Does Set Theory have anything to do with Mathematics?

Fields Institute
November 8, 2012

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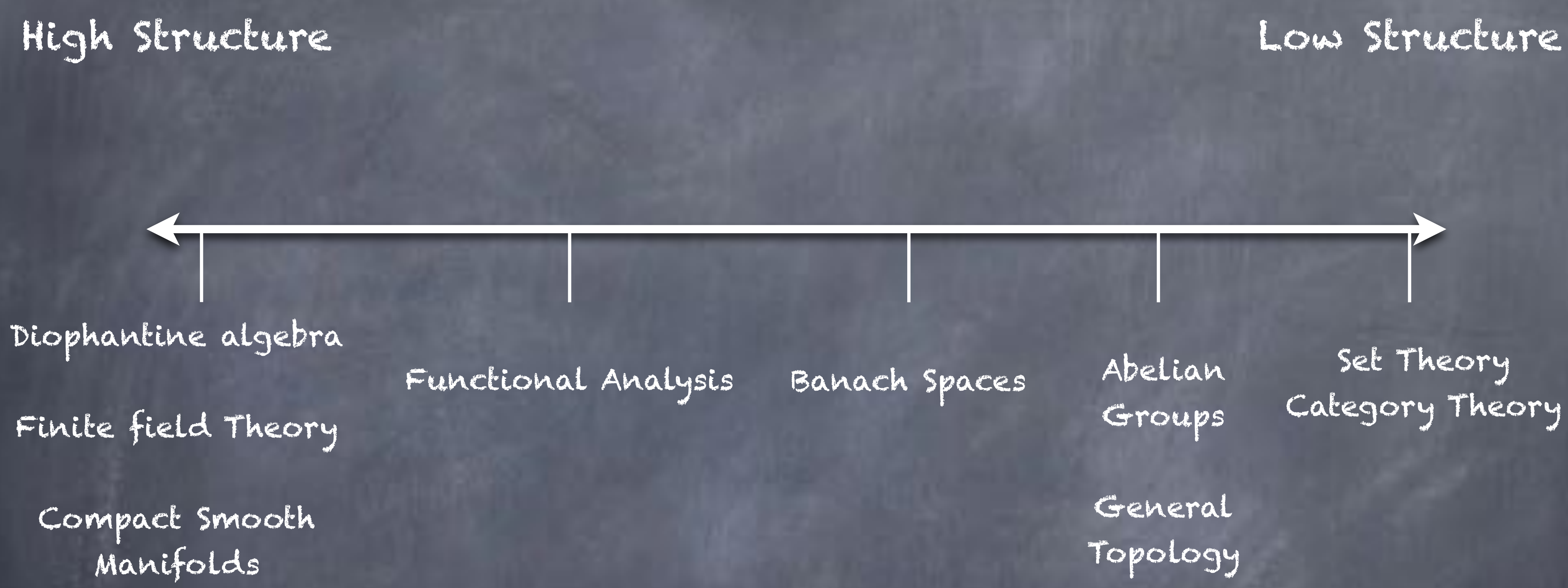
Quasi-Historical Survey

Continuum of Structure



Not Drawn to Scale!

Continuum of Structure

High Structure ⟷ Low Structure

Three inexorable pressure in mathematics that tend to push from left to right:

- Generalization
- Abstraction
- reduction to combinatorics

Plan of the talk

- Give examples from both combinatorial and descriptive set theory
- Talk about one example I've been involved with

Birth of Set Theory as a distinct subarea of math

- When working on sets of uniqueness for trigonometric series Cantor discovered that there were different sizes of infinity
- Is |{real numbers}| the first uncountable cardinal?

Closely related: non-constructive existence principles

- Well ordering principle (Axiom of Choice)
- Hahn Banach Theorem

This tradition continues to this day with the label "Combinatorial Set Theory"

Issues arising from basic questions in measure theory

- "Complexity" hierarchies of sets (open/closed sets used to generate the Borel sets by transfinite induction)
- Continuous images of closed subsets of Polish Spaces

“Descriptive Set Theory”

- Borel, Baire, Lebesgue
- Egorov, Luzin, Suslin

These remain the two main
streams of set theory
may fertile interactions

Investigations of the AC and CH led to the work of Godel and Cohen showing that

- the CH is independent of ZFC.
- The AC is independent of ZF.

Solovay's Theorem

- Assuming an innocuous large cardinal exists it is consistent to have:

ZF + Countable Axiom of Choice +

"All subsets of the real numbers are Lebesgue Measurable"

The Borel Conjecture

A set $A \subseteq \mathbb{R}$ has strong measure zero iff for all $\langle \varepsilon_i : i \in \mathbb{N} \rangle$ of positive numbers there are intervals $\langle I_i : i \in \mathbb{N} \rangle$ such that

$$A \subseteq \cup I_i.$$

The Borel Conjecture:

Every strong measure zero set is countable.

The Borel Conjecture is independent

- Luzin: Assuming CH the Borel Conjecture is False
- Laver (1976): It is consistent with ZFC that the Borel Conjecture is True

Rich Laver
1942-2012

Marczewski's Question

Let X be a Polish space and A a subset of X. Then A is

perfectly meager

iff $A \cap P$ is meager inside every perfect set P.

Marczewski's Question

In 1935 Marczewski asked:

Are perfectly meager sets closed under products?

Marczewski's Question

- Reclaw (1991): CH implies "no"
- Bartoszynski (2000): Consistently "yes"

Kaplansky: Banach Algebras

In 1947 Kaplansky asked whether every algebraic homomorphism of $C(X)$ to a Banach algebra B is necessarily continuous.

- In 1978, 1979 Dales and Esterelle independently showed that: If the CH holds it is possible to construct a counterexample.
- Solovay (1979, using an important Lemma by Woodin) showed that it is consistent that the answer is "Yes"
- Woodin (early 1980's) showed it is consistent that with MA that the answer is "Yes".

Ramsey Theory:

Combinatorial Set Theory

Version 1: Combinatorial Set Theory

This kind of Ramsey theory is exemplified by people like Erdos and Hajnal. Full use of the Axiom of Choice (and any convenient cardinal arithmetic).

Clearest example is the Erdos-Rado theorem

- (finite Ramsey's Theorem) For all k, m there is an $l > k$ such that if D is a set of size l and $f : [D]^2 \to \{0, \ldots m-1\}$ then there is an $H \subseteq D$ of size k such that f is constant on $[H]^2$. In symbols:

$$l \to (n)^2_m.$$

- (Erdos-Rado Theorem) For all κ, μ there is a $\lambda > \kappa$ such that

$$\lambda \to (\kappa)^2_\mu.$$

NOT a straightforward generalization.

Version 2: Descriptive Set Theory

Let $[\mathbb{N}]^{\mathbb{N}}$ be the collection of infinite subsets of $\mathbb{N}$. This has a natural topology, the Ellentuck Topology.

Galvin-Prikry Theorem: If $B \subset [\mathbb{N}]^{\mathbb{N}}$ is Borel then there is an infinite $H \subseteq \mathbb{N}$ such that either

- $[H]^{\mathbb{N}} \subseteq B$ or
- $[H]^{\mathbb{N}} \cap B = \emptyset$.

The descriptive Ramsey theory played an important role in Gower's Dichotomy theorem(s) in Banach spaces.

Descriptive Ramsey theory was developed into a very powerful general theory by Todorcevic.

We note that many of the combinatorial ideas come from forcing type results. (Prikry Forcing, Mathias Forcing.)

Abelian Groups

The set theoretic development of Abelian Group Theory was pushed far by people like Eklof and Shelah. Probably the most emblematic results is on the Whitehead Conjecture:

Suppose that A is an Abelian group with

$\mathrm{EXT}^1(A, \mathbb{Z}) = 0.$

Then A is free Abelian

Shelah showed that Whitehead's conjecture is independent of ZFC.

General Topology

Dow, Steprans, Tall, Watson, Weiss, Juhasz, Soukup

General Topology in many ways, adds the minimal amount of structure to naked sets. Developed before WW1 by Hausdorf (as part of set theory) it was promoted by Hilbert and others as a general way of understanding many phenomena.

Moore-Mrowka Problem

Is every compact countably tight Hausdorff space sequential?

- Ostaszewski (1976): Assuming Diamond, there is a counterexample.
- Balogh (after Todorcevic): PFA imlies "yes"

One of the most famous Problems is the Normal Moore Space conjecture:

Every Normal Moore Space is metrizable

- (Fleissner) If CH is true then the Normal Moore Space Conjecture is false.
- (Kunen, Nykos) If there is a supercompact cardinal then it is consistent that the Normal Moore Space Conjecture is true.

Another Nice Toronto Example of this phenomenon:

Classification of Linear Operators on Hilbert Spaces

Cast of Characters:

- H a separable infinite dimensional Hilbert Space
- $B(H)$ the collection of bounded operators on H.
- $K(H)$ the ideal of compact operators
- $Q(H) = B(H)/K(H)$ the "Calkin Algebra"

Why Q(H)?

By reducing modulo the compact operators one gets a structure theory:

Berg-Weyl-von Neumann theorem

By reducing "random noise" the conjugacy relation becomes tractable.
"Compalent" equivalence relation

Essentially Normal Operators

Essentially normal operators are those that commute with their adjoints in the Calkin algebra (i.e. mod compact operators)

Are these classifiable?

For example: is it possible that for A,B essentially normal operators:

- A is compalent to B iff
- there is an automorphism Φ of $Q(H)$ with $\Phi([A]) = [B]$

Inner vs. Outer

Inner automorphisms preserve much more structure on Q(H):

for example they preserve "Fredholm index."

First question: Is every automorphism inner?

Familiar Pattern

- Phillips and Weaver (2007): CH implies there is an automorphism that is not inner
- Farah (2010): PFA implies all automorphisms are inner.

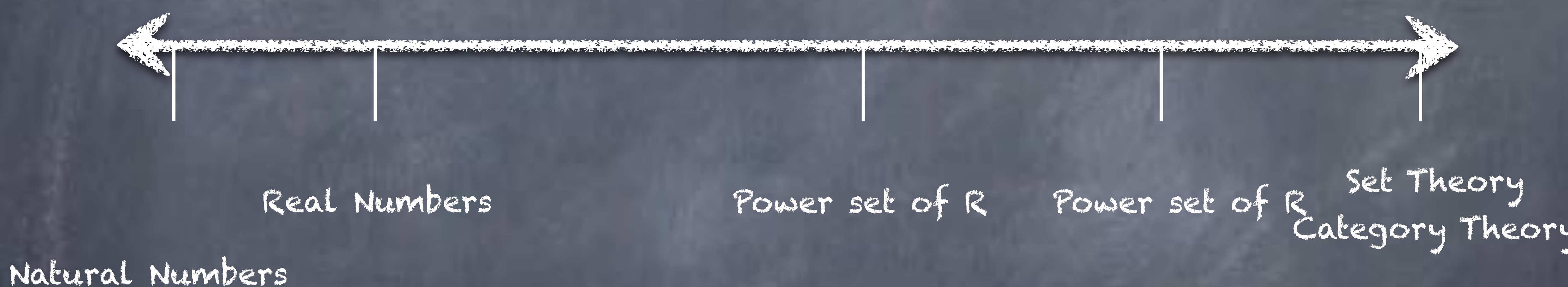
Continuum of Structure

High Structure — Low Structure

- Diophantine algebra
- Finite field Theory
- Compact Smooth Manifolds
- Functional Analysis
- Banach Spaces
- Abelian Groups
- General Topology
- Set Theory
- Category Theory

Not Drawn to Scale!

Another Continuum: Set Theoretic Complexity



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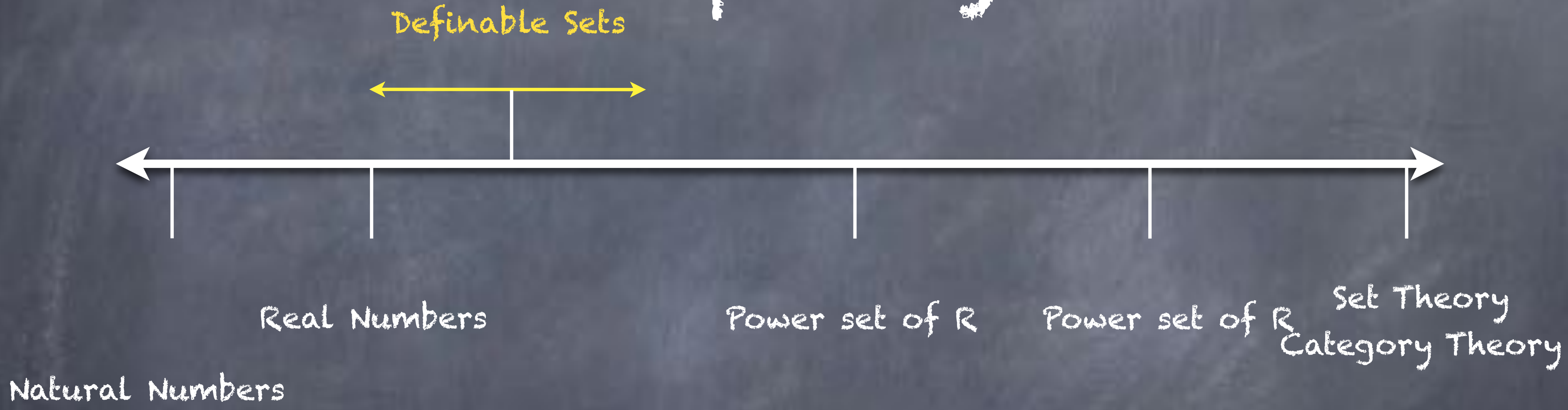

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Hierarchy of Definable sets

Open/closed sets Borel Sets Analytic Sets Σ^1_2-sets Projective sets Definable Sets

These are strict hierarchies!

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Descriptive Set Theory

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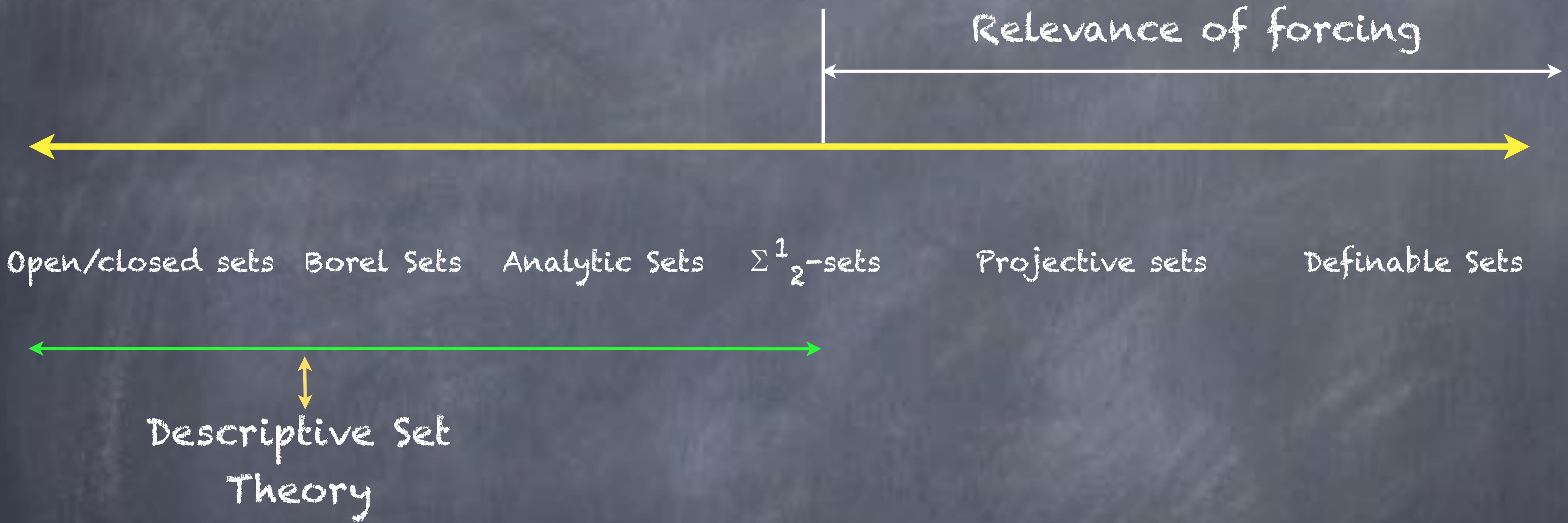

These are strict hierarchies!

Hierarchy of Definable sets

Relevance of forcing

Open/closed sets Borel Sets Analytic Sets Σ^1_2-sets Projective sets Definable Sets

Descriptive Set Theory

Descriptive Set Theory Assuming Large Cardinals

These are strict hierarchies!

Classification Problems

When can you take one problem
(an equivalence relation relation you
are trying to characterize)
and reduce it to another
(the equivalence relation "invariants")

Definition Let X and Y be Polish spaces and $E \subseteq X \times X$, $F \subseteq Y \times Y$ be equivalence relations.

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Then E is **Borel reducible to F** iff there is a Borel function $f : X \to Y$ such that

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In symbols $E \leq_B F$.

The general classification program from the DST point of view:

Consider mathematical classification problems and place them in the ZOO of equivalence relations under $\leq_B$.

Important Benchmarks:

1. Countable equivalence relations (i.e. those with countable classes)
2. Equivalence relations induced by S^{∞}-actions
3. Equivalence relations induced by Polish Group actions

S_∞-actions

S_∞ actions play a special role: they characterize the equivalence relations that correspond to countable "algebraic" invariants (up to isomorphism)

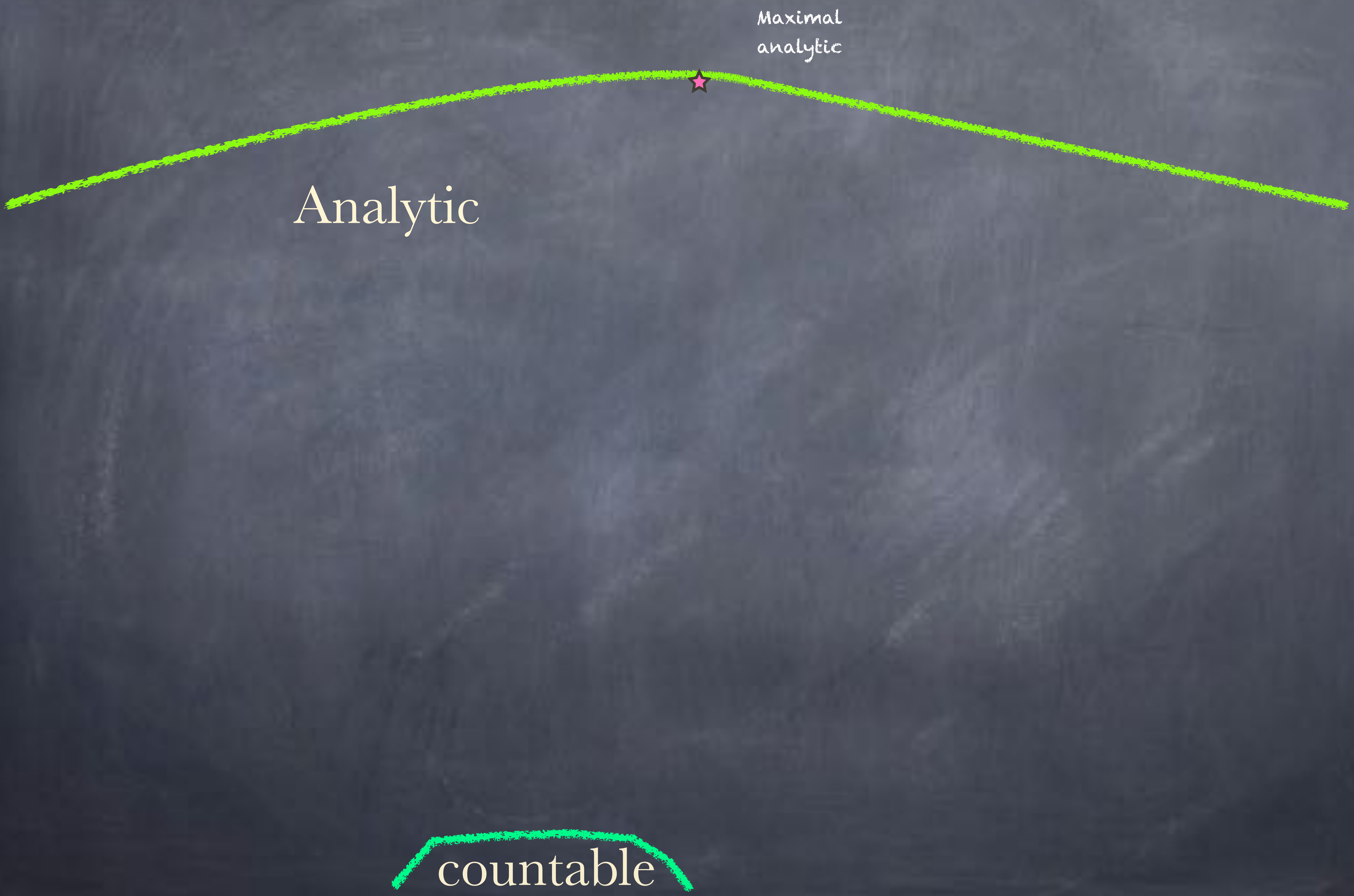
Facts

- (Harrington) There is a $\leq_B$-maximal analytic equivalence relation
- There is a $\leq_B$-maximal Polish Group action

for the problems we discuss, this will be an upper bound on the complexity

The Zoo

The Zoo



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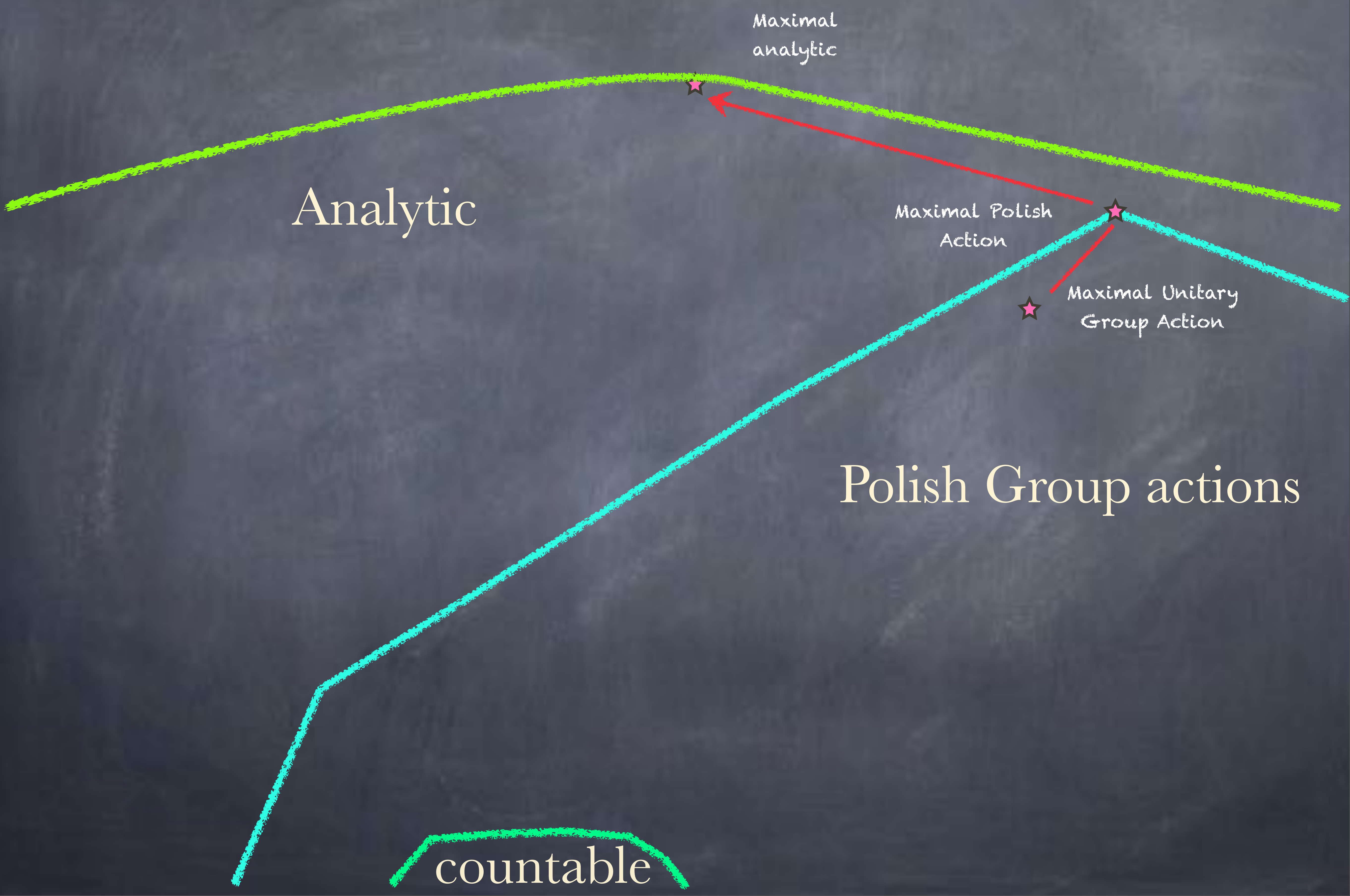
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Polish Group actions
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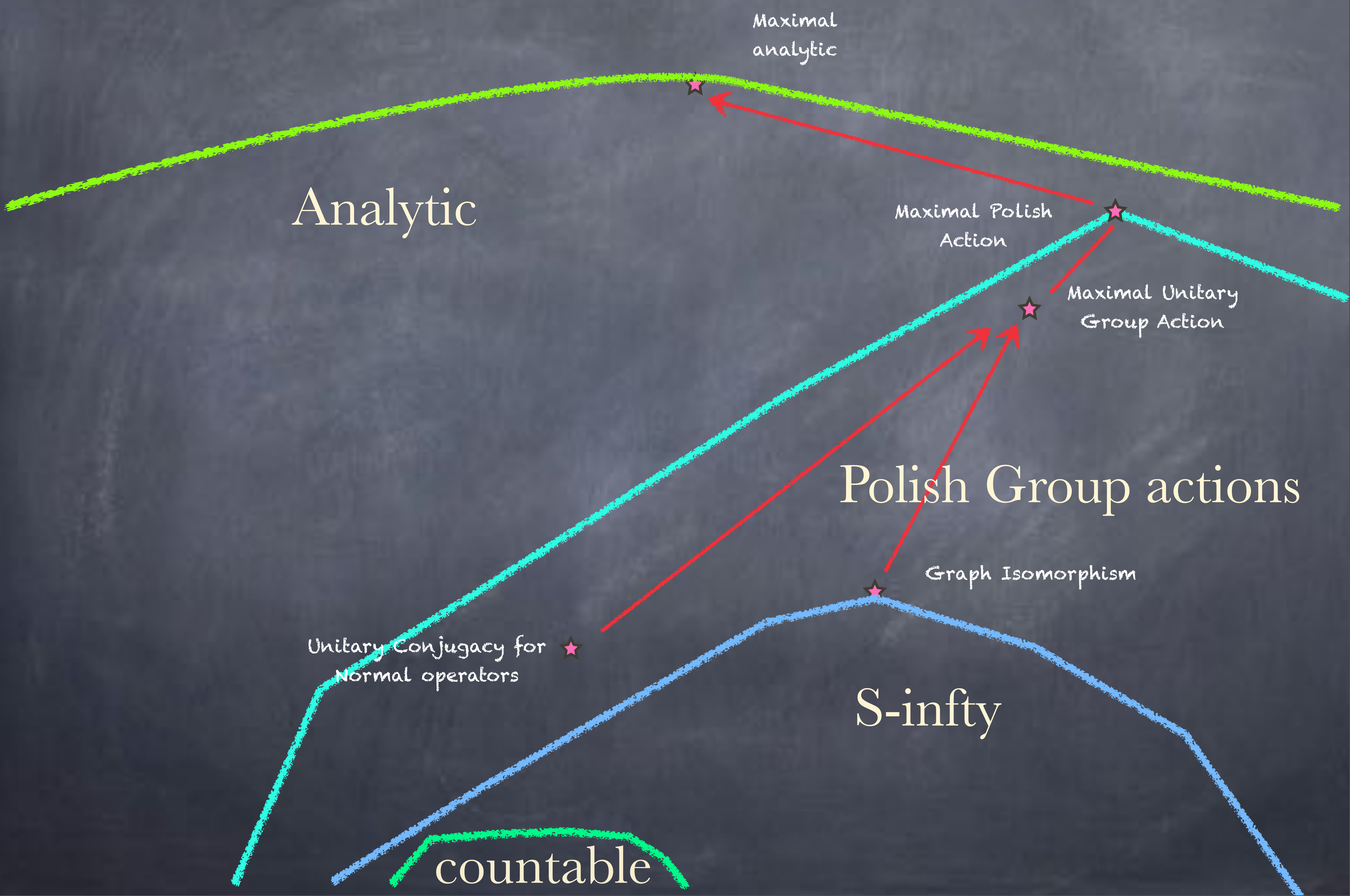

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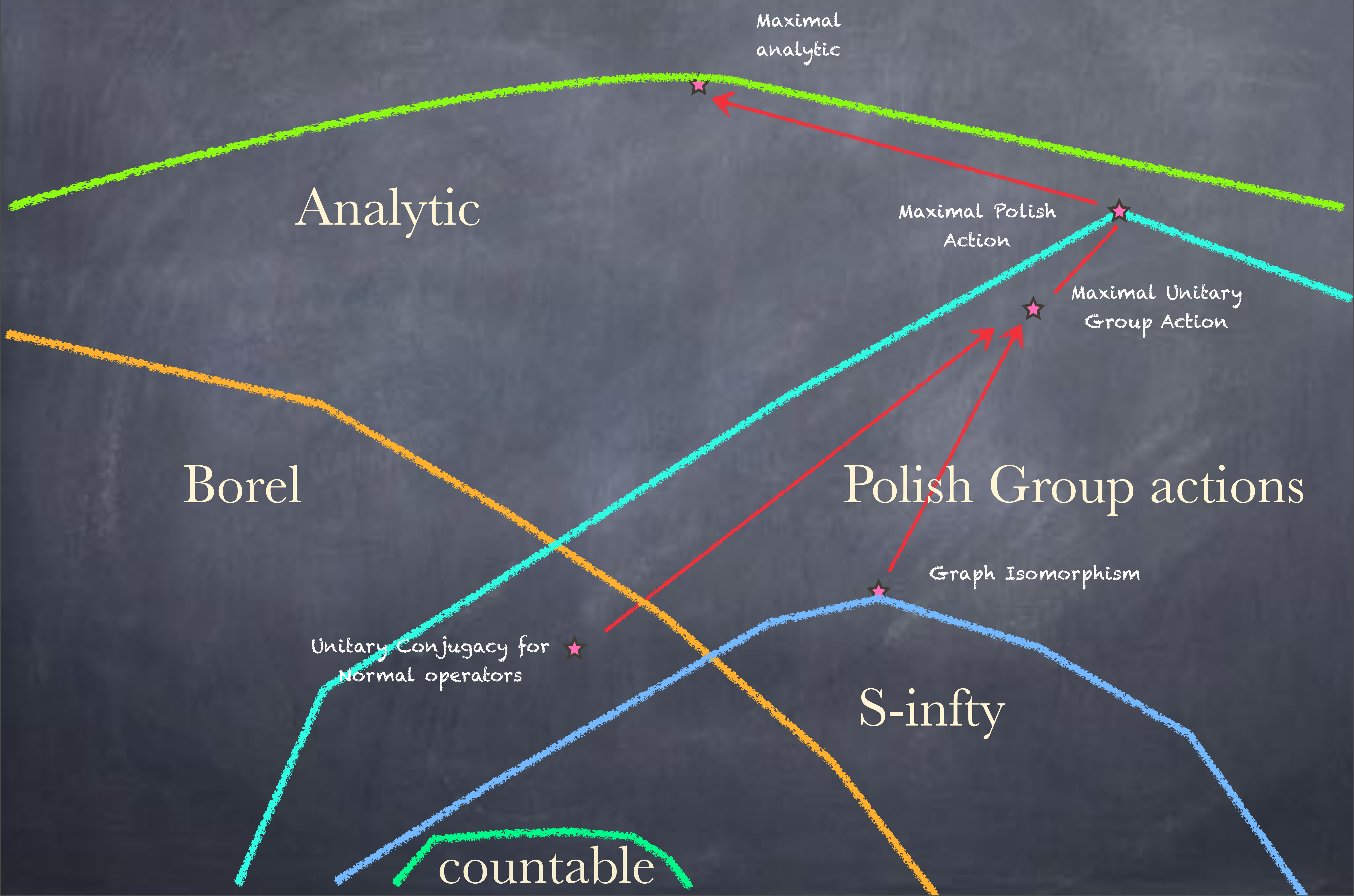
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What are examples of these equivalence relations?

At the top

- (Ferenczi-Louveau-Rosendal) Isomorphism of separable Banach Spaces is the maximal analytic equivalence relation.
- (Becker-Kechris-Hjorth-Mackey) There is an action of Iso(U) which gives a maximal Polish Group Action.

Hjorth's Turbulance

Turbulance is a wonderful property of some Polish Group actions. It is very powerful generalization of topological 0-1 laws.

The main consequence of an equivalence relation begin turbulant is that no generic subset is reducible to an S^∞ action

Elliot Classification Program

Idea: Classify separable, Unital, simple, nuclear C^*-algebras using K- theoretic invariants.

The Elliot invariants didn't turn out to be a complete invariant (Rordam and Toms), but there are other difficulties ...

Classification complexity:

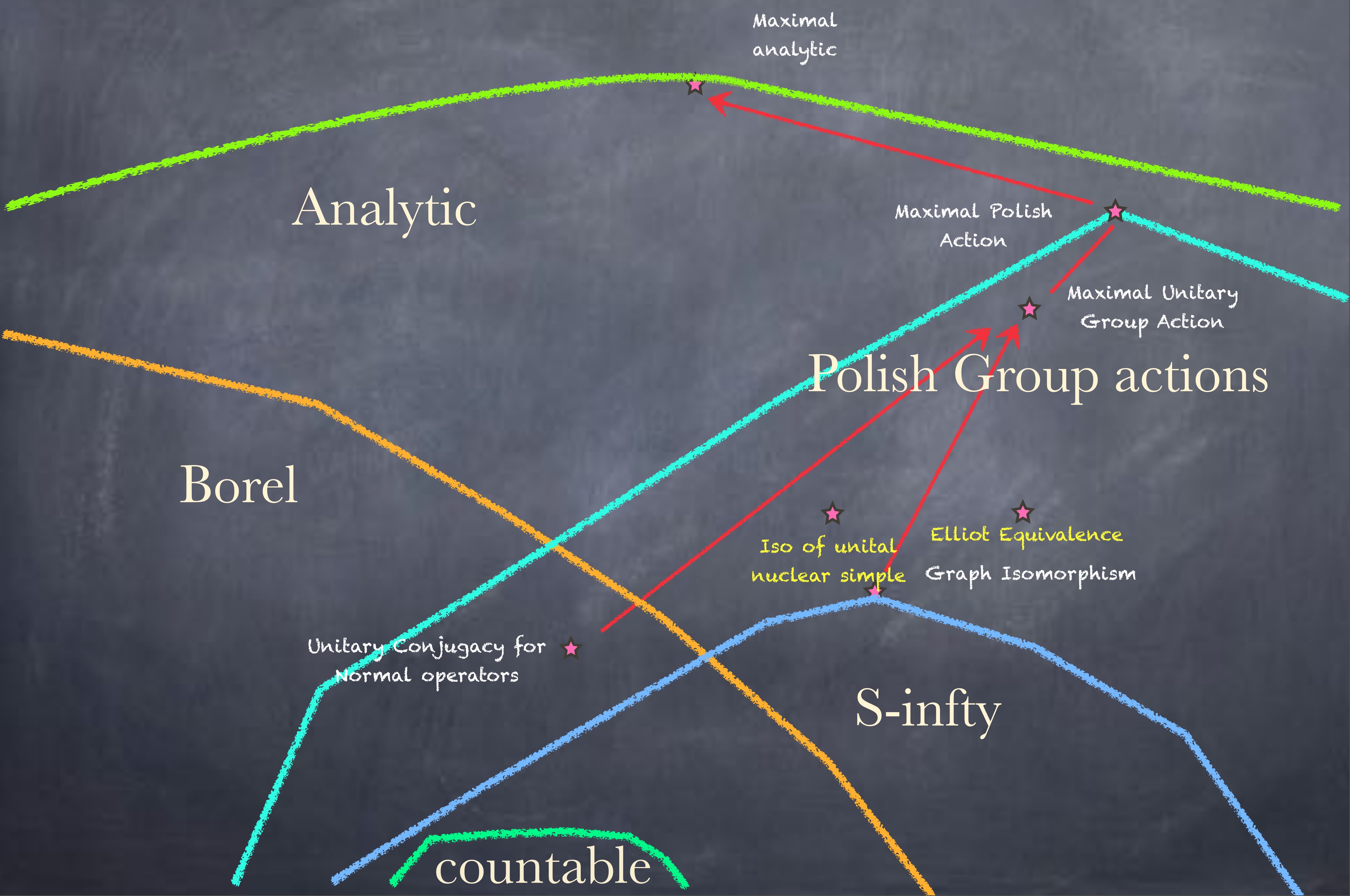
Elliot, Farah, Paulsen, Rosendal, Toms, Tornquist

- Isometry is below a Polish Group action.

- The invariant is turbulent (so not reducible to an S^{∞}-action.)

- The classification problem itself is turbulent!

The Zoo



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Elliot Equivalence
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S-infty
countable

The group of Measure Preserving Transformations

- Many dynamical systems admit an invariant probability measure on the underlying spaces. Necessary for standard "statistics".
- These systems can be paradoxical: even concrete completely deterministic systems exhibit provably random behavior.

Canonical Model

- Every non-atomic separable probability measure space is isomorphic to LM on [0,1]
- Hence all of the "statistical" dynamical behavior is exhibited in the group of invertible measure preserving transformations of [0,1]. (I call this MPT.)

von Neumann Classification Program

In 1932 von Neumann proposed classifying the measure preserving transformations up to isomorphism.

Isomorphism corresponds to the conjugacy equivalence relation in MPT.

von Neumann Classification Program

Measure preserving transformations can be glued together from the basic building blocks: ergodic measure preserving transformations.

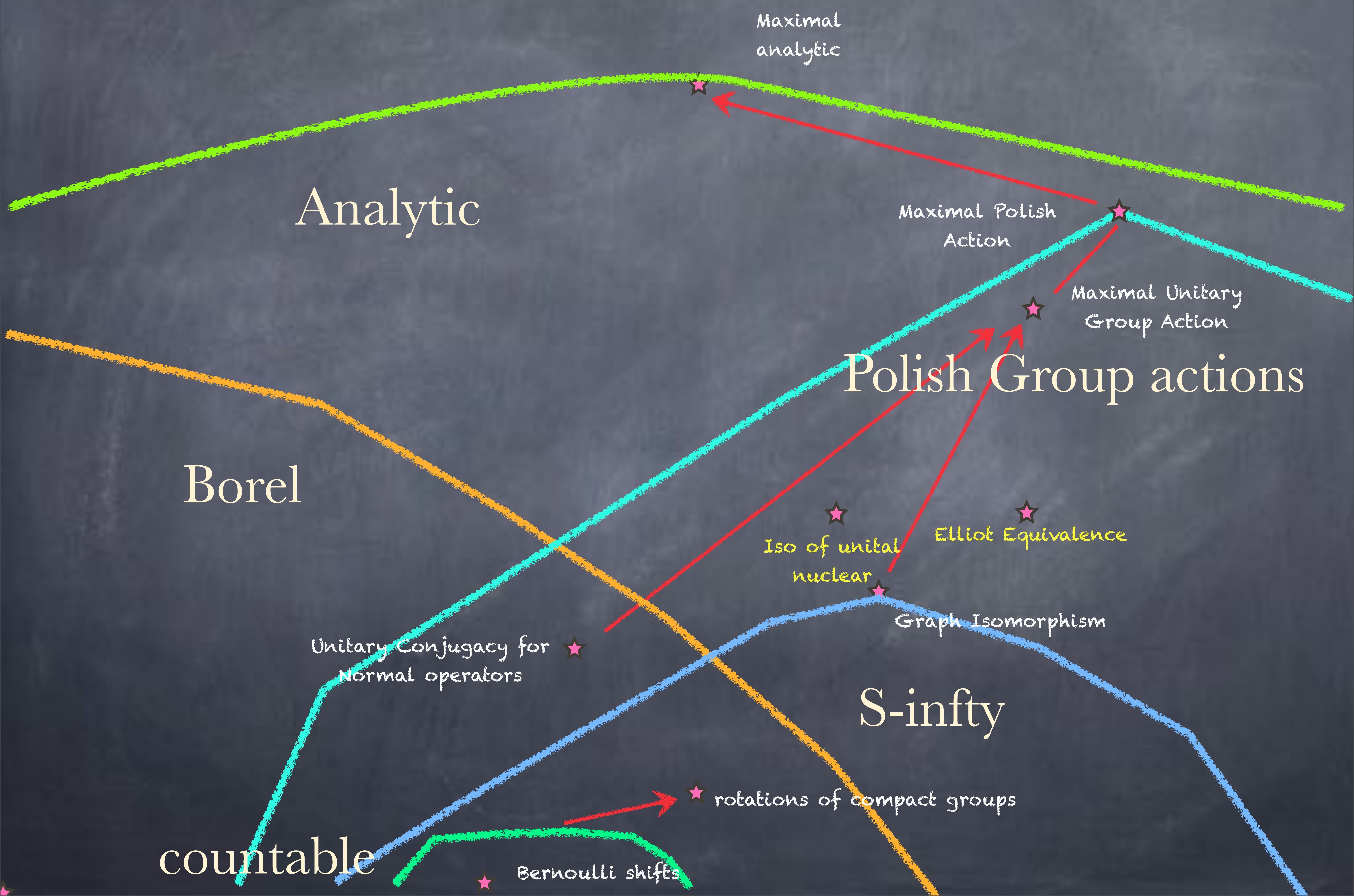
vN program usually stated as classifying the ergodic transformations.

Positive Results

- Halmos-von Neumann proved that translations on compact groups can be characterized entirely by their spectrum (pps)
- Orstein showed that entropy is a complete invariant for Bernoulli shifts

- The spectrum of an operator associated with an ergodic MPT is a countable subgroup of the unit circle.
- Entropy is a number.

The Zoo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S-infty
rotations of compact groups
countable
Bernoulli shifts

What about the general classification problem?

After all: Bernoulli Shifts and rotations on compact groups are 1st category subsets of the space of ergodic MPT's

Hjorth's Work

- Hjorth showed that the general equivalence relation of isomorphism for MPT's was NOT Borel

- Isomorphism for Rank 2 distal flows was not reducible to an S^∞ action

Generic Classes of actions

- (Foreman-Weiss) The isomorphism relation of ergodic MPT's is turbulent.

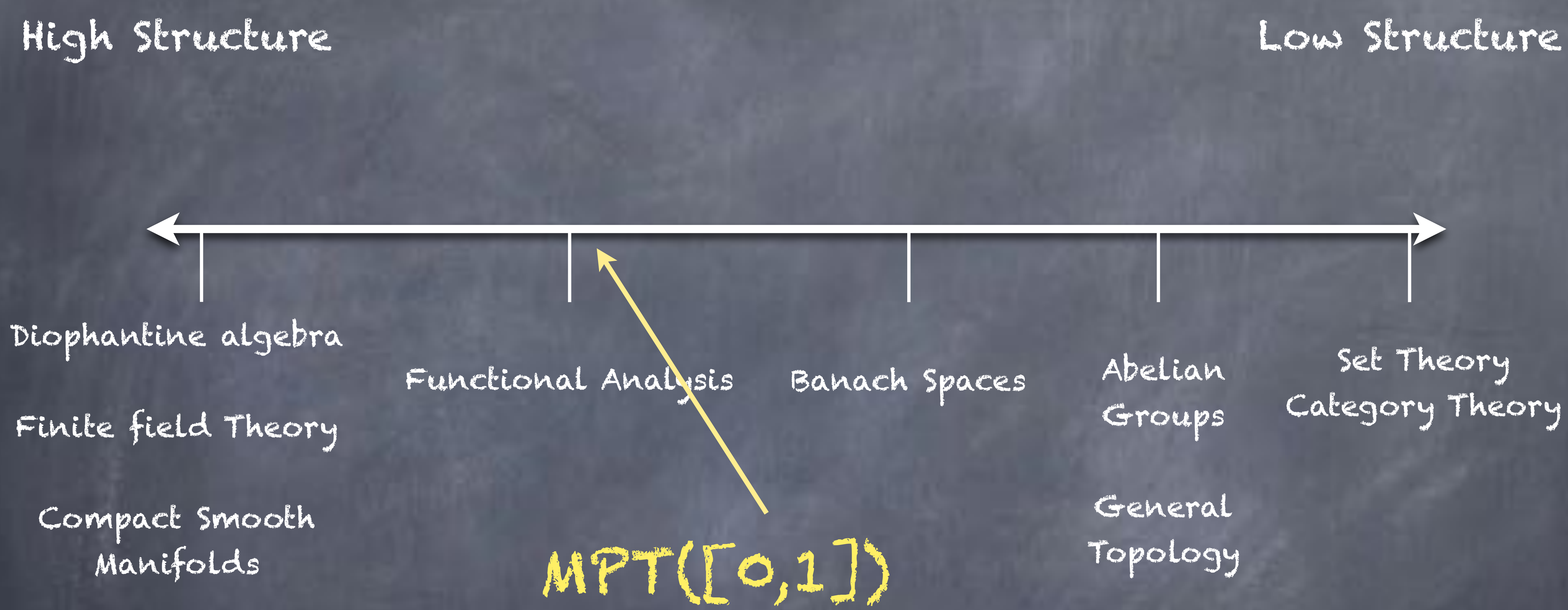
- Consequently no generic class can be classified algebraicly. (i.e. by S^∞ actions)

But is the relation even Borel?

(Foreman, Rudolph, Weiss) The collection of T such that T is isomorphic to its inverse is complete analytic. Thus

$\{(S,T) : S$ and T are ergodic and S iso to $T\}$ is not Borel

Continuum of Structure



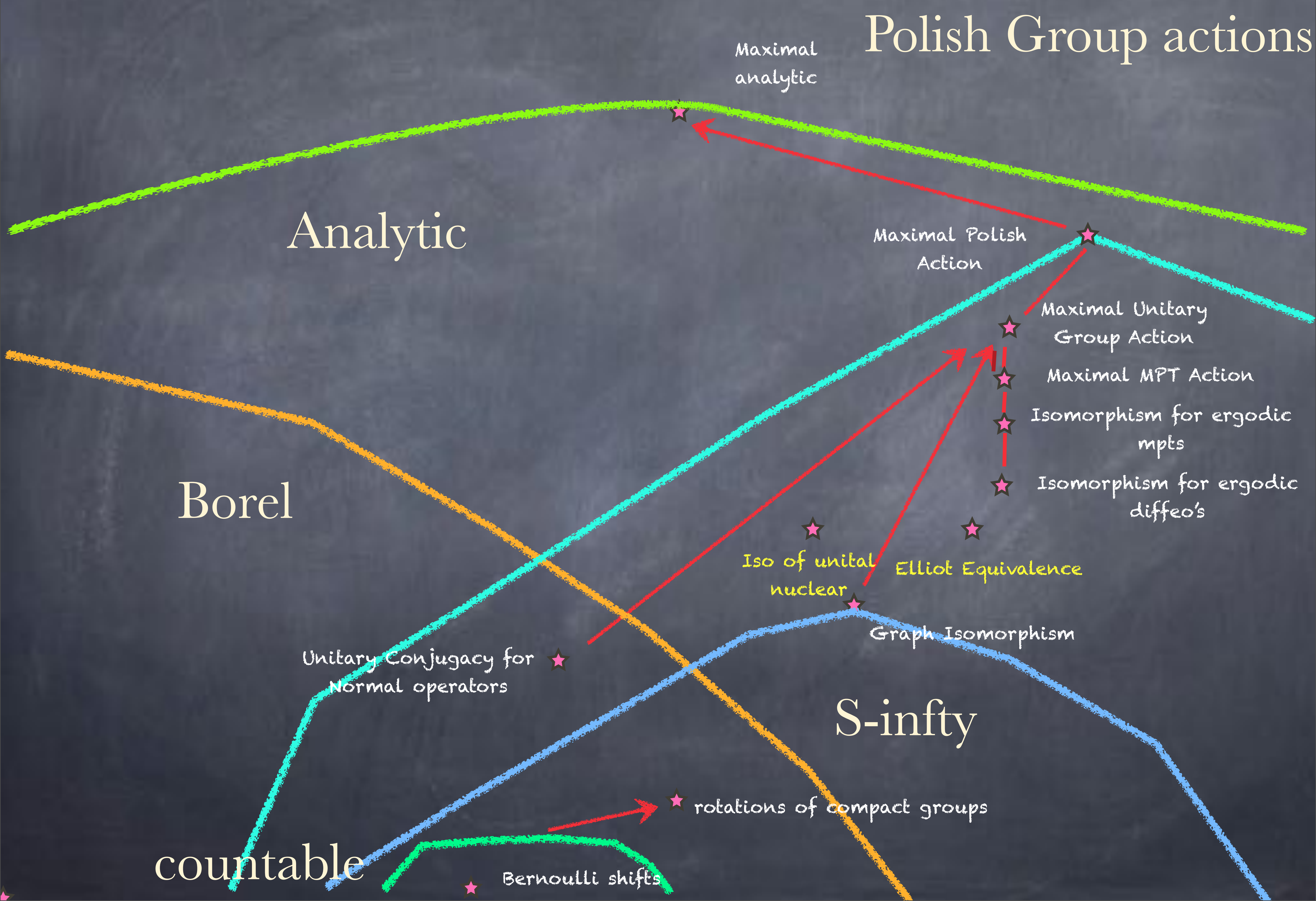
Not Drawn to Scale!

Diffeomorphisms of smooth compact manifolds

(Foreman, Weiss 2010) Let M be the 2-torus. Let S be the space of C^k, measure preserving and ergodic diffeomorphism ($1<k \leq \infty$) of M. Then the isomorphism relation on S is complete analytic.

Even for concrete diffeo's on the torus, classification is inherently impossible.

The Zoo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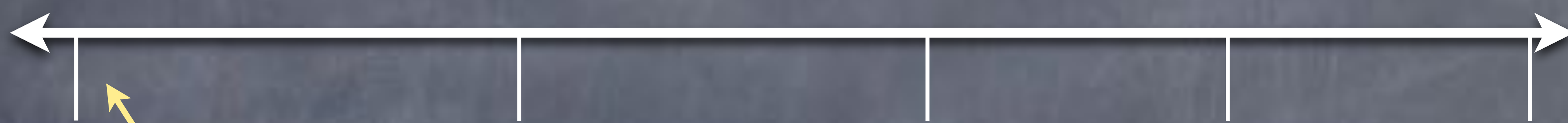
Continuum of Structure

High Structure

Low Structure



Diophantine algebra

Finite field Theory

Compact Smooth Manifolds

Functional Analysis

Banach Spaces

Abelian Groups

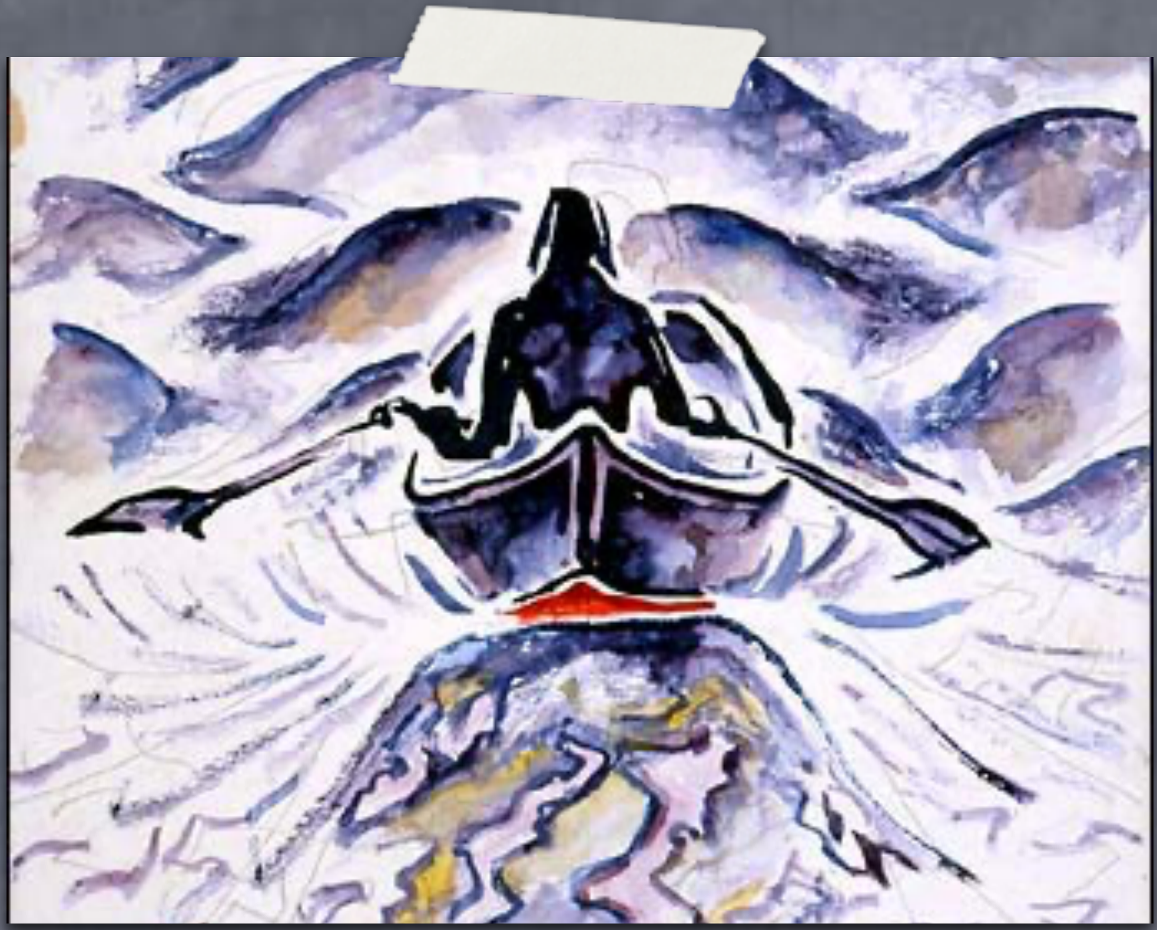
General Topology

Set Theory

Category Theory

Even down here there are examples of high set theoretic complexity

Not Drawn to Scale!



Thank you!