# The generalized Kummer construction and cohomology rings 

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- I.A. Taimanov: A canonical basis of two-cycles on a K3 surface. Sb. Math. 209:8 (2018), 1248-1256.
- I.A. Taimanov: The generalized Kummer construction and cohomology rings of $G_{2}$-manifolds. Sb. Math. 209:12 (2018), 1803-1811.


## Holonomy of a Riemannian manifold

$M^{d}$ - a connected Riemannian manifold. There is defined a parallel translation of vector fields along paths.
Let $p \in M^{d}$ and we consider loops with which start and end at $p$. For simplicity, we assume that $M^{d}$ is simply-connected.

The translation $h_{\gamma}$ of tangent vectors at $p$ along a loop $\gamma$ determines the automorphism of the tangent space

$$
h_{\gamma}: T_{p} M \rightarrow T_{p} M
$$

which preserves the inner product. All automorphisms of type $h_{\gamma}$ form the closed Lie subgroup $\mathrm{Hol}\left(\mathrm{M}^{d}\right)$ of $\mathrm{SO}(\mathrm{d})$ - the holonomy group of $M^{d}$. Up to conjugations this subgroup does not depend on the base point $p$.

## Berger's Theorem

If a simply-connected Riemannian manifold $M^{d}$ is not locally symmetric (i.e. the identity $\nabla_{i} R_{j k l m}=0$ does not hold) and the action of $\mathrm{Hol}\left(M^{d}\right)$ on $T_{p} M^{d}$ is irreducible (in particular, $M^{d}$ is not a product), then its holonomy group belongs to the following list:

- $S O(n) \quad(\operatorname{dim} M=n)$;
- U(n) $\quad(\operatorname{dim} M=2 n$, Kähler);
- $S U(n) \quad(\operatorname{dim} M=2 n$, Calabi-Yau);
- $\operatorname{Sp}(n) \quad(\operatorname{dim} M=4 n$, hyperkähler);
- $\operatorname{Sp}(n) \operatorname{Sp}(1) \quad(\operatorname{dim} M=4 n$, quaternionic-Kähler);
- $G_{2} \quad(\operatorname{dim} M=7)$;
- $\operatorname{Spin}(7) \quad(\operatorname{dim} M=8)$.
$S U(n), S U(n), S p(n), S p(n) S p(1), G_{2}$, and $\operatorname{Spin}(7)$ are called special holonomy groups.

The original Berger list contained the group $\operatorname{Spin}(9) \subset S O(16)$, but it was showed that manifold with such the holonomy group are locally symmetric (Alekseevsky, Brown-Gray).
Since $S p(k) \subset S U(2 k) \subset U(2 k)$, manifolds with $S p(n)$ or $S U(n)$ holonomy are Kähler manifolds.

Manifolds with $S U(n), S p(n), G_{2}$, or $\operatorname{Spin}(7)$ holonomy are Ricci-flat, i.e. satisfy Einstein's equations (in vacuum):

$$
R_{i k}=0
$$

These are the only known examples of simply-connected Ricci-flat compact manifolds.

If $\operatorname{dim} M=7$ or 8 , then $M$ admits a nontrivial parallel spinor field if and only if its holonomy group is $G_{2}$ or $\operatorname{Spin}(7)$.

Explicit examples of Ricci-flat metric on these manifolds are unknown.
All known results concern only the existence of such metrics:
Yau's theorem (on Calabi-Yau manifolds, 1977), the generalized Kummer construction: Joyce (1995: $T^{N} / \Gamma$ for $G_{2}$ and $\operatorname{Spin}(7), 1999: C Y^{8} / \Gamma$ for $\left.\operatorname{Spin}(7)\right)$
the twisted connected sum and its generalizations (for constructing $G_{2}$ manifolds): Kovalev (2003), Kovalev-Lee (2011), Corti-Haskins-Nordström-Pacini (2015) and etc.
Compact Calabi-Yau manifolds are distinguished by the condition $c_{1}=0$ (follows from Yau's theorem).

Qauternionic-Kähler manifolds are Einstein manifolds:

$$
R_{i k}-\frac{1}{n} R g_{i k}=0, \quad R=\text { const } \neq 0
$$

All known examples of compact quaternionic-Kähler manifolds are locally symmetric.

For every simple Lie group there exists a symmetric quaternionic-Kähler manifold of positive scalar curvature: $R>0$ (Wolf spaces).

Conjecture: Wolf spaces are exactly all complete quaternionic-Kähler manifolds of positive scalar curvature (LeBrun-Salamon).

## Minimal models by Sullivan

$\mathcal{M}=\mathcal{M}_{0} \oplus \mathcal{M}_{1} \oplus \ldots$ - a free graded-commutative algebra over
$\mathbb{Q}$ with homogeneous generators $x_{1}, \ldots$ such that
$1 \leq \operatorname{deg} x_{i} \leq \operatorname{deg} x_{j}$ for $i \leq j$ and all $\mathcal{M}_{k}$ are finite-dimensional. We assume that $\mathcal{M}_{0}=\mathbb{Q}$, i.e. an algebra is connected.
Such an algebra with a differential $d: \mathcal{M}_{i} \rightarrow \mathcal{M}_{i+1}, i \geq 1$, is minimal if $d x_{i} \in \bigwedge\left(x_{1}, \ldots, x_{i-1}\right)$ for $i \geq 1$.

A minimal algebra $\mathcal{M}$ is the minimal model of $\mathcal{A}$ if there is a homomorphism of d.g.a $f: \mathcal{M} \rightarrow \mathcal{A}$ which induces an isomorphism $f^{*}: H^{*}(\mathcal{M}) \xrightarrow{=} H^{*}(\mathcal{A})$.

Theorem [Sullivan]
For every compact simply-connected manifold or nilmanifold $X$ the algebra $\mathcal{A}(X)$ of $\mathbb{Q}$-polynomial forms on $X$ there is a minimal algebra $\mathcal{M}_{X}$ and a homomorphism $f: \mathcal{M}_{X} \rightarrow \mathcal{A}_{X}$ which induces an isomorphism in cohomology. The algebra $\mathcal{M}_{X}$ determines the rational homotopy type of $X$ and

$$
\operatorname{Hom}\left(\pi_{*}(X), \mathbb{Q}\right)=\mathcal{M}_{X} / \mathcal{M}_{X} \wedge \mathcal{M}_{X}
$$

A minimal algebra $\mathcal{M}$ is formal if there exists a homomorphism of d.g.a $f:(\mathcal{M}, d) \rightarrow\left(H^{*}(\mathcal{M}), 0\right)$ which induces an isomorphism of cohomology rings.

## Formal spaces

A space $X$ is formal if its minimal model $\mathcal{M}_{X}$ is formal, i.e. $\mathcal{M}_{X}$ is the minimal model of $\left(H^{*}(\mathcal{M}), 0\right)$. Therewith the rational homotopy type of $X$ is reconstructed from the cohomology ring. a differential manifold is geometrically formal if a product of harmonic forms is harmonic. This property implies formality.

Examples of formal spaces:

1) symmetric spaces (they are geometrically formal);
2) $k$-symmetric spaces (Kotschick-Terzic);
3) closed simply-connected manifolds of dimension $\leq 6$;
4) Kähler manifolds (Deligne-Griffiths-Morgan-Sullivan).

There are nonformal simply-connected symplectic manifolds
(Babenko-I.T., 1998).

## Massey products

$\alpha_{1}, \ldots, \alpha_{n}$ are homogeneous cohomology classes from $H^{*}(\mathcal{M})$. $\left[a_{1}\right]=\alpha_{1}, \ldots,\left[a_{n}\right]=\alpha_{n}, a_{i} \in \mathcal{M}$, are their representatives meeting the "Maurer-Cartan equation":

$$
d A-\bar{A} \wedge A=\left(\begin{array}{cccc}
0 & \ldots & 0 & \tau \\
0 & \ldots & 0 & 0 \\
& \ldots & & \\
0 & \ldots & 0 & 0
\end{array}\right)
$$

where

$$
A=\left(\begin{array}{cccccc}
0 & a_{1} & * & \ldots & * & * \\
0 & 0 & a_{2} & \ldots & * & * \\
& & & \ldots & & \\
0 & 0 & 0 & \ldots & 0 & a_{n} \\
0 & 0 & 0 & \ldots & 0 & 0
\end{array}\right)
$$

and $\bar{a}=(-1)^{k} a$ for $a \in \mathcal{M}_{k}$. The set of classes $[\tau] \in H^{*}(\mathcal{M})$ for all solutions and choices of $a_{i}$ forms the $n$-tuple Massey product $\left\langle\alpha_{1}, \ldots, \alpha_{n}\right\rangle$.

## Massey products and formality

The simplest example: $\left\langle\alpha_{1}, \alpha_{2}, \alpha_{3}\right\rangle$. In this case it is necessary that

$$
\alpha_{1} \wedge \alpha_{2}=\alpha_{2} \wedge \alpha_{3}=0
$$

Then take $U$ and $V$ such that $d U=a_{1} a_{2}$ and $d V=a_{2} a_{3}$ and put $[\tau]=(-1)^{p+q-1}\left[a_{1} \wedge V+(-1)^{p-1} U \wedge a_{3}\right]$, where $\alpha_{1} \in H^{p}, \alpha_{2} \in H^{q}$.

The Massey product $\left\langle\alpha_{1}, \ldots, \alpha_{n}\right\rangle$ is trivial if it contains the zero.
If there is a nontrivial Massey product in $\mathcal{M}$ then $\mathcal{M}$ is not formal.

Question: are compact simply connected manifolds with special holonomy groups formal?

Kähler manifolds $(U(n), S U(n), S p(n))$ - yes, quaternionic-Kähler manifolds, $G_{2}$ and $\operatorname{Spin}(7)$ - ?

Possible obstructions to formality for $G_{2}$ and $\operatorname{Spin}(7)$ manifolds are triple Massey products $\left\langle\alpha_{1}, \alpha_{2}, \alpha_{3}\right\rangle$ of classes of degree
$(2,2,2)$ for $G_{2}$ manifolds,
$(2,2,2)$ и $(2,2,3)$ for $\operatorname{Spin}(7)$ manifolds.

The Kummer construction:
$T^{4} /\langle\sigma\rangle, \quad \sigma: x \rightarrow-x, \quad \sigma^{2}=1, \quad\langle\sigma\rangle=\mathbb{Z}_{2}$.
16 fixed points of the involution $\sigma$ give 16 conic (over $\mathbb{R} P^{3}$ ) points in $T^{4} / \mathbb{Z}_{2}$.
By resolution of singularities we replace these cones by fibrations over $\mathbb{C} P^{1}=S^{2}$ with fibers diffeomorphic to the 2-disc.
Every copy of $\mathbb{C} P^{1}$ gives a two-cycle [z] with self-intersection equal to -2 :

$$
[z] \cap[z]=-2
$$

The projections of two-tori lead to three copies of $2 H=\left(\begin{array}{ll}0 & 2 \\ 2 & 0\end{array}\right)$ in the intersection form.
In terms of these cycles the intersection form (over $\mathbb{Q})$ is as follows

$$
\underbrace{(-2) \oplus \cdots \oplus(-2)}_{16} \oplus(2 H) \oplus 2 H \oplus 2 H .
$$

Therefore, the signature of the Kummer surface is equal to -16 and $b_{2}=22$.

Milnor (1958):
an even undetermined unimodular form over $\mathbb{Z}$ is uniquely defined by its rank $r$ and the signature $\tau$ :

$$
\left(\frac{-\tau}{8}\right) E_{8}(-1) \oplus\left(\frac{r+\tau}{2}\right) H
$$

where

$$
E_{8}(-1)=(-1) \cdot\left(\begin{array}{cccccccc}
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{array}\right) .
$$

The intersection form of the $K 3$ is equal to

$$
E_{8}(-1) \oplus E_{8}(-1) \oplus H \oplus H \oplus H
$$

The explicit construction of a canonical basis (without using the theory of lattices) (T.,2017).

For a closed oriented $n$-dimensional manifold $X$ the intersection form (for cycles)

$$
H_{k}(X ; \mathbb{Z}) \times H_{l}(X ; \mathbb{Z}) \xrightarrow{\cap} H_{k+l-n}(X ; \mathbb{Z})
$$

is defined by

$$
u \cap v=D^{-1}(D u \cup D v),
$$

where

$$
D: H_{i}(X ; \mathbb{Z}) \rightarrow H^{n-i}(X ; \mathbb{Z}), \quad i=0, \ldots, n
$$

- the Poincare duality operator. The intersection ring
(Poincare-Lefschetz-Pontryagin) is dual to the cohomology ring.
If cycles $u$ are $v$ realized by transversally intersections submanifolds
$Y$ and $Z$, then their intersection is a smooth submanifold $W$ which realizes the cycle $w$ such that

$$
u \cap v=w, \quad u \cap v=(-1)^{(n-k)(n-l)} v \cap u
$$

## Example of the Joyce $G_{2}$ manifold

Take on $T^{7}=\mathbb{R}^{7} / \mathbb{Z}^{7}$ the involutions

$$
\begin{gathered}
\alpha\left(\left(x_{1}, \ldots, x_{7}\right)\right)=\left(-x_{1},-x_{2},-x_{3},-x_{4}, x_{5}, x_{6}, x_{7}\right), \\
\beta\left(\left(x_{1}, \ldots, x_{7}\right)\right)=\left(b_{1}-x_{1}, b_{2}-x_{2}, x_{3}, x_{4},-x_{5},-x_{6}, x_{7}\right), \\
\gamma\left(\left(x_{1}, \ldots, x_{7}\right)\right)=\left(c_{1}-x_{1}, x_{2}, c_{3}-x_{3}, x_{4}, c_{5}-x_{5}, x_{6},-x_{7}\right),
\end{gathered}
$$

which pairwise commute

$$
\alpha \beta=\beta \alpha, \quad \alpha \gamma=\gamma \alpha, \quad \beta \gamma=\gamma \beta
$$

and therefore for all $b_{1}, b_{2}, c_{1}, c_{3}, c_{5}$ induce an action of $\Gamma=\mathbb{Z}_{2}^{3}$ on $T^{7}$.
Take the following constants

$$
b_{1}=c_{5}=0, \quad b_{2}=c_{1}=c_{3}=\frac{1}{2}
$$

which correspond to a simply-connected manifold $M^{7}$.

- $\Gamma$ acts on $H^{*}\left(T^{7}\right)$ by involutions, $H^{1}\left(T^{7}\right)$ and $H^{2}\left(T^{7}\right)$ have no nontrivial invariant subspaces and the invariant subspace of $H^{3}\left(T^{7}\right)$ is generated by the forms
$d x_{2} \wedge d x_{4} \wedge d x_{6}, \quad d x_{3} \wedge d x_{4} \wedge d x_{7}, \quad d x_{5} \wedge d x_{6} \wedge d x_{7}$,

$$
\begin{array}{ll}
d x_{1} \wedge d x_{2} \wedge d x_{7}, & d x_{1} \wedge d x_{3} \wedge d x_{6} \\
d x_{1} \wedge d x_{4} \wedge d x_{5}, & d x_{2} \wedge d x_{3} \wedge d x_{5}
\end{array}
$$

This implies that

$$
b^{1}\left(T^{7} / \Gamma\right)=b^{2}\left(T^{7} / \Gamma\right)=0, \quad b^{3}\left(T^{7} / \Gamma\right)=7
$$

Since the 7-form $d x_{1} \wedge \cdots \wedge d x_{7}$ is $\Gamma$-invariant, *: $H^{k}\left(T^{7}\right) \rightarrow H^{7-k}\left(T^{7}\right)$ maps invariant forms into invariant ones, and, therefore,

$$
b^{6}\left(T^{7} / \Gamma\right)=b^{5}\left(T^{7} / \Gamma\right)=0, \quad b^{4}\left(T^{7} / \Gamma\right)=7
$$

- The action of $\Gamma=\mathbb{Z}_{2}^{3}$ is not free. For every involution $\alpha, \beta$, or $\gamma$ its fixed points set consists of 16 three-tori.
- $\Gamma / \mathbb{Z}_{2}$ acts by nontrivial permutations on the point sets of $\alpha, \beta$, and $\gamma$ and a $\Gamma$-orbit of every such a torus consists of four tori.
- Products of different elementary involutions, i.e. $\alpha \beta$ and etc., have no fixed points.
- Every involution $\alpha, \beta$, or $\gamma$ acts on $T^{7}$ such that

$$
T^{7} / \mathbb{Z}_{2}=T^{3} \times\left(T^{4} / \mathbb{Z}_{2}\right)
$$

where $T^{4} / \mathbb{Z}_{2}$ is a singular Kummer surface. Moreover $\pi_{1}\left(T^{7} / \Gamma\right)=0$.

- The singular set in $T^{7} / \Gamma$ splits into 12 three-tori. For every singular torus there is a neighborhood homeomorphic to

$$
U=T^{3} \times(D /\langle-1\rangle)
$$

where $D=\left\{|z| \leq \tau: \quad z \in \mathbb{C}^{2}\right\}$.

- From topological point of view, $M^{7}$ is constructed from $T^{7} / \Gamma$ by fiberwise resolution of singularities in $D / \mathbb{Z}_{2}$.
- The rational homology groups of $M^{7}$ have the following form: $b^{2}=12$ and the generators are given by 12 cycles $c_{\delta i}$ corresponding to submanifolds of the form $\mathbb{C} P^{1}$ which appear after the resolution of singularities;
$b^{3}=43$ and the generators are given by 7 cycles $t_{k}$ represented by three-tori corresponding to invariant 2-forms on $T^{7}$ and by 12 families of products of $\mathbb{C} P^{1}$ and generating 1-cycles in singular tori: $\lambda_{\delta i j}$.

Here $\delta \in\{\alpha, \beta, \gamma\}, i=1,2,3,4, j=1,2,3, k \in\{\alpha, \beta, \gamma, 1,2,3,4\}$.

## Theorem.

The rational homology groups $H_{*}\left(M^{7} ; \mathbb{Q}\right)$ have the following generators of dimension $\leq \operatorname{dim} M^{7}=7$ :

$$
\begin{array}{ll}
\operatorname{dim}=2: & c_{\delta i} ; \quad \operatorname{dim}=3: \quad c_{\delta i j}, t_{\delta}, \\
\operatorname{dim}=4: \\
\operatorname{dim} & c_{\delta i j}^{\prime}, \quad t_{\delta}^{\prime}, t_{i}^{\prime} ; \quad \operatorname{dim}=5: \quad c_{\delta i}^{\prime}
\end{array}
$$

where $\delta \in\{\alpha, \beta, \gamma\}, i=1, \ldots, 4, j=1,2,3$. Nontrivial intersections are as follows:

$$
\begin{gathered}
c_{\delta i} \cap c_{\delta i}^{\prime}=-2, \quad c_{\delta i j} \cap c_{\delta i j}^{\prime}=-2, \quad t_{\delta} \cap t_{\delta}^{\prime}=4, \quad t_{i} \cap t_{i}^{\prime}=8, \\
c_{\delta i}^{\prime} \cap c_{\delta i}^{\prime}=-2 t_{\delta}, \quad t_{\delta}^{\prime} \cap c_{\delta i}^{\prime}=4 c_{\delta i} .
\end{gathered}
$$

## Remarks.

1) For describing the cohomology ring it is enough to correspond to every generator $a \in H_{*}, \operatorname{dim} a=k$ a generator $\bar{a} \in H^{*}, \operatorname{deg} \bar{a}=n-k$, and every relation of the form $a \cap b=c$ replace by a relation $\bar{a} \cup \bar{b}=\bar{c}$.
2) The first line in Theorem describes the pairing of the groups $H_{k}$ and $H_{n-k}$ (the Poincare duality), which takes the form $a \rightarrow a^{\prime}$.
3) $M^{7}$ has no nontrivial Massey products.
4) Joyce's examples obtained by the generalized Kummer construction, roughly speaking, split into three types. For certain examples of other types the intersection rings were calculated by our students I.V. Fedorov and V.E. Todikov in their diploma works.
