Displaying the cohomology of toric line bundles (Klaus Altmann)

(Toronto / Berlin, 5/14/20)

w/ $\boxed{\text{Jarek Buczyński, Lars Kastner, David Ploog, Annalena Wirz}}$

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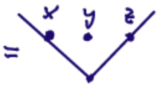
① $X/\mathbb{C}$ algebraic variety $\qquad M_{\mathbb{R}} = M \otimes \mathbb{R}$, $N_{\mathbb{R}} = N \otimes \mathbb{R}$

$T \cong (\mathbb{C}^*)^d \curvearrowright X \qquad M, N \ (\cong \mathbb{Z}^d)$; $M \times N \to \mathbb{Z}$

$\sigma \subseteq N_{\mathbb{R}} \rightsquigarrow$ affine $\underset{\text{Spec } \mathbb{C}[\sigma^\vee \cap M]}{\overset{\|}{TV(\sigma)}}$, e.g.: $\sigma =$  $\Rightarrow TV(\sigma) = \mathbb{C}^2$

Σ fan in $N_{\mathbb{R}}$ (e.g.: complete) $\rightsquigarrow X = TV(\Sigma) := \bigcup_{\sigma \in \Sigma} TV(\sigma)$ e.g.:  $\rightsquigarrow \mathbb{P}^2$

Singularities are possible: $\sigma^\vee =$ 

$\rightsquigarrow$ equation $xz = y^2$

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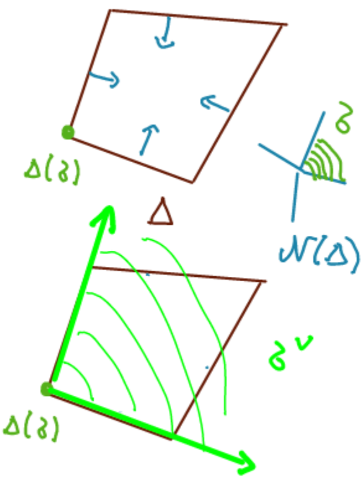
② Divisors: a) Weil divisors: $\rho \in \Sigma(1) \rightsquigarrow D_\rho = \overline{orb}(\rho)$

$\hookrightarrow D = \sum_{\rho \in \Sigma(1)} \lambda_\rho \cdot D_\rho$, $\lambda_\rho \in \mathbb{Z} \rightsquigarrow \mathcal{O}_X(D) \subseteq \mathbb{C}(X)$

reflexive, rank one

b) Cartier divisors (nef ✓): $\Delta \subseteq M$ polytope with $\Sigma \leq \mathcal{N}(\Delta)$



$\sigma \in \Sigma$ full-dimensional $\rightsquigarrow$ full cone in $\mathcal{N}(\Delta)$

$\rightsquigarrow$ vertex $\Delta(\sigma) \in \Delta \cap M$

$\rightsquigarrow x^{\Delta(\sigma)} \cdot \mathbb{C}[\sigma^\vee \cap M] \subseteq \mathbb{C}[M]$

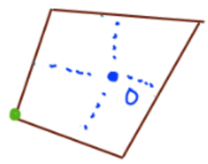
$\qquad \overset{!!}{} \qquad\qquad\qquad \|$

$\mathcal{O}_\sigma(\Delta) \subseteq j_* \mathcal{O}_T$

$\Downarrow \qquad$ via $j: T \hookrightarrow X$ open

glues to $\mathcal{O}_X(\Delta)$

c) Relation: Δ given $\rightsquigarrow D(\Delta) = \sum_{\rho \in \Sigma(1)} \underbrace{\overbrace{-\min \langle \Delta, \rho \rangle}^{\lambda_\rho}}_{\text{lattice distance of } 0 \leftrightarrow \rho\text{-facet}} \cdot D_\rho$



d) $\operatorname{Pic} TV(\Sigma) \supseteq \operatorname{Nef} = \{\Delta \,/\, \Sigma \leq \mathcal{N}(\Delta)\}$ with Minkowski addition

$\operatorname{Pic} TV(\Sigma) = \operatorname{Nef} - \operatorname{Nef}$ (Grothendieck group)

$\qquad \Psi$

$(\Delta^+, \Delta^-) = \Delta^+ - \Delta^- \rightsquigarrow \mathcal{O}_X(\Delta^+ - \Delta^-) = \mathcal{O}_X(\Delta^+) \otimes \mathcal{O}_X(\Delta^-)^{-1}$

③ Cohomology

D = divisor $\rightsquigarrow$ $H^i(X, \mathcal{O}(D))$ is M-graded

$m \in M \rightsquigarrow H^i(X, \mathcal{O}(D))(m)$ = f.d. $\mathbb{C}$-vector space

- Weil divisor: $D = \sum \lambda_\rho \cdot D_\rho$

$$\rightsquigarrow V_{D,m} := \bigcup_{\sigma \in \Sigma} \operatorname{conv}\{\rho \in \sigma(1) \mid \langle m, \rho \rangle < -\lambda_\rho\} \subseteq N_{\mathbb{R}} \setminus \{0\}$$

(Σ = smooth $\rightsquigarrow \exists$ deformation retract: subcomplex of $\Sigma \setminus 0$)

Ⓦ Theorem: $H^i(X, \mathcal{O}(D))(m) = \tilde{H}^{i-1}(V_{D,m})$ (>50 years old ...)

$$\boxed{\tilde{H}^{-1}(Z) = \begin{cases} 0 & \text{if } Z \neq \emptyset \\ \mathbb{C} & \text{if } Z = \emptyset \end{cases}}$$

- Cartier divisor: $\Delta = \Delta^+ - \Delta^-$

Ⓒ Theorem: $H^i(X, \mathcal{O}(\Delta^+ - \Delta^-))(m) = \tilde{H}^{i-1}((\Delta^- + m) \setminus \Delta^+)$
(2018/19)

$$(\Delta^- + m) \setminus \Delta^+ \cong \Delta^- \setminus (\Delta^+ - m) \quad \text{(shift)}$$

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④ Examples

(i) $H^0(\Delta^+ - \Delta^-)(m) = \begin{cases} 0 & \text{if } (\Delta^- + m) \setminus \Delta^+ \neq \emptyset \\ \mathbb{C} & \text{if } (\Delta^- + m) \setminus \Delta^+ = \emptyset \iff \Delta^- + m \subseteq \Delta^+ \end{cases}$

i.e.: $H^0(\Delta^+ - \Delta^-) \triangleq$ lattice points $m \in M$ of the polytope

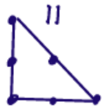
$$\{m \in M_{\mathbb{R}} \mid \Delta^- + m \subseteq \Delta^+\} =: (\Delta^+ - \Delta^-)$$

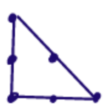
(like an ideal quotient)

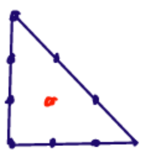
e.g.: $\mathcal{O}(\Delta)$ usual $\triangleq (\Delta, \bullet) \Rightarrow (\Delta - \bullet) = \Delta$

(ii)  $\Sigma = \mathcal{N}(\Delta)$ $\rightsquigarrow \mathcal{O}(-2) \triangleq (\bullet, 2\Delta)$

$\mathbb{P}^2$

2Δ = 

$\rightsquigarrow H^2(\mathcal{O}(-2))_m \triangleq H^1($  $\setminus \overset{(-m)}{\bullet}) = 0$

but: $\mathcal{O}(-3) \rightsquigarrow$  has a cycle for exactly one $m \in \mathbb{Z}^2$!

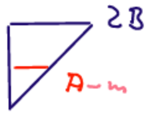
(iii) $X = (\mathbb{P}^2)' = \mathbb{F}(1)$ = del Pezzo of degree 8: 

$\Sigma = \mathcal{N}($  $)$

$\|$

$(X \to \mathbb{P}^1)$ $(X \to \mathbb{P}^2)$

; $\mathrm{Pic} = \mathbb{Z}\cdot A \oplus \mathbb{Z}\cdot B$

$\mathcal{O}(A-2B)$: $H^1 \mathrel{\hat=}$  $= 2B \setminus (A-m)$

We have seen that $H^1(\mathcal{O}(A) \otimes \mathcal{O}(2B)^{-1}) = \mathbb{C}$

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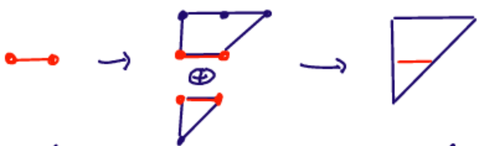
$\mathrm{Ext}^1(2B, A)$ ⇝ ↳ corresponds to a non-trivial extension $0 \to \mathcal{O}(A) \to * \to \mathcal{O}(2B) \to 0$

- Work in progress: (Amelie Flatt, Lutz Hille)



(exact sequence of polytopes)

- interesting example: $A =$  $B' =$ } $H^1(A-B')$:  ⇝ the 2 components are <u>not</u> lattice polytopes

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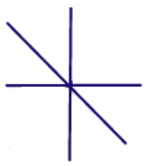
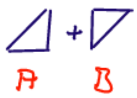
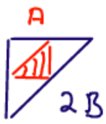
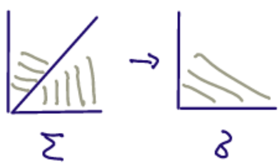
(iv) $\Sigma =$  $= \mathcal{N}($  $) \Rightarrow TV(\Sigma) = (\mathbb{P}^2)''' =$ del Pezzo of deg 6

 (general $\mathrm{Pic} = \mathbb{Z}^4$)

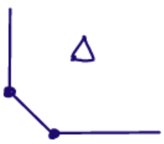
$H^1(A-2B) = \mathbb{C}^2$:  (has three components)

(v) The theorem does also work for semiprojective (projective over affine) varieties

$\tilde{\mathbb{A}}^2 \to \mathbb{A}^2$ ⥯ Σ with $|\Sigma| = \delta$ = convex cone



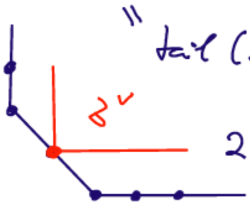
(in $N_{\mathbb{R}}$)

in $M_{\mathbb{R}}$:  ⇝ $\Delta \mathrel{\hat=} (-E) = $ nef

$\delta^\vee \mathrel{\hat=} 0$-divisor

$\|$

tail(Δ)

$H^1(2E)$ ⇝ $2E = (\delta^\vee, 2\Delta)$ ⇝ $2\Delta \setminus (\delta^\vee - m)$ 

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$\mathbb{C}$ (and there is a huge $H^0 \mathrel{\hat=} 2\Delta \subseteq \delta^\vee - m$)

(5) Proof of the theorem

w.l.o.g.: $m = 0$

1st variant: (A) $\longrightarrow$ (B)

$$\begin{array}{ccc} \tilde H^\bullet(V_{0,D}) & \overset{?}{=} & \tilde H^\bullet(\Delta^- \setminus \Delta^+) \\ \cap & & \cap \\ N_{\mathbb{R}} & & M_{\mathbb{R}} \end{array}$$

π_N, π_M

– we need a homotopy equivalence

$\rightsquigarrow$ construct $W \subseteq M_{\mathbb{R}} \oplus N_{\mathbb{R}}$ such that π_M, π_N have contractible fibers

($\hat{=}$ Fourier transformation)

2nd variant: [direct approach via Čech cohomology]

- $\mathcal{L} = \mathcal{O}(\Delta^+ - \Delta^-)$ is quasi coherent $\rightsquigarrow$ use the acyclic covering via the affine $U_\sigma := TV(\sigma) \subseteq X$

$$H^0(U_\sigma, \Delta^+ - \Delta^-) = \begin{cases} 0 & \text{if } 0 \notin (\Delta^+(\sigma) - \Delta^-(\sigma)) + \sigma^\vee \\ \mathbb{C} & \text{if } 0 \in (\Delta^+(\sigma) - \Delta^-(\sigma)) + \sigma^\vee \iff \Delta^-(\sigma) \in \Delta^+(\sigma) + \sigma^\vee \end{cases}$$

$\rotatebox{90}{$\ni$}$ $(x^m \in \mathbb{C}[M]$ with $m = 0)$

- $\Delta^+ \setminus \Delta^-$ is covered by $S(\sigma) := \Delta^- \setminus (\Delta^+ + \sigma^\vee)$
  $= \Delta^- \setminus (\Delta^+(\sigma) + \sigma^\vee)$

  either • retractable to $\Delta^-(\sigma) \in S(\sigma)$ $\iff 0$

  or • $\emptyset$ $(\iff \Delta^-(\sigma) \in \Delta^+(\sigma) + \sigma^\vee)$ $\iff \mathbb{C}$

  $\Big\} = H^0(\Delta^-, S(\sigma))$