Φ) is a *toric symplectic manifold*.
In the moment polytope $\Phi(M) \subset \mathbb{R}^k$ the k edges meeting at each vertex form a basis of $\mathbb{Z}^k$ over $\mathbb{Z}$. Hence its normal fan corresponds to a compact smooth toric variety. For example,



moment polytope for the toric action $(S^1)^2 \circlearrowright (\mathbb{CP}^2, \omega_{\mathsf{FS}})$.

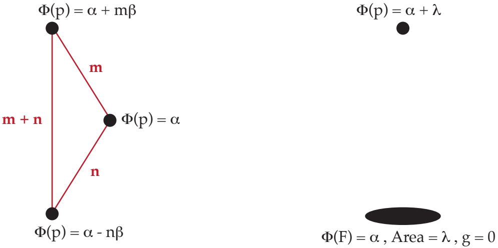
Delzant (88) classified toric symplectic manifolds by their moment polytopes.

Special case: Decorated graphs

If

$$k = \dim T = \frac{1}{2}\dim M - 1$$

the action is of *complexity one*. Karshon (99) classified complexity one spaces of dimension 4 by their *decorated graphs*.
For example,



decorated graphs for two $S^1 \circlearrowright (\mathbb{C}P^2, \omega_{\mathrm{FS}})$, with only isolated fixed points on the left, and with a fixed surface on the right.

Relating combinatorial and algebraic structures

By Masuda (2008), in the toric case, $H_T^*(M)$ as a module over $H_T^*(\mathrm{pt})$ is related to the fan defined by the moment polytope.

Here we look at the $\dim 4$ complexity one case.

Generators and relations description of $H_{S^1}^{2*}(M^4)$, Holm-K 2020

The generators correspond to the fat vertices and edges of the decorated graph, and the relations are read from the adjacency relation and edge-labels in the decorated graph. We also express the module structure in terms of the generators.

Theorem (Holm-K 2020)

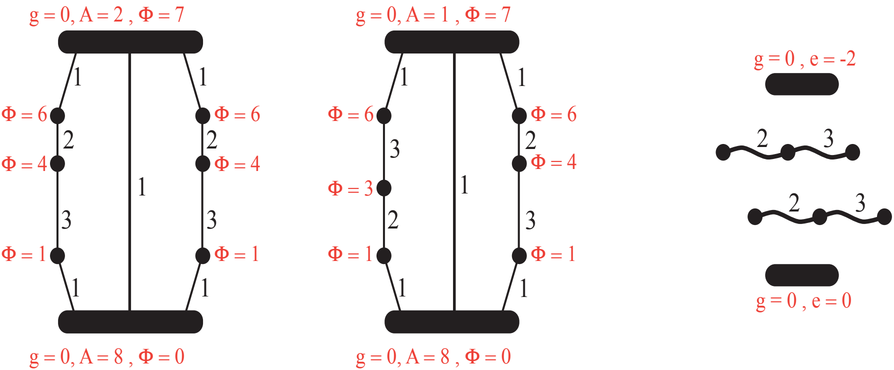
$H^{2*}_{S^1}(M)$, as a module over $H^*_{S^1}(\mathrm{pt})$, and $\dim H^1_{S^1}(M)$ are determined by the dull graph of $S^1 \circlearrowright (M, \omega)$.

The *dull graph* is obtained from the decorated graph by omitting the height and area labels, and adding a label for the self intersection of a fixed surface.

Two Hamiltonian $S^1 \circlearrowright (M^4, \omega)$ have the same dull graph iff their extended decorated graphs differ by a finite composition of

- the flip of the whole graph;
- a positive rescaling of edge-lengths and fat vertex-areas;
- a flip of a chain that begins and ends with an edge of label 1.

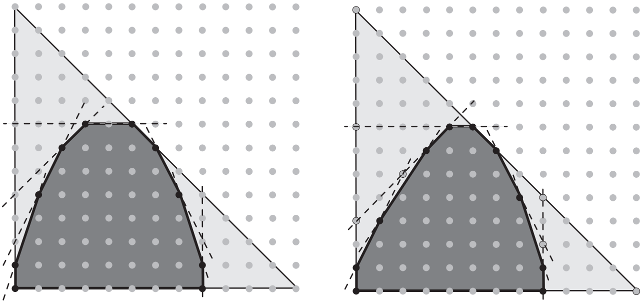
Example: different S^1-manifolds with the same dull graph



On the left are two (extended) decorated graphs that differ by a flip of one chain. On the right is the dull graph.

The toric picture of the example

The S^1-actions are obtained by precomposing the inclusion $S^1 \hookrightarrow (S^1)^2$ sending $s \mapsto (1, s)$ on the following toric actions:



Both toric actions are obtained from the toric action on $(\mathbb{CP}^2, 12\omega_{\mathsf{FS}})$ by a sequence of 7 equivariant blowups, of sizes $(5, 4, 3, 2, 2, 1, 1)$ in the left and $(5, 4, 4, 2, 2, 1, 1)$ in the right. The polytopes define different fans, corresponding to different toric varieties.

There is no equivariant diffeomorphism preserving a generic or integrable compatible almost complex structure

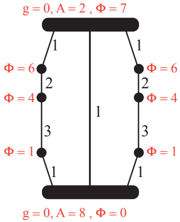
Let $F\colon M \to N$ be an equivariant diffeomorphism of S^1-manifolds, J_M an invariant almost complex structure on M, and

$$J_N := F_* J_M = dF \circ J_M \circ dF^{-1}.$$

Then

- F sends a fixed sphere to a fixed sphere with the same self intersection, and an invariant J_M-holomorphic sphere to an invariant J_N-holomorphic sphere.
- Moreover, F preserves or negates *simultaneously* the weights of the complex representations at the poles of an invariant J_M-holomorphic sphere.

Assume that J_M is ω_M-compatible and generic or integrable. Then such F preserves or negates *simultaneously* the weights of the complex representations at every point in the fixed spheres, hence at every isolated vertex connected to a fat vertex by an edge (of label 1) in both chains, and hence at all the isolated vertices in both chains.



If J_M and J_N are compatible with ω_M and ω_N, the weights of the complex representations can be read from the graphs:

$$\{\{-1,3\},\{-3,2\},\{-2,1\},\{-1,3\},\{-3,2\},\{-2,1\}\}$$

at the isolated fixed points in M, and

$$\{\{-1,2\},\{-2,3\},\{-3,1\},\{-1,3\},\{-3,2\},\{-2,1\}\}$$

at the isolated fixed points in N.
Since these sets are not equal nor differ by negation, there cannot be such a diffeomorphism.

But the equivariant cohomology modules are isomorphic

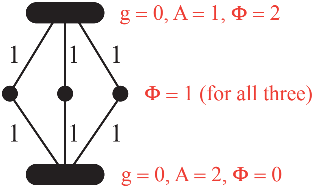
However, the two S^1-manifolds have the same dull graph, thus, by our theorem, their cohomologies are isomorphic as modules. (Here, each of the odd-dimensional equivariant cohomology groups is trivial.)

Corollary: the finiteness theorem

Theorem

The number of maximal Hamiltonian circle actions on (M^4, ω) *is finite.*

A Hamiltonian circle action is *maximal* if it does not extend to a Hamiltonian action of a strictly larger torus.



A maximal S^1-action on a 4-blowup of $\mathbb{CP}^2$.

Sketch of Proof

The proof is analogous to the proof of McDuff and Borisov (2011) for the finiteness of **toric** actions on a symplectic manifold. The key is the application of the *Hodge index theorem*.

The $\dim 4$-complexity one case is not as rigid as the toric one: $S^1 \circlearrowright (M^4, \omega)$ is not algebraic. However it is rigid enough for the proof to hold.

Theorem (Karshon 99)

A Hamiltonian $S^1 \circlearrowright (M^4, \omega)$ *admits an integrable complex structure* J *such that* (M^4, ω, J) *is Kähler, and the action is holomorphic.*

For a Hamiltonian action s, the fat vertices and edges of the decorated graph are images of holomorphic curves in (M, J). Hence the set X_s of their Poincaré duals is contained in $H^{1,1}(M, J) \cap H^2(M; \mathbb{Z})$.

We can now apply Hodge index theorem.

By Hodge index theorem, the intersection form

$$\langle \alpha, \beta \rangle := \int_M \alpha \wedge \beta$$

on $H^{1,1}(M,J) \cap H^2(M;\mathbb{R})$ is negative definite on the orthogonal complement to $[\omega]$.
For $x \in X_s$, write

$$x = y + r[\omega], \text{ where } \langle y, \omega \rangle = 0 \text{ and } r \in \mathbb{R}.$$

By Hodge index theorem

$$\langle y, y \rangle \leq 0.$$

Since x is the dual of the class of a symplectic sphere or surface S,

$$r\langle \omega, \omega \rangle = \langle x, \omega \rangle = \int_S \omega > 0.$$

We show that there are constants N, C and A, that depend only on (M, ω), such that for every s in the set S of maximal Hamiltonian $S^1 \circlearrowright (M, \omega)$, the set X_s is contained in the bounded subset

$$\{y + r[\omega] \,:\, 0 \leq -\langle y, y \rangle \leq NC^2 - A \text{ and } 0 < r \leq C\}$$

of $H^2(M; \mathbb{R})$. Therefore, the set $\cup_{s \in S} X_s \subset H^2(M; \mathbb{Z})$ is finite.

It follows from our generators and relations description of $H^{2*}_{S^1}(M)$ that a maximal Hamiltonian S^1-action s on (M, ω) is determined by the set X_s.

We conclude that the set of maximal Hamiltonian $S^1 \circlearrowright (M, \omega)$ is finite.

A soft proof for a soft property

Note that this proof of the soft finiteness property is *soft*; it does not use *hard* pseudo-holomorphic tools.
This is in contrast to the deduction of the finiteness from the characterization of the Hamiltonian circle actions on (M, ω) in Karshon-K-Pinsonnault (2015) and in Holm-K (2019), which use pseudo-holomorphic curves.