Canonical bases and collective integrable systems

Jeremy Lane
Joint with Benjamin Hoffman (in preparation)

McMaster University

May 16, 2020

Q: What is the symplectic analogue of canonical bases?

A1: Gelfand-Zeitlin *collective* integrable systems.... (GS)

But they don't work for all groups!

Neither do Gelfand-Zeitlin canonical bases for that matter!

A2: Integrable systems constructed by toric degenerations of projective spherical varieties.... (NNU,HK)

But many Hamiltonian manifolds aren't projective or spherical!

And this doesn't produce collective systems!

A3: Partial tropicalization of dual Poisson-Lie groups (AM,AD,APS1,APS2, ALL,ABHL1,ABHL2,AHLL1,AHLL2 ...).

A4: This talk.

Three versions of the same story

"If you start to feel confused at any point, look at this photo of Jeff Goldblum from Jurassic Park."

- Chris Manon (paraphrasing).

Version #1: representation theory

Let G a connected reductive complex algebraic group and V a G-module. There are canonical isomorphisms:

$$\overset{G}{\underset{V}{\curvearrowright}} \underset{G}{\cong} \left(V^{N_-} \otimes \left(\overset{G \times H}{\curvearrowright} \bigoplus_{\lambda \in \Lambda_+} V(\lambda)^* \otimes v(\lambda) \right) \right)^H \underset{H \times H}{\cong} \left(V^{N_-} \otimes \left(\overset{H \times H}{\curvearrowright} \bigoplus_{b \in \mathcal{B}^*} \mathbb{C} \cdot b \right) \right)^H \qquad (1)$$

The first isomorphism is the *isotypic decomposition*.

The second isomorphism is via Kashiwara and Lusztig's *dual canonical bases*.

The basis $\mathcal{B}^*$ can be made a weight basis for an action of $\mathbb{H} = (\mathbb{C}^\times)^m \times H$ that extends the action of $H \times H$ (with some choice) such that:

V is a multiplicity-free G-module $\Leftrightarrow$ V is a multiplicity-free $\mathbb{H}$-module.

Interlude: base affine space

Recall that

$$\bigoplus_{\lambda \in \Lambda_+} V(\lambda)^* \otimes v(\lambda) \cong_{G \times H} \mathbb{C}[G]^N.$$

Base affine space is the (singular) affine $G \times H$-variety $G /\!\!/ N$ with

$$\mathbb{C}[G /\!\!/ N] = \mathbb{C}[G]^N.$$

If $\mathcal{B}^*$ has been made a weight basis for $\mathbb{H}$ as above in a sufficiently nice way (e.g. with a string parameterization), then we can construct multiplicative filtrations of $\mathbb{C}[G]^N$ that give rise to *$H \times H$-equivariant flat degenerations* (first observed by Caldero and Alexeev-Brion)

$$G /\!\!/ N \underset{H \times H}{\Rightarrow} X.$$

The 0-fibers of these these degenerations are affine toric $\mathbb{H}$-varieties X with

$$\mathbb{C}[X] = gr(\mathbb{C}[G]^N) \cong \mathbb{C}[\mathsf{S}]$$

where S is the semigroup of weights of $\mathbb{H} \curvearrowright \mathbb{C}[G]^N$.

Version #2: algebraic geometry

Let Y be an affine G-variety. There are flat degenerations:

$$\begin{array}{ccccc} G & & G \times H & & \mathbb{H} = (\mathbb{C}^\times)^m \times H \\ \curvearrowright & & \curvearrowright & & \curvearrowright \\ Y & \underset{G}{\Rightarrow} & (Y /\!\!/ N_- \times G /\!\!/ N) /\!\!/ H & \underset{H \times H}{\Rightarrow} & (Y /\!\!/ N_- \times X) /\!\!/ H \end{array} \qquad (2)$$

The left degeneration, *horospherical contraction*, is due to Popov.

The right degenerations arise from the previous slide.

Y is a spherical G-variety $\Leftrightarrow$ $(X /\!\!/ N_- \times X) /\!\!/ H$ is a toric $\mathbb{H}$-variety.

Interlude: symplectic implosion

Guillemin-Jeffrey-Sjamaar tell us that *symplectic implosion*, $\mathcal{E}M$, is the way to take "highest weights" of a Hamiltonian K-manifold M.

The *universal symplectic implosion* is $\mathcal{E}T^*K$. Guillemin-Jeffrey-Sjamaar also tell us that as singular Hamiltonian $K \times T$-manifolds,

$$\mathcal{E}T^*K \cong G /\!\!/ N.$$

Rep. Th.	Alg. Geo.	Symp. Geo.
$G \curvearrowright V$	$G \curvearrowright X$	$K \curvearrowright M$
$H \curvearrowright V^N$	$H \curvearrowright X /\!\!/ N$	$T \curvearrowright \mathcal{E}M$
$G \times H \curvearrowright \bigoplus_{\lambda \in \Lambda_+} V(\lambda)^*$	$G \times H \curvearrowright G /\!\!/ N$	$K \times T \curvearrowright \mathcal{E}T^*K$

Let M a Hamiltonian K-manifold. There are commuting diagrams:

$$\begin{array}{ccc} K & & K \times T \\ \curvearrowright & & \curvearrowright \\ M & \xrightarrow[K]{} & (\mathcal{E}M \times \mathcal{E}T^*K) /\!\!/_0 \, T \\ \downarrow & & \downarrow \\ \mathfrak{t}^* & \xrightarrow{=} & \mathfrak{t}^* \end{array} \tag{3}$$

The left square, *symplectic contraction*, is due to Hilgert-Martens-Manon. It is independent of any choices.

Version #3: symplectic geometry

Let M a Hamiltonian K-manifold. There are commuting diagrams:

$$
\begin{array}{ccccc}
K & & K \times T & & \mathbb{T} = (S^1)^m \times T \\
\curvearrowright & & \curvearrowright & & \curvearrowright \\
M & \xrightarrow[K]{} & (\mathcal{E}M \times \mathcal{E}T^*K) /\!\!/_0\, T & \xrightarrow[T \times T]{} & (\mathcal{E}M \times X) /\!\!/_0\, T \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{k}^* & \xrightarrow{=} & \mathfrak{k}^* & \longrightarrow & \mathrm{Lie}(\mathbb{T})^*
\end{array}
\qquad (4)
$$

The left square, *symplectic contraction*, is due to Hilgert-Martens-Manon. It is independent of any choices.

We construct the right squares.

M is a multiplicity-free as a Hamiltonian K-manifold $\Leftrightarrow$ The $\mathbb{T}$-action is completely integrable.

Q: What is the symplectic analogue of Lusztig and Kashiwara's canonical bases?

Theorem (Hoffman-L.)

Let K a compact connected Lie group, M a Hamiltonian K-manifold.

We construct commuting diagrams

$$\begin{array}{ccc} M & \xrightarrow{\phi_M} & X_M \\ \downarrow & & \downarrow \\ \mathfrak{t}^* & \xrightarrow{\Psi} & \mathrm{Lie}(\mathbb{T})^* \end{array} \tag{5}$$

where:

- $X_M = (\mathcal{E}M \times X) /\!\!/_0 T$ *is a singular Hamiltonian* $\mathbb{T}$*-space.*
- ϕ_M *is a continuous, proper,* T*-equivariant, surjective map that is a symplectomorphism from a dense subset onto its image.*

Moreover:

- *The diagram generates a Hamiltonian* $\mathbb{T}$*-action on a dense subset of* M*.*
- M *is multiplicity-free* $\Leftrightarrow$ *the action of* $\mathbb{T}$ *is completely integrable.*
- *If* M *is compact, then the image is a convex polytope in* $\mathrm{Lie}(\mathbb{T})^*$*.*

How do we construct the right squares?

$$\begin{array}{ccccc} M & \xrightarrow[K]{} & (\mathcal{E}M \times \mathcal{E}T^*K) /\!\!/_0\, T & \xrightarrow[T\times T]{} & (\mathcal{E}M \times X) /\!\!/_0\, T \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{k}^* & \xrightarrow{=} & \mathfrak{k}^* & \longrightarrow & \mathrm{Lie}(\mathbb{T})^* \end{array} \tag{6}$$

Theorem (Hoffman-L.)

There exists a continuous, proper map, $T \times T$-equivariant surjective map

$$\phi \colon \mathcal{E}T^*K \to X$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E}T^*K & \xrightarrow{\phi} & X \\ \Big\downarrow /T & & \Big\downarrow \\ \mathfrak{t}^* & \xrightarrow{\Psi} & \mathrm{Lie}(\mathbb{T})^* \\ \Big\downarrow & & \Big\downarrow \\ \mathfrak{t}^* \times \mathfrak{t}^*_+ & \xrightarrow{=} & \mathfrak{t}^* \times \mathfrak{t}^*_+ \end{array} \tag{7}$$

Moreover:

- *The image in* $\mathrm{Lie}(\mathbb{T})^*$ *is the convex rational polyhedral cone generated by the semigroup* S.

Theorem (Hoffman-L.)

There exists a continuous, proper map, $T \times T$-equivariant surjective map

$$\phi \colon \mathcal{E}T^*K \to X$$

such that the following diagram commutes:

$$\begin{array}{ccccc} K/[K_\sigma, K_\sigma] \times \sigma & \subset & \mathcal{E}T^*K & \xrightarrow{\phi} & X \\ \big\downarrow /T & & \big\downarrow /T & & \big\downarrow \\ K/K_\sigma \times \sigma & \subset & \mathfrak{k}^* & \xrightarrow{\Psi} & \mathrm{Lie}(\mathbb{T})^* \\ \big\downarrow /K & & \big\downarrow /K & & \big\downarrow \\ \sigma & \subset & \mathfrak{t}^*_+ & \xrightarrow{=} & \mathfrak{t}^*_+ \end{array} \tag{8}$$

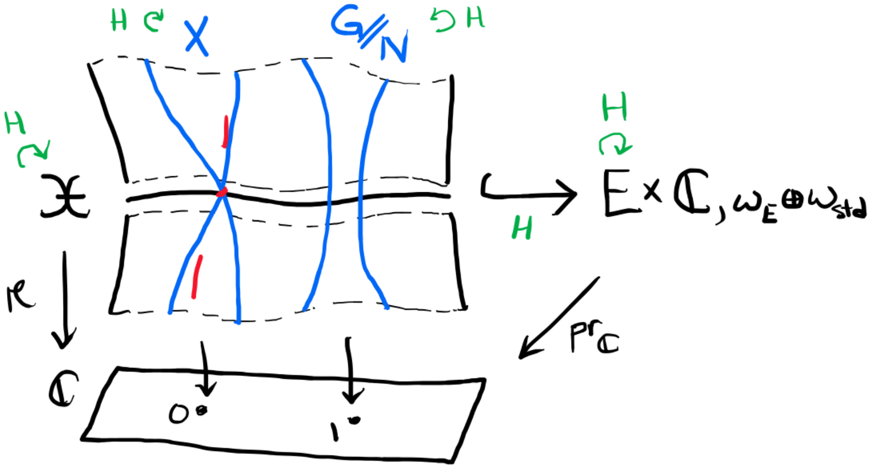
Moreover:

- *For all strata of $\mathcal{E}T^*K$, there is a dense subset such that ϕ defines a symplectomorphism onto its image.*

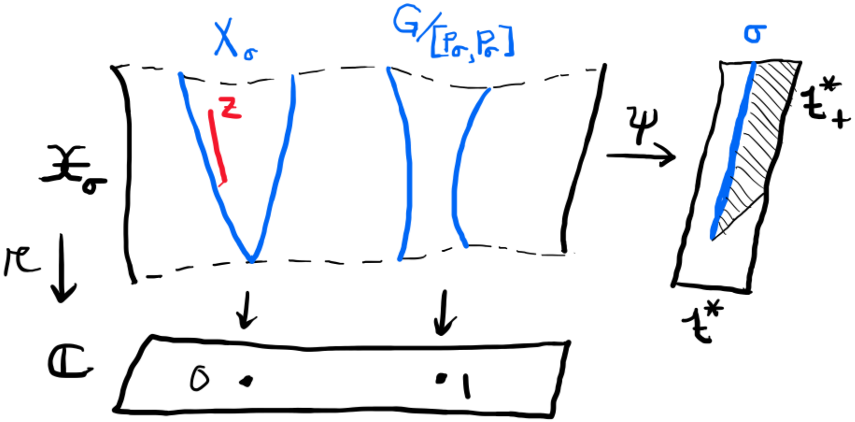
How do we construct ϕ?

Some boring details to help you believe me (but I will skip them):

- Let S the saturated affine semigroup of a chosen parameterization of $\mathcal{B}^*$ and let X the associated normal affine toric variety (e.g. Berenstein-Zelevinksy string parameterization).
- Let $v\colon \mathbb{C}[G]^N \to$ S be a valuation with 1-dim. leaves that coincides on $\mathcal{B}^*$ with the parameterization (cf. Kaveh).
- Let
$$E = \bigoplus_{\lambda \in \Pi} V(\lambda)$$
where Π generates Λ_+ and equip E with a $G \times H$-invariant Hermitian inner product.
- $K \times T$-equivariantly embed $G /\!\!/ N \cong \mathcal{E}T^*K \hookrightarrow E$ so that $\omega = \omega_E|_{\mathcal{E}T^*K}$ (cf. Guillemin-Jeffrey-Sjamaar).
- Assume the pre-image of Π generates S.
- Use v to construct an orthonormal basis for E^* on which $\mathbb{T}$ acts by weights (cf. Harada-Kaveh). $\mathbb{T}$-equivariantly embed $X \hookrightarrow E$.
- Do a standard Rees algebra construction of a $H \times H$ degeneration. Embeds $H \times H$-equivariantly into $E \times \mathbb{C}$ by construction (Caldero, Anderson, etc.).

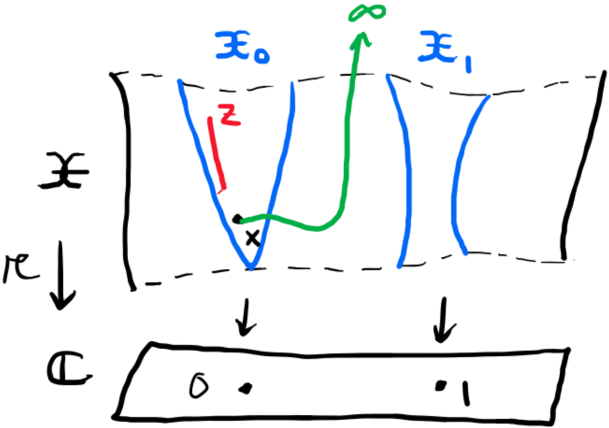


The degeneration $G /\!\!/ N \Rightarrow_H X$ with its embedding into $E \times \mathbb{C}$.



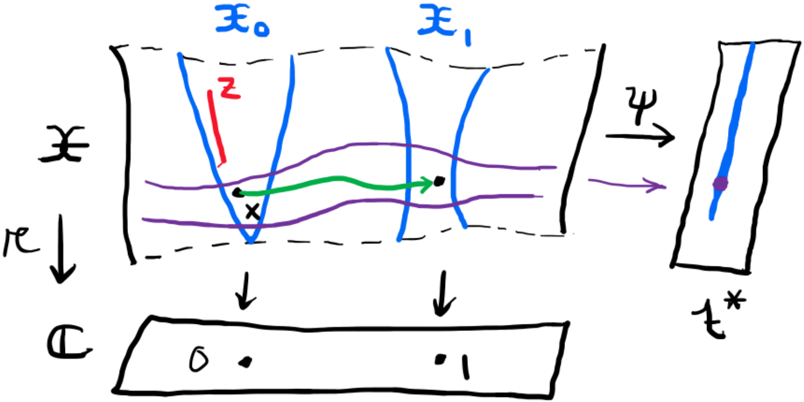
A subfamily of the degeneration corresponding to a G-orbit in $G /\!\!/ N$.

ψ is the moment map for the globally defined action of T.



The general situation of a degeneration with non-compact fibers.

Flows starting on the 0 fiber may blow up in finite time.



How to use moment maps to integrate gradient-Hamiltonian flows when there are non-compact fibers.

Suppose the action of T preserves fibers of π.

By Noether's theorem, the flow preserves level sets of the moment map.

Theorem (Hoffman-L.)

Let $\pi\colon \mathfrak{X} \to \mathbb{C}$ *be a family as on the previous slide.*

Assume:

1. *There is a Hamiltonian* T*-action on* $\mathfrak{X} \setminus Z$ *with moment map given by the restriction to* $\mathfrak{X} \setminus Z$ *of a continuous map* $\psi\colon \mathfrak{X} \to \mathfrak{t}^*$ *such that the action of* T *preserves the fibers of* π *and the Kähler metric.*
2. *The map* $(\pi, \psi)\colon \mathfrak{X} \to \mathbb{C} \times \mathfrak{k}^*$ *is proper as a map to its image.*

Then, the time -1 *gradient-Hamiltonian flow exists for all* $x \in \mathfrak{X}_0 \setminus Z$ *and defines a map of Hamiltonian* T*-manifolds,*

$$\varphi_{-1}\colon \mathfrak{X}_0 \setminus Z \to \mathfrak{X}_1.$$

Assume in addition that:

1. $\mathfrak{X}_1$ *is connected and the Duistermaat-Heckman measures of* $\mathfrak{X}_0 \setminus Z$ *and* $\mathfrak{X}_1$ *are equal.*

Then, $\varphi_{-1}\colon \mathfrak{X}_0 \setminus Z \to \mathfrak{X}_1$ *is a symplectomorphism onto a dense subset of* $\mathfrak{X}_1$*.*

It is not at all obvious from what I've said above that this defines a continuous map $\phi\colon G /\!\!/ N \to X$.

We resolve these issues directly.

Do the integrable systems $\Psi\colon \mathfrak{k}^* \to \mathrm{Lie}(\mathbb{T})^*$ have Flaschka-Ratiu analogues on the dual Poisson-Lie group K^*?

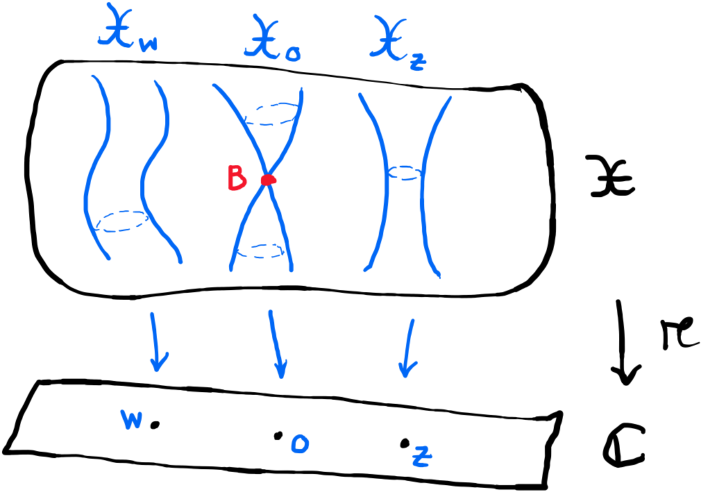
Can Alekseev-Meinrenken style Ginzburg-Weinstein isomorphisms be recovered from the integrable systems $\Psi\colon \mathfrak{k}^* \to \mathrm{Lie}(\mathbb{T})^*$?

Thank you!

Appendix

How to prove the last theorem of the talk. This does not go into the more complicated issue of continuity between strata of the degeneration.

Gradient-Hamiltonian vector fields



Our setting: an algebraic map from a complex algebraic variety $\mathfrak{X}$ to $\mathbb{C}$.

Gradient-Hamiltonian vector fields

Definition

The *bad set* $B \subset \mathfrak{X}$ is the union of $\mathfrak{X}_{sing}$ and the critical set of π.

Assume:

- $\pi \colon \mathfrak{X} \setminus B \to \mathbb{C}$ is a submersion *onto* $\mathbb{C}$,
- $B \subset \mathfrak{X}_0$, and
- $\mathfrak{X} \setminus B$ is Kähler. i.e.

$$g(-, J-) = \omega(-, -)$$

Then:

- $\mathfrak{X} \setminus B$ is a complex manifold and $\pi \colon \mathfrak{X} \setminus B \to \mathbb{C}$ is a holomorphic map.
- The fibers of $\pi \colon \mathfrak{X} \setminus B \to \mathbb{C}$ are non-empty Kähler submanifolds of complex codimension 1.

Gradient-Hamiltonian vector fields

Given $f\colon \mathfrak{X}\setminus B \to \mathbb{R}$,

Gradient vector field: $\quad g(\nabla f, \cdot) = df,$

Hamiltonian vector field: $\quad \omega(X_f, \cdot) = df.$

Lemma

Denote $\pi = \Re\pi + \sqrt{-1}\Im\pi$. *On* $\mathfrak{X}\setminus B$,

$$X_{\Im\pi} = -\nabla(\Re\pi).$$

Proof.

$$\begin{aligned}\omega(\nabla(\Re\pi), \cdot) &= g(\nabla(\Re\pi), J\cdot)\\ &= d(\Re\pi)\circ J\\ &= -d(\Im\pi) \quad \text{since } \pi \text{ is holomorphic}\\ &= \omega(-X_{\Im\pi}, \cdot)\end{aligned}$$

□

Gradient-Hamiltonian vector fields

Definition

The *gradient-Hamiltonian vector field* of π is

$$V_\pi := \frac{X_{\Im\pi}}{||X_{\Im\pi}||^2} = \frac{-\nabla\Re\pi}{||\nabla\Re\pi||^2}.$$

Since $\pi\colon \mathfrak{X}\setminus B \to \mathbb{C}$ is a submersion, V_π is defined everywhere on $\mathfrak{X}\setminus B$.

Definition

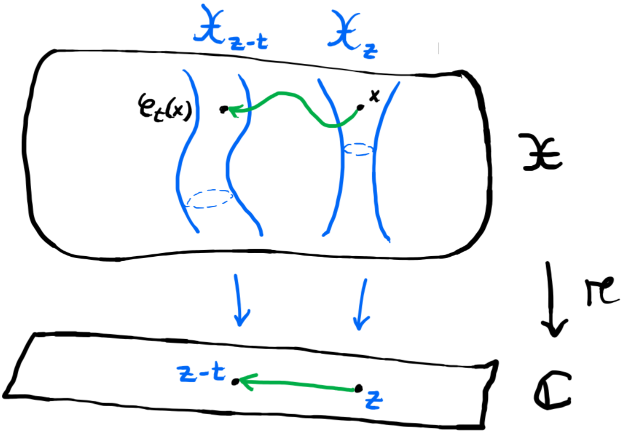
The *gradient-Hamiltonian flow* is the maximal flow

$$\varphi\colon J \to \mathfrak{X}\setminus B, \quad J \subset \mathbb{R}\times(\mathfrak{X}\setminus B),$$

of the vector field V_π. i.e.

$$\frac{d}{dt}\varphi_t(x) = V_\pi(\varphi_t(x)), \quad \varphi_0(x) = x.$$

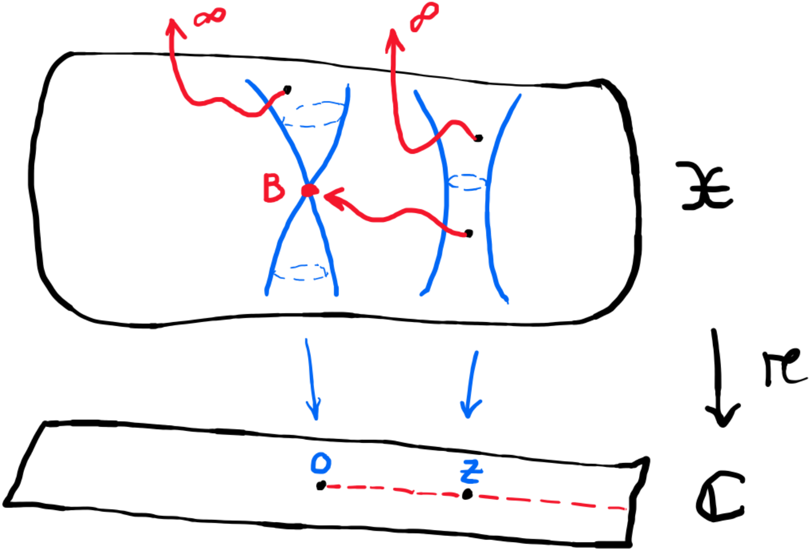
Gradient-Hamiltonian vector fields



Reason for normalization in the definition of V_π:

- If $x \in \mathfrak{X}_z$ and $\varphi_t(x)$ is defined, then $\varphi_t(x) \in \mathfrak{X}_{z-t}$.
- $\varphi_t^* \omega_{z-t} = \omega_z$ where $\omega_z = \omega|_{\mathfrak{X}_z}$.

Gradient-Hamiltonian vector fields

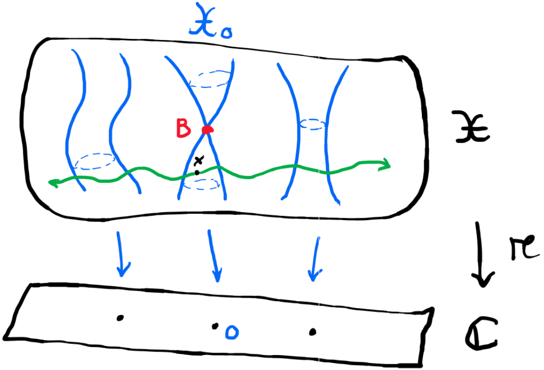


Two things can go wrong.

Gradient-Hamiltonian vector fields

Lemma

If $\pi\colon \mathfrak{X} \to \mathbb{C}$ is proper and $x \in \mathfrak{X}_0 \setminus B$, then $\varphi_t(x)$ is defined for all $t \in \mathbb{R}$.



I first learned this from:

Harada, Kaveh. Integrable systems, toric degenerations, and Okounkov bodies. 2015.

Gradient-Hamiltonian vector fields

Lemma

Suppose that K acts on $(\mathfrak{X} \setminus B, \omega)$ with moment map $\psi \colon \mathfrak{X} \setminus B \to \mathfrak{k}^*$.

If the action of K preserves

- *the fibers of π, and*
- *the Kähler metric*

then, the flow of V_π

- *preserves the fibers of ψ, and*
- *is K-equivariant.*

The K-equivariant part of this lemma (which was the inspiration for this approach) appears in:
Hilgert, Manon, and Martens. Contraction of Hamiltonian K-spaces. 2016.

Gradient-Hamiltonian vector fields

Proof that φ preserves fibers of ψ.

$$\begin{aligned}\mathcal{L}_{V_\pi}\langle\psi,\xi\rangle &= d(\langle\psi,\xi\rangle)(V_\pi) = \omega(\underline{\xi}, V_\pi)\\ &= \frac{1}{||X_{\Im\pi}||^2}\omega(\underline{\xi}, X_{\Im\pi}) = \frac{-1}{||X_{\Im\pi}||^2}d(\Im\pi)(\underline{\xi}) = 0.\end{aligned}$$

□

Gradient-Hamiltonian vector fields

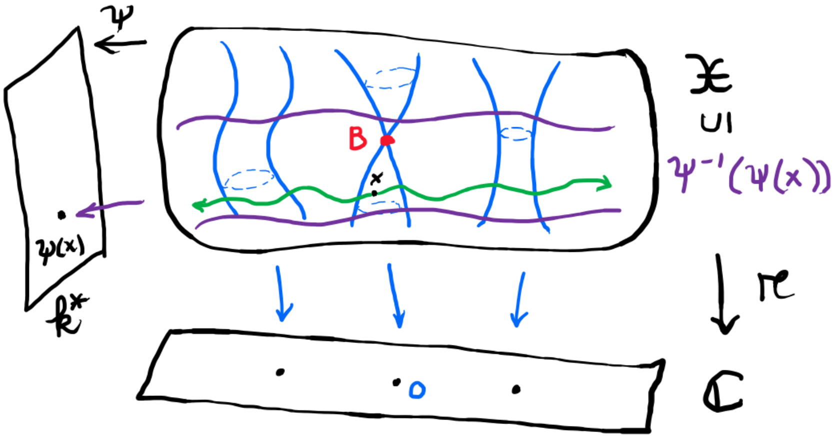
Proof that φ is equivariant.

For all $\xi \in \mathfrak{k}$,

$$
\begin{aligned}
[\underline{\xi}, V_\pi] &= \left[\underline{\xi}, \frac{1}{||X_{\Im\pi}||^2} X_{\Im\pi}\right] \\
&= \underline{\xi}\left(\frac{1}{||X_{\Im\pi}||^2}\right) X_{\Im\pi} + \frac{1}{||X_{\Im\pi}||^2}[\underline{\xi}, X_{\Im\pi}] \\
&= -\frac{(\mathcal{L}_{\underline{\xi}} g)(X_{\Im\pi}, X_{\Im\pi}) + 2g([\underline{\xi}, X_{\Im\pi}], X_{\Im\pi})}{||X_{\Im\pi}||^4} X_{\Im\pi} \\
&\quad + \frac{1}{||X_{\Im\pi}||^2}[\underline{\xi}, X_{\Im\pi}] \\
&= 0.
\end{aligned}
$$

□

Gradient-Hamiltonian vector fields



If $\pi\colon \psi^{-1}(\psi(x)) \to \mathbb{C}$ is proper, then we can use the same idea as above to integrate $\varphi_t(x)$.

Gradient-Hamiltonian vector fields

Theorem (Hoffman, L., Part I)

Let $\pi\colon \mathfrak{X} \to \mathbb{C}$ *as above and assume that:*

1. *A connected Lie group* K *acts on* $\mathfrak{X} \setminus B$ *with moment map* $\psi\colon \mathfrak{X} \setminus B \to \mathfrak{k}^*$*.*
2. *The action of* K *preserves the fibers of* π *and the Kähler metric.*
3. ψ *extends to a continuous map* $\psi\colon \mathfrak{X} \to \mathfrak{k}^*$*.*
4. $(\pi, \psi)\colon \mathfrak{X} \to \mathbb{C} \times \mathfrak{k}^*$ *is proper as a map to its image.*

Then,

- $\varphi_t(x)$ *is defined for all* $x \in \mathfrak{X}_0 \setminus B$ *and* $t \in \mathbb{R}$*. For* $t \neq 0$ *fixed,*

$$\varphi_{-t}\colon (\mathfrak{X}_0 \setminus B, \omega_0, \psi) \to (\mathfrak{X}_t, \omega_t, \psi)$$

  is a map of Hamiltonian K*-manifolds.*

Proof.

Combine the previous slides.

Gradient-Hamiltonian vector fields

Theorem (Hoffman, L., Part II)

Continuing from above.

Let $t \neq 0$ *fixed. Assume further that:*

1. $K = T$ *is a compact torus,*
2. $\mathfrak{X}_t$ *is connected, and*
3. *the Duistermaat-Heckman measures of* $(\mathfrak{X}_0 \setminus B, \omega_0, \psi)$ *and* $(\mathfrak{X}_t, \omega_t, \psi)$ *are equal.*

Then,

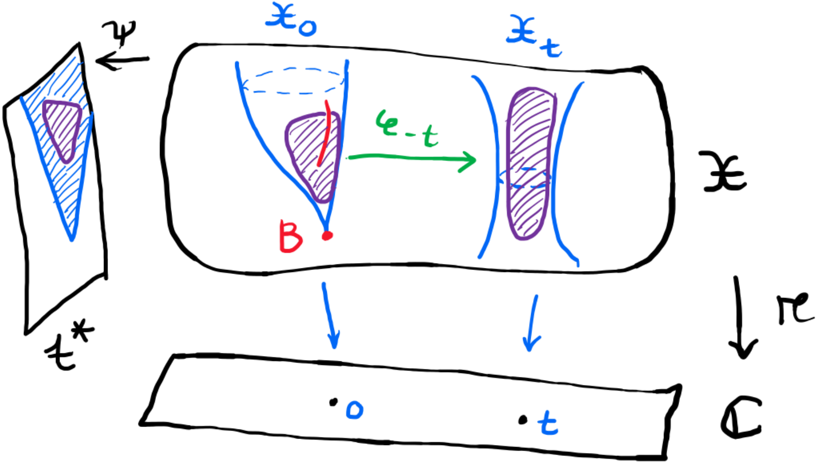
$$\varphi_{-t} \colon (\mathfrak{X}_0 \setminus B, \omega_0, \psi) \to (\mathfrak{X}_t, \omega_t, \psi)$$

is a symplectomorphism onto a dense subset of $\mathfrak{X}_t$*.*

Proof.

See the image on the following slide. Use the convexity theorem for Hamiltonian torus actions! □

Gradient-Hamiltonian vector fields



Using the Duistermaat-Heckman measure to prove the image is dense.

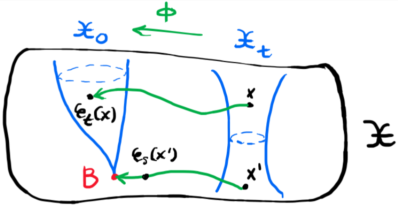
Gradient-Hamiltonian vector fields

Theorem (Hoffman, L., Part III)

For all $x \in \mathfrak{X}_t$,

$$\lim_{s \to t^-} \varphi_s(x)$$

exists and defines a continuous, proper, surjective map $\phi\colon \mathfrak{X}_t \to \mathfrak{X}_0$ that is a symplectomorphism from a dense subset of $\mathfrak{X}_t$ onto its image.



Proof.

Same argument as in
Harada, Kaveh. Integrable systems, toric degenerations, and Okounkov bodies. 2015. □