Cyclical consistency and cyclical monotonicity

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joint work with **Olga Kudryavtseva, Tigran Nagapetyan**

Rationalizability problem and revealed preferences

P. Samuelson (1938), H.S. Houthakker (1955)

We are given n goods and collection of $2N$ vectors from $\mathbb{R}^n_+$ which are interpreted as

Observations $x_1, \cdots, x_N$

Prices $p_1, \cdots, p_N$

Every observation

$$x_i = (x_i^1, \cdots, x_i^n), \; x_i^j \geq 0$$

corresponds to a choice of goods made by customer

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Rational choice

The choice of goods (x_i, p_i) is rational if there exists **utility function** u satisfying

$$u(y) < u(x_i)$$

for all i and every $y \in \mathbb{R}^n_+$ such that

$$\langle y, p_i \rangle > \langle x_i, p_i \rangle$$

Observation: u must have convex superlevel sets $\{u > c\}$.

Problem

Find necessary and sufficient condition for rationalizability of $\{(x_i, p_i)\}$.

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Cyclical consistency axiom

Choose a subset of the data (denote again $x_1, x_2, \cdots$)

x_i is directly prefered to x_j

$$x_i \succ x_j$$

if

$$\langle x_j, p_i \rangle > \langle x_i, p_i \rangle$$

Equivalently

$$a_{ij} = \langle x_j - x_i, p_i \rangle > 0.$$

Cyclical consistency axiom

The following cycle is not possible

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$$a_{12} = a_{23} = \cdots = a_{k1} = 0.$$

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(Houthakker) *Cyclical consistency is equivalent to rationalizability.*

Another assumption which implies cyclical consistency: there exists a positive function c on $\mathbb{R}_n^+$ satisfying

$$c(p_1)a_{12} + c(p_2)a_{23} + \cdots + c(p_k)a_{k1} \leq 0$$

for every subset $\{x_i, p_i\}$ of D.

Rearranging the terms we get

$$c(p_1)\langle x_2, p_1\rangle + c(p_2)\langle x_3, p_2\rangle + \cdots + c(p_k)\langle x_1, p_k\rangle \\ \leq c(p_1)\langle x_1, p_1\rangle + c(p_2)\langle x_2, p_2\rangle + \cdots + c(p_k)\langle x_k, p_k\rangle.$$

This is exactly the **cyclical monotonicity** assumption for the cost function

$$h(x, y) = -c(y)\langle x, y\rangle.$$

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Does cyclical consistency imply cyclical monotonicity for some function c?

Discrete case: yes

Theorem

(Afriat) Given a finite cyclically consistent vector field $D = \{x_i, p_i\}$, $1 \leq i \leq N$ there exist numbers c_i such that $\{x_i, c_i \cdot p_i\}$ is cyclically monotone

$$h(x, y) = -\langle x, y \rangle.$$

By the Rockafellar theorem, there exists a **concave** utility function u such that $u(x_j) \leq u(x_i) + c_i \langle x_j - x_i, p_i \rangle$.

Ekeland, Galichon (2012). Interpretation of the rationalizability problem as a dual to the housing problem of Shapley and Scarf.

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What happens in continuous case?

Additional assumption: the field is homogeneous

$$\{x_i, p_i\} \in D \Longrightarrow \{t \cdot x_i, p_i\},\ \ t \geq 0$$

(H. Varian) Every homogeneous cyclically consistent vector field satisfies the following axiom (HARP):

$$\langle x_1, p_1\rangle \cdots \langle x_k, p_k\rangle \geq \langle x_2, p_1\rangle \cdots \langle x_1, p_k\rangle$$

Proof of HARP for $k = 2$.

Find t such that $\langle x_1, p_1\rangle = t\langle x_2, p_1\rangle = \langle tx_2, p_1\rangle$. Cyclical consistency: $\langle tx_2, p_2\rangle \geq \langle x_1, p_2\rangle$. Substituting $t = \frac{\langle x_1, p_1\rangle}{\langle x_2, p_1\rangle}$ into the latter inequality we get the claim.

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Taking logarithm we get that this condition is equivalent to cyclical monotonicity for $h(x, y) = -\log\langle x, y\rangle$.

Theorem

Every (in general non-discrete) homogeneous cyclically consistent vector field $\{(x, p(x))\} \subset \mathbb{R}^n_+ \times \mathbb{R}^n_+$, $|p| = 1$ *solves optimal transportation problem for every couple of probability measures* μ, $\nu = \mu \circ p^{-1}$ *and cost function*

$$c(x, y) = -\log\langle x, y\rangle.$$

provided transport plan is finite cost plan.

Important: optimality always implies cyclical monotonicity but the converse is not always true.

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Geometric interpretation

Alexandrov problem

Find a convex surface F with given Gauss curvature $K(n)$, where $n : F \to S^{n-1}$ is the Gauss normal map.

Theorem

(Oliker, 2007) Denote by σ the normalized Hausdorff measure on the unit sphere S^{d-1}. The Alexandrov problem can be stated as an optimal transportation problem for the cost function

$$c(x, y) = -\log\langle x, y\rangle$$

on $S^{n-1} \times S^{n-1}$ and measures σ, $K(n) \cdot \sigma$.
The potential functions h, ρ in the corresponding dual problem can be interpreted as the support and the radial function of F. They satisfy

$$\log h(n) - \log \rho(x) \geq \log\langle x, y\rangle.$$

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Extension of the Varian's result

Let A, B be two convex sets contaning zero. Let $u = t$ on $\partial(A + Bt)$, where the sum is understood in the Minkowski sense. The corresponding vector field $p(x) = \frac{\nabla u}{|\nabla u|}$ is c-monotone for the cost function

$$c(x, y) = -\log\langle x - n_A^{-1}(y), y\rangle, \ \ y \in S^{n-1},$$

where n_A^{-1} is the inverse Gauss map for ∂A.

General continuous case

Assume we are given a cyclically consistent vector field $p(x) \in \mathbb{R}^n_+ \cap S^{n-1}, x \in \mathbb{R}^n_+$ and a corresponding utility function u_0.
Any corresponding utility function u is a composition

$$u = f(u_0),$$

where f is increasing. We want $f(u_0)$ to be concave.
Equivalently, if u has *convex* sublevel sets $\{u \leq c\}$ we are looking for increasing f such that $f(u)$ is *convex*.

It is known that the Afriat's theorem does not hold for general continuous case.

First results: De Finetti (1949), Fenchel (1953).

Counterexamples

Functions

$$x + \sqrt{x + y^2}$$

$$\frac{2x}{2 - y},\ 0 < x, y \le 1$$

have hyperplanes for level sets and are non-convexifiable.

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Necessary and sufficient conditions

$$\alpha(x_1, x_2, x_3) = \sup_{y_i \sim x_i} \frac{|y_2 - y_1|}{|y_3 - y_2|},$$

y_i collinear, y_2 between y_1, y_3.
Y. Kannai: a cyclically consistent vector field p is convexifiable if and only if

$$\sup \Big[\sum_{k=1}^{n} \sum_{i=k}^{n-1} \alpha(x_{i-1}, x_i, x_{i+1})\Big]^{-1} \sum_{k=1}^{j} \sum_{i=k}^{n-1} \alpha(x_{i-1}, x_i, x_{i+1}) < 1$$

where $p_n \succ \cdots \succ p_2 \succ p_1 \succ p_0$, p_n is maximal, $p_j = p$, $j < n$.

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One-point condition (Fenchel) necessary and suffient conditions for existence of *twice differentiable* f such that $f(u)$ is convex.

For every fixed $\nu \in S^{n-1}$ consider a family of points Γ_ν where the field $p(x)$ coinsides with ν (this is *inverse Gauss map*. Assume that every Γ_ν is a continuously differentiable curve. Natural parametrization $t \to \gamma_\nu(t)$, unit speed tangent vector $\omega = \frac{d}{dt}\gamma_\nu(t)$.

Theorem

Let p be a cyclically consistent unit vector field on $\mathbb{R}^n_+$. Assume that p, ω are continuous and satisfies the following properties:

- *$p|_{x_i=0}$ does not depend on x_i for every $1 \le i \le n$ and has zero for its i-th component*
- *The projection of the acceleration $\nabla_\omega \omega(x)$ onto the hyperplane orthogonal to $p(x)$ is a continuous vector field with has a positive first component for every $x \notin \{te_1, t \ge 0\}$.*

Then the rationalizing function u satisfying $u(te_1) = t$ is convex.

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n=2

For $n = 2$ one can get a more precise statement:
Assume that the curvatures of all γ_ν are bounded from below by a number $K \leq 0$. Let $\alpha \in [0, \frac{\pi}{2})$ be the angle between n and ω.
Assume that there is an upper bound $\alpha \leq \alpha_0 < \frac{\pi}{2}$. Finally, assume that $p(x, 0) = 1$
Then there exists a universal function f on $[0, \frac{\pi}{2})$ such that u is convex provided

$$u_{xx}(t, 0) \geq -Ku_x^2(t, 0)\frac{f(\alpha_0)}{\min_t |u'(t)|}.$$