Characterizing unipolar flag manifolds by their varieties of minimal rational tangents

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Fano manifolds

Joint project with Jun-Muk Hwang and Qifeng Li (KIAS, Seoul)

Ground field is $\mathbb{C}$.

Definition

A *Fano manifold* is a smooth projective algebraic variety X such that $\bigwedge^n \mathcal{T}_X$ is ample ($n = \dim X$).

In other words: $\exists$ closed embedding $X \hookrightarrow \mathbb{CP}^N$ and a global tensor field $\sigma \in H^0(X, (\bigwedge^n \mathcal{T}_X)^{\otimes m})$ such that

$$\{x \in X \mid \sigma(x) = 0\} = X \cap H, \qquad H \subset \mathbb{CP}^N \text{ is a general hyperplane.}$$

Fano manifolds are of importance in algebraic geometry (Minimal Model Program, etc).

Picard group $\operatorname{Pic} X \simeq H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^\rho$, $\rho = \rho_X$, *Picard number*.

Restriction: $\rho = 1$ (*unipolar* Fano manifolds)

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Flag manifolds

Example

Generalized flag manifold $X = G/P$ is Fano.

Here G is a semisimple Lie group, P is a parabolic subgroup.

$\rho_X = \dim P/[P,P] =$ maximal length of $(P = P_1 \subset P_2 \subset \cdots \subset P_\rho \subset G)$

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Grassmannians

Unipolar flag manifolds: examples

1. Grassmannians:

$$G = SL_n, \quad P = \begin{array}{c|cc|} & k & \\ \hline k & * & * \\ \hline & 0 & * \\ \hline \end{array}, \quad X = \mathrm{Gr}_k(\mathbb{C}^n) = \{U \subset \mathbb{C}^n \mid \dim U = k\}$$

2. Symplectic (isotropic) Grassmannians:

$$G = Sp_n = Sp(\mathbb{C}^n, \omega) \quad (n \text{ even}, \omega \text{ a symplectic form})$$

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Rational curves

Assume: X Fano, $\rho_X = 1$.

Degree of an algebraic curve $C \subset X$:

$$\deg C \in H_2(X, \mathbb{Z}) \simeq \mathbb{Z}$$

$$\deg C = \frac{1}{i_X} |C \cap H|, \qquad X \subset \mathbb{CP}^N, \quad H \subset \mathbb{CP}^N \text{ is a general hyperplane,}$$

$$i_X \in \mathbb{N}, \ \textit{Fano index.}$$

Rational curves: $C \simeq \mathbb{CP}^1$.

X is covered by rational curves. Countably many families of such curves:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\mu} & X, \\ {\scriptstyle\pi}\big\downarrow {\scriptstyle \mathbb{CP}^1\text{-bundle}} & & \\ \mathcal{K} \text{ quasiprojective, irreducible} & & \end{array} \qquad C = \mu(\pi^{-1}(c)), \quad c \in \mathcal{K}.$$

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Minimal rational curves:

Rational curves play an important role in the geometry of Fano manifolds (goes back to S. Mori).

Families of *minimal* rational curves:

1. $\mu(\mathcal{P}) \supset X^0$, a dense open subset of X;
2. $\deg C = \min$ over all families $\mathcal{K}$ with (1) (=*dominating families*).

Analogy (Yum-Tong Siu et al):

Geodesic curves in Riemannian geometry	Minimal rational curves in the geometry of Fano manifolds
$\exists$ geodesic curve through any point in any direction	Not true in general

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VMRT

$x \in X^0 \rightsquigarrow \mathcal{K}_x = \pi(\mu^{-1}(x))$
parameterizes minimal rational curves through x.

Tangent map (proper, birational)

$$\begin{aligned} \tau_x : \mathcal{K}_x &\longrightarrow \mathbb{P}(\mathcal{T}_{X,x}) \\ C = \mu(\pi^{-1}(c)) &\longmapsto \mathbb{P}(\mathcal{T}_{C,x}) \end{aligned}$$

Variety of minimal rational tangents (VMRT) $\mathcal{C}_x = \operatorname{Im} \tau_x \subset \mathbb{P}(\mathcal{T}_{X,x})$, embedded projective variety.

Particular case: $X \subset \mathbb{CP}^N$, $\forall x \in X^0$ $\exists$ line $C \subset X$, $C \ni x$.

Then: minimal rational curves = lines in X $\implies$ $\tau_x : \mathcal{K}_x \xrightarrow{\sim} \mathcal{C}_x$.

Holds for $X = G/P$ (unipolar flag manifold); lines in X form a single family $\mathcal{K}$.

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Example: VMRT of Grassmannians

Example 1

$X = \mathrm{Gr}_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ (Plücker embedding)
$x = U = \langle v_1, \ldots, v_k \rangle \mapsto \langle v_1 \wedge \cdots \wedge v_k \rangle$

Lines:
$C = \{U \mid U' \subset U \subset U''\}$; U', U'' fixed, $\dim U' = k-1$, $\dim U'' = k+1$.

VMRT: $\mathcal{C}_x = \mathbb{P}(U^*) \times \mathbb{P}(\mathbb{C}^n/U) \xhookrightarrow{\text{Segre}} \mathbb{P}(U^* \otimes \mathbb{C}^n/U) = \mathbb{P}(\mathcal{T}_{X,x})$.

Example 2

$X = \mathrm{IGr}_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\bigwedge^k \mathbb{C}^n)$

Lines: as in Example 1 with $\omega(U', U'') = 0$. This equation cuts out $\mathcal{C}_x \subset \mathbb{P}(U^* \otimes \mathbb{C}^n/U)$.

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$C = \{U \mid U' \subset U \subset U''\};\ \ U', U''$ fixed, $\dim U' = k-1$, $\dim U'' = k+1$.

VMRT: $\mathcal{C}_x = \mathbb{P}(U^*) \times \mathbb{P}(\mathbb{C}^n/U) \xhookrightarrow{\text{Segre}} \mathbb{P}(U^* \otimes \mathbb{C}^n/U) = \mathbb{P}(\mathcal{T}_{X,x})$.

Example 2

$X = \mathrm{IGr}_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\bigwedge^k \mathbb{C}^n)$

Lines: as in Example 1 with $\omega(U', U'') = 0$. This equation cuts out $\mathcal{C}_x \subset \mathbb{P}(U^* \otimes \mathbb{C}^n/U)$.

Example: VMRT of Grassmannians

Example 1

$X = \mathrm{Gr}_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ (Plücker embedding)
$x = U = \langle v_1, \dots, v_k \rangle \mapsto \langle v_1 \wedge \dots \wedge v_k \rangle$

Lines:
$C = \{U \mid U' \subset U \subset U''\}$; U', U'' fixed, $\dim U' = k-1$, $\dim U'' = k+1$.

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Fano manifolds via VMRT

Principle (J.-M. Hwang, Ngaiming Mok, 90's)

Geometry of a unipolar Fano manifold X is controlled by projective geometry of its VMRT $\mathcal{C}_x \subset \mathbb{CP}^{n-1}$ at a general point.

Program

Recognize X by $\mathcal{C}_x$.

Theorem (S. Mori, 1979; K. Cho and Y. Miyaoka, 1998)

$\mathcal{C}_x = \mathbb{CP}^{n-1} \implies X = \mathbb{CP}^n$

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Main theorem

Main Theorem (J.-M. Hwang, N. Mok, J. Hong, Q. Li, D.T.)

Suppose $Y = G/P$ is a unipolar flag manifold, $y \in Y$.

If X is a unipolar Fano manifold such that, for $x \in X^0$,

$$\begin{array}{ccc} \mathcal{C}_x & \xrightarrow{\sim} & \mathcal{C}_y \\ \bigcap & & \bigcap \\ \mathbb{P}(\mathcal{T}_{X,x}) & \xrightarrow{\sim} & \mathbb{P}(\mathcal{T}_{Y,y}), \end{array}$$

then $X \simeq Y$.

Remark

$\mathcal{C}_y$ is locally rigid.

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$\mathcal{C}_y$ is locally rigid.

Scheme of the proof

Proof combines ideas and techniques from: pure algebraic geometry, differential geometry, Lie algebras and algebraic groups, spherical varieties.

Scheme of the proof:

$$\begin{aligned}\text{VMRT} &\rightsquigarrow \text{differential-geometric structure on } X\\ &\rightsquigarrow G' \curvearrowright X \supset X^0 = G'/P',\ \operatorname{codim}(X \setminus X^0) \geq 2\\ &\implies G' = G,\ P' = P,\ X = Y\end{aligned}$$

May assume $G = (\operatorname{Aut} Y)^\circ$, $P = P(\alpha_k)$, α_k is a simple root of G.

Two cases:

1. α_k long; example: $Y = \mathrm{Gr}_k(\mathbb{C}^n)$;
2. α_k long; example: $Y = \mathrm{IGr}_k(\mathbb{C}^n)$, $1 < k < n/2$.

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Step 1: VMRT of Y

Lie algebra grading:

$$\mathfrak{g} = \operatorname{Lie} G = \overbrace{\underbrace{\mathfrak{g}_{-d} \oplus \cdots \oplus \mathfrak{g}_{-1}}_{\mathfrak{p}_u} \oplus \mathfrak{g}_0}^{\mathfrak{p} = \operatorname{Lie} P} \oplus \underbrace{\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_d}_{\mathfrak{p}_u^+ \simeq \mathcal{T}_{Y,y}}$$

1. Long root case: $\mathcal{C}_y =$ the unique closed orbit of $G_0 \curvearrowright \mathbb{P}(\mathfrak{g}_1)$.
2. Short root case: $\mathcal{C}_y = \mathcal{C}_0 \sqcup \mathcal{C}_1 \subset \mathbb{P}(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$, $\mathcal{C}_0 =$ the open P-orbit, $\mathcal{C}_1 = \mathcal{C}_y \cap \mathbb{P}(\mathfrak{g}_1) =$ the closed P-orbit.

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VMRT of Y: examples

Examples

1. $Y = \mathrm{Gr}_k(\mathbb{C}^n)$, $\quad \mathfrak{g} = \mathfrak{sl}_n = \begin{array}{c|cc} & k & \\ \hline k & 0 & -1 \\ & 1 & 0 \end{array}$,

$$\mathcal{C}_y = \begin{array}{|c|c|} \hline 0 & 0 \\ \hline \mathrm{rk} = 1 & 0 \\ \hline \end{array} = \mathbb{P}(\mathbb{C}^k)^* \times \mathbb{P}(\mathbb{C}^n/\mathbb{C}^k).$$

2. $Y = \mathrm{IGr}_k(\mathbb{C}^n, \omega)$ $(1 < k < n/2)$, $\quad \mathfrak{g} = \mathfrak{sp}_n = \begin{array}{c|ccc|c} & k & & & \\ \hline k & 0 & -1 & -2 & \\ & 1 & 0 & -1 & \\ & 2 & 1 & 0 & k \\ \hline & & & k & \end{array}$,

$$\mathcal{C}_y = \{(U', U'') \mid \omega(U', U'') = 0\},$$
$$\mathcal{C}_0 = \{\text{U'' non-isotropic}\}, \quad \mathcal{C}_1 = \{\text{U'' isotropic}\}.$$

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$$\widehat{\mathcal{C}}_x \subset \mathcal{T}_{X,x} \rightsquigarrow \mathcal{W}_x = \langle \widehat{\mathcal{C}}_x \rangle \subset \mathcal{T}_{X,x} \rightsquigarrow \mathcal{W} \subset \mathcal{T}_{X^0}$$

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$$\mathcal{W}^k = \mathcal{W}^{k-1} + [\mathcal{W}^{k-1}, \mathcal{W}] \cdot \mathcal{O}_X$$

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Proposition

$\mathcal{W}^d = \mathcal{T}_{X^0}$ *(unipolarity is essential!)*

Definition

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Symbol algebra: properties

- $\mathfrak{g}_x$ depends on $\mathcal{C}_x \simeq \mathcal{C}_y$ only;
- $\mathfrak{g}_x \simeq \mathfrak{p}_\mathrm{u}^+$ in the long root case.

Short root case: $\mathfrak{g}_x$ sometimes too coarse $\implies$ modify the definition.

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Fine symbol algebra $\mathfrak{g}_x = \mathcal{U}_x^1 \oplus \mathcal{U}_x^2/\mathcal{U}_x^1 \oplus \cdots$, where:

$$\mathcal{U}^1 \subset \mathcal{U}^2 = \mathcal{W}^1 \subset \mathcal{U}^3 \subset \mathcal{U}^4 = \mathcal{W}^2 \subset \dots$$

$\mathcal{U}_x^1 = \langle \widehat{\mathcal{C}}_1 \rangle, \quad \mathcal{U}^{2k+1} = \mathcal{W}^k + [\mathcal{W}^k, \mathcal{U}] \cdot \mathcal{O}_X.$

Properties of the fine symbol algebra:

- $\mathfrak{g}_x$ may depend on X, not only on $\mathcal{C}_x$;
- $\mathfrak{g}_x \simeq \mathfrak{p}_{\mathrm{u}}^+$ or its degeneration: finitely many possibilities.

Step 3: universal prolongation

Problem

Given a positively graded finite-dimensional nilpotent Lie algebra $\mathfrak{h}^+ = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_d$ embed it into a $\mathbb{Z}$-graded Lie algebra $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_{-1} \oplus \mathfrak{h}_{-2} \oplus \cdots$ such that $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{h}^+) \subset \mathfrak{h}^+$ and $\mathfrak{h}_0$ is prescribed.

Definition

We call such an $\mathfrak{h}$ a *prolongation* of $(\mathfrak{h}^+, \mathfrak{h}_0)$.

Definition-Proposition

Among all prolongations $\mathfrak{h}$ there exists the biggest one, called the *universal prolongation* of $(\mathfrak{h}^+, \mathfrak{h}_0)$. (May happen $\dim \mathfrak{h} = \infty$!)

Criterion

A prolongation $\mathfrak{h}$ is universal iff $H^1(\mathfrak{h}^+, \mathfrak{h})_{\deg<0} = 0$.

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Universal prolongation of the symbol algebra

Our situation:

$\mathfrak{g}_x$ (coarse or fine) $\hookrightarrow$ universal prolongation

$$\mathfrak{g}' = \mathfrak{g}_x \oplus \overbrace{\mathfrak{g}'_0 \oplus \mathfrak{g}'_{-1} \oplus \mathfrak{g}'_{-2} \oplus \cdots}^{\mathfrak{p}'},$$

$$\mathfrak{g}'_0 = \mathrm{Lie}\,\mathrm{Aut}(\mathfrak{g}_x, \widehat{\mathcal{C}}_x).$$

Proposition

- $\dim \mathfrak{g}' < \infty$
- *Long root case:* $\mathfrak{g}' = \mathfrak{g}$, $\mathfrak{p}' = \mathfrak{p}$.
- *Short root case:* $(\mathfrak{g}', \mathfrak{p}') = (\mathfrak{g}, \mathfrak{p})$ *or its degeneration.*
- *Any case:* $\dim G'/P' = \dim G/P = \dim X$.

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Step 4: Cartan connection

Definition

A *Cartan connection* of type $(\mathfrak{g}', \mathfrak{p}')$ on X^0 is a principal P'-bundle $\mathcal{P} \to X^0$ together with a $\mathfrak{g}'$-valued 1-form $\gamma : \mathcal{T}_{\mathcal{P}} \to \mathfrak{g}'$ such that:

- γ is P'-equivariant;
- $\gamma : \mathcal{T}_{\mathcal{P},p} \xrightarrow{\sim} \mathfrak{g}'$, $\forall p \in \mathcal{P}$;
- $\gamma : \mathcal{T}_{\mathcal{P}_x} \xrightarrow{\sim} \mathfrak{p}'$ is the Maurer–Cartan form, $\forall x \in X^0$.

Proposition

- $\exists$ *Cartan connection of type* $(\mathfrak{g}', \mathfrak{p}')$ *on a dense open subset* $X^0 \subset X$;
- *curvature* $K = d\gamma + \frac{1}{2}\gamma \wedge \gamma = 0$.

Proof: inductive construction starting from a principal G_0'-bundle $\mathcal{P}^0 \to X^0$ associated with $\widehat{\mathcal{C}} \to X^0$, based on N. Tanaka's method. Obstructions are sections of a vector bundle with fiber $H^2(\mathfrak{g}_-, \mathfrak{g}')_{\deg<0}$.

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Step 5: Transitive group action

Proposition

$\exists$ action $G' \curvearrowright X$ with a dense open orbit $X^0 \simeq G'/P'$, $\operatorname{codim}(X \setminus X^0) \geq 2$.

Proof based on flatness of the Cartan connection and Cartan–Fubini type extension theorem (J.-M. Hwang, N. Mok, 2001).

Case 1: $G' = G, \ P' = P \implies X^0 = Y \implies X = X^0$.

Case 2: G' non-reductive (only short root case) $\implies G' = G'_{\mathrm{uni}} \rtimes G'_{\mathrm{red}}$, Borel subgroup $B \subset G'_{\mathrm{red}}$ acts on X^0 with a dense open orbit, i.e., X^0 is a spherical G'_{red}-variety.

Theory of spherical varieties $\implies X^0$ admits no G'_{red}-equivariant compactifications with small boundary.

This excludes Case 2 and completes the proof.

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