Equivariant unitary bordism for torus groups

(Based on the joint work with Jun Ma and Wei Wang)

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Outline

- Notations and background

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- Question

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- Question
- Main results

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- Main results
- Proofs

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- Main results
- Proofs
- Further discussion and problem

Unitary manifolds

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A **unitary manifold** M is a compact, oriented, smooth manifold whose tangent bundle admits a stably almost complex structure (i.e.,

$$J : TM \oplus \underline{\mathbb{R}}^l \longrightarrow TM \oplus \underline{\mathbb{R}}^l$$

such that $J^2 = -id$).

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Milnor and Novikov: classifying all closed unitary manifolds up to unitary bordism.

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Ω_*^U forms a ring with the following addition and multiplication

$$[M_1] + [M_2] = [M_1 \sqcup M_2]$$

$$[M] \cdot [N] = [M \times N]$$

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Remark

Different sets of generators for Ω_*^U can be reconstructed.

- Buchstaber–Ray: a set of generators from complex projective toric manifolds.
- L–Panov: a set of generators from quasitoric manifolds.
- Hirezbruch problem

More history backgrounds for various bordisms.

Bordism	Structure group	invariant	ring
unoriented (Thom)	$O(n)$	Stiefel–Whitney numbers	Ω_*^O
orientable (Wall et al.)	$SO(n)$	Stiefel–Whitney numbers Pontryagin numbers	Ω_*^{SO}
unitary (Milnor, Novikov)	$U(n)$	**Chern numbers**	Ω_*^U
special unitary (Conner–Floyd et al.)	$SU(n)$	Chern numbers KO-theory char. numbers	Ω_*^{SU}
spin (Anderson, Brown, Peterson et al.)	Spin(n)	Stiefel–Whitney numbers KO-theory char. numbers	Ω_*^{Spin}
⋮	⋮	⋮	⋮

Equivariant case

G: compact Lie group

Definition

A unitary G-manifold is a unitary manifold with a G-action preserving the unitary structure (i.e., there exists the following commutative diagram

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}}^l & \xrightarrow{J} & TM \oplus \underline{\mathbb{R}}^l \\ g\downarrow & & \downarrow g \\ TM \oplus \underline{\mathbb{R}}^l & \xrightarrow{J} & TM \oplus \underline{\mathbb{R}}^l \end{array}$$

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where $J^2 = -id$ and $g \in G$.

Example: Quasi-toric $(2n)$-manifolds are closed unitary T^n-manifolds.

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$\Omega_*^{U,G}$ also forms a ring.

§2 Question

Natural question

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- The determination of the ring structure of $\Omega_*^{U,G}$

On the complete invariant

Theorem (tom Dieck, 1971)

Let $G = T^k \times \mathbb{Z}_m$. Then $[M]_G = 0$ in $\Omega_*^{U,G} \Longleftrightarrow$ all equivariant K-theoretic Chern numbers of M vanish.

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Theorem (Guillemin–Ginzburg–Karshon, 2002)

Let $G = T^k$. Then a closed unitary T^k-manifold M with only isolated fixed points represents the zero element in $\Omega_*^{U,T^k} \Longleftrightarrow$ all equivariant cohomology Chern numbers of M vanish.

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Without the restriction of isolated fixed-points, Guillemin–Ginzburg–Karshon posed

Conjecture (Guillemin–Ginzburg–Karshon, 2002)

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Remark

In their book [Moment maps, cobordisms, and Hamiltonian groups actions, Math. Sur. and Mono. 98, AMS, 2002] , Guillemin–Ginzburg–Karshon discussed the problem of calculating the ring $\mathcal{H}_*^{T^k}$ of equivariant Hamiltonian bordism classes of all unitary Hamiltonian T^k-manifolds.

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Then Guillemin – Ginzburg – Karshon constructed a monomorphism

$$\mathcal{H}_*^{T^k} \longrightarrow \Omega_{*+2}^{U,T^{k+1}}$$

On the complete invariant

Let G be a compact Lie group.

Conjecture (Bix–tom Dieck)

All G-equivariant K-theoretic Chern numbers form a full system of invariants of G-equivariant unitary bordism $\Longleftrightarrow$ $G = T^k \times \mathbb{Z}_m$.

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Remark

Bix-tom Dieck showed that when G is finite, the above conjecture is true.

On the structure of $\Omega_*^{U,G}$

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On the module structure

Evenness Conjecture posed by Uribe in 2018 ICM

$\Omega_*^{U,G}$ is a free Ω_*^{U}-module on even-dimensional generators whenever G is a compact Lie group.

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Corollary

Mixed equivariant characteristic numbers separate equivariant Hamiltonian bordism.

Proof of Theorem A

Key points

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Key points

- Kronecker pairing between bordism and cobordism
- Universal toric genus

–Kronecker pairing between bordism and cobordism

Notions-homotopic bordism and cobordism

- Homotopic bordism

$$\begin{aligned} MU_*(X) &= \lim_{r\to\infty} [S^{2r+*}, X_+ \wedge MU(r)] \\ &= \lim_{r\to\infty} \pi_{2r+*}(X_+ \wedge MU(r)) \end{aligned}$$

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where $X_+ = X \cup \{pt\}$, $MU(r)$: Thom space of universal complex r-dim. vector bundle over $BU(r)$.

- Homotopic cobordism

$$MU^*(X) = \lim_{r\to\infty}[S^{2r-*} \wedge X_+, MU(r)] = (\text{or} \lim_{r\to\infty}[\Sigma^{2r-*}(X_+), MU(r)])$$

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Geometric interpretations for elements in $MU_*(X)$ and $MU^*(X)$

- Homotopic bordism: Thom-Pontryagin construction tells us that

$$MU_*(X) \cong \Omega_*^U(X)$$

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- Homotopic bordism: Thom-Pontryagin construction tells us that
$$MU_*(X) \cong \Omega_*^U(X)$$
where $\Omega_*^U(X)$ is formed by the bordism classes of singular manifolds $f : M \longrightarrow X$ for M: closed unitary manifold
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Quillen gave a geometric interpretation

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Quillen's geometric interpretation of elements in $MU^*(X)$

Each element $\alpha \in MU^{\pm n}(X)$ can be represented by an oriented complex map $f : M \longrightarrow X$,

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If n is odd, E is replaced by $E \times \mathbb{R}$.

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$\alpha \in MU^n(X)$: represented by a map $\Sigma^{2r-n}(X_+) \longrightarrow MU(r)$

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$\Longrightarrow \langle \alpha, \beta \rangle$: represented by the composite map

$$S^{2r+2k+m-n} \xrightarrow{\Sigma^{2r-n}\beta} \Sigma^{2r-n}(X_+) \wedge MU(k) \xrightarrow{\alpha \wedge id} MU(r) \wedge MU(k) \longrightarrow MU(r+k)$$

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Then $\langle \alpha, \beta \rangle$ is the bordism class of the pull-back $\widetilde{f}^*(E)$

$$\begin{array}{ccc} \widetilde{f}^*(E) & \xrightarrow{\widetilde{f}} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

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- Re-defined by Buchstaber–Panov–Ray in a geometric way as follows:

$$[M]_{T^k} \longmapsto [\pi : ET^k \times_{T^k} M \longrightarrow BT^k]$$

equivariant Chern class and number

$\pi : EG \to BG$ is the universal principal G-bundles.

The Borel construction gives us $EG \times_G \tau_M$ over $EG \times_G M$.

- ***G equivariant Chern class***

$$c^G(M) := c(EG \times_G \tau_M).$$

- ***G equivariant Chern number***

  The constant map gives $p_! : H^*_G(M) \to H^*(BG)$. Then

$$c^G_\omega[M] := p_!(c^G_\omega(M))$$

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By universal toric genus and Kronecker pairing,

$$\langle \Phi([M]_{T^k}), [f : N \longrightarrow BT^k] \rangle = [\widetilde{f}^*(ET^k \times_{T^k} M)] \in MU_* = \Omega_*^U$$

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Remark: $\widetilde{f}^*(ET^k \times_{T^k} M)$ is a closed unitary manifold of dimension=$\dim M + \dim N$.

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$\Longrightarrow [M]_{T^k} = 0$ since Φ is injective.

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Step I:(continued)

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$\Longrightarrow$

$$\mathcal{T}\widetilde{f}^*(ET^k \times_{T^k} M) \cong \pi'^*\mathcal{T}N \oplus \widetilde{f}^*(ET^k \times_{T^k} \mathcal{T}M).$$

$\Longrightarrow$

$$c_\omega(\widetilde{f}^*(ET^k \times_{T^k} M)) = \widetilde{f}^*(c_\omega^{T^k}(M)) + \pi'^*(c_\omega(N)) + \sum \pi'^*(\beta_j) \cdot \widetilde{f}^*(\gamma_j)$$

where $\omega = (i_1, ..., i_s)$ is a partition with $|\omega| = \dim \widetilde{f}^*(ET^k \times_{T^k} M)$

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$$\begin{aligned} & c_\omega[\widetilde{f}^*(ET^k \times_{T^k} M)] = (p\pi')_!(c_\omega(\widetilde{f}^*(ET^k \times_{T^k} M))) \\ = \; & (p\pi')_!(\widetilde{f}^*(c_\omega^{T^k}(M))) + (p\pi')_!(\pi'^*(c_\omega(N))) \\ & +(p\pi')_!(\sum \pi'^*(\beta_j) \cdot \widetilde{f}^*(\gamma_j)) \\ = \; & p_! f^*(\pi_!(c_\omega^{T^k}(M))) + p_!(\pi'_!\pi'^*(c_\omega(N))) \\ & +p_!(\sum \pi'_!\pi'^*(\beta_j) \cdot \pi'_!\widetilde{f}^*(\gamma_j)) \\ = \; & 0 \end{aligned}$$

where $g_! : H^*(X) \longrightarrow H^*(Y)$ is the Gysin map for a map $g : X \longrightarrow Y$.

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Note: Clearly if $|\omega| < m$, then $\pi_!(c_\omega^{T^k}(M)) = 0$.

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For each J, choose $N = \mathbb{C}P^{j_1} \times \cdots \times \mathbb{C}P^{j_k}$, we can obtain that $n_J = 0$.

Another approach

For $G = T^k$, consider

$$\Omega_*^{U,G} \xrightarrow{\Phi} MU^*(BG) \xrightarrow{B} H^*(BG) \otimes \mathbb{Z}[[\mathbf{a}]]$$

where B is the Boardman map, simply given by

$$B\Phi([M]_G) = B([EG \times_G M \longrightarrow BG]) = \sum_{\omega} S_\omega^G[M] b_\omega$$

with $(1 + b_1 t + b_2 t^2 + \cdots) \cdot (1 + a_1 t + a_2 t^2 + \cdots) = 1$.

Theorem

B is injective. Thus,

$$[M]_G = 0 \Leftrightarrow S_\omega^G[M] = 0 \text{ for all } \omega.$$

Localization

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\begin{array}{ccccc}
\Omega_*^{U,G} & \xrightarrow{\Phi} & MU^*(BG) & \xrightarrow{B} & H^*(BG)\otimes\mathbb{Z}[[a]] \\
\downarrow \Lambda & & \downarrow \Lambda' & & \downarrow \Lambda'' \\
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By direct calculation,

$$(S^{-1}B)(S^{-1}\Phi)\ \Lambda([M]_G) = \sum_F \frac{\overline{v}(\nu_F)\cdot\overline{v}(\tau_F)}{e^G(\nu_F)} \in H^*(BG)\otimes\mathbb{Z}[[a]].$$

Corollary

$\{\nu_F \to F\}$ is the fixed point data of an unitary G-manifold if and only if

$$\sum_F \frac{\overline{v}(\nu_F)\cdot\overline{v}(\tau_F)}{e^G(\nu_F)}[F] \in H^*(BG)\otimes\mathbb{Z}[[a]].$$

In particular, for the case in which F is finite (i.e., some isolated fixed points), we have that

Corollary

Some complex T^k-representations W^1, ..., W^s of dimension $2n$ are the fixed point data of an unitary G-manifold if and only if for any symmetric homogeneous polynomial $f(x)$ over $\mathbb{Z}$ in n variables,

$$\sum_{r=1}^{s} \frac{f(x_1^r, \dots, x_n^r)}{x_1^r \cdots x_n^r} \in H^*(BG),$$

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This answers the following question

Buchstaber–Panov–Ray Problem in 2010

For any set of complex T^k-representations W_x, is there a necessary and sufficient conditions for the existence of a tangentially stably complex T^k manifold with the given representations as fixed point data?

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Like non–equivariant case, to give the criterion for detecting the generators of Ω_*^{U,T^k}.

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Like non–equivariant case, to give the criterion for detecting the generators of Ω_*^{U,T^k}.

Answer: No result.

Thank You!