Two-sided investments and matching with multi-dimensional cost types and attributes

Deniz Dizdar\textsuperscript{1}

\textsuperscript{1}Department of Economics, University of Montréal

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Investments and matching
agents from both sides of a large two-sided economy have to make costly investments before they compete for partners in a matching market
agents from both sides of a large two-sided economy have to make costly investments before they compete for partners in a matching market

Examples

- individuals and firms
- sellers and buyers
Investments and matching

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Main features
- investments affect the surplus/gains from trade that can be generated in future matches
- agents cannot bargain and contract with potential partners before they invest
- when agents choose investments, they take into account their costs and the payoff they expect to get in the matching market
- the prospect of competition provides incentives to invest
Investments and matching
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How efficient are investments and matching patterns from an ex-ante perspective?
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Focus of the present paper

- economies with a competitive (continuum, frictionless) one-to-one matching market
- consequences of market incompleteness
A sketch of the model
A sketch of the model

- continuum of heterogeneous buyers and sellers with quasi-linear utility functions: each agent is characterized by a cost type
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Two stages

- at stage 1, all agents simultaneously and non-cooperatively choose investments
- at stage 2, agents compete in a one-to-one matching market
  - sunk investments determine the match surplus
  - the market is an assignment game: matching is frictionless and utility is transferable $\Rightarrow$ based on their investments, buyers and sellers match efficiently
A sketch of the model

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Cole, Mailath and Postlewaite (2001a)
- investments are one-dimensional and match surplus is supermodular
- cost types are one-dimensional and cost functions are submodular
Equilibrium concept and results of Cole, Mailath and Postlewaite (2001a)
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In an **ex-post contracting equilibrium**, any investment must “best-reply” to the correctly anticipated trading possibilities and payoffs in the endogenous market.

- Investment choices are not directed by a complete system of Walrasian payoffs for all ex-ante possible investments: there are **market** payoffs only for investments that exist at stage 2.

- An agent who deviates to an otherwise non-existent investment can match with any marketed investment from the other side, leave the market payoff to the partner and keep the remaining surplus.
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Cole, Mailath and Postlewaite (2001a)

- An efficient equilibrium always exists.

- Two examples of inefficient equilibria with coordination failures.
Contributions (I)

Motivation

- the sets of possible investments are multi-dimensional in most interesting environments
- multi-dimensional cost types are needed to model ex-ante heterogeneity
- general forms of surplus and cost functions
Contributions (I)

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- the sets of possible investments are multi-dimensional in most interesting environments
- multi-dimensional cost types are needed to model ex-ante heterogeneity
- general forms of surplus and cost functions

I verify that efficient ex-post contracting equilibria exist in a general assignment game framework

Main contribution

- I shed light on what enables/constrains/precludes the existence of inefficient equilibria, both in environments with one-dimensional and with multi-dimensional heterogeneity
Contributions (II)
Two kinds of inefficiency

- **inefficiency of joint investments**
- **mismatch** of buyers and sellers from an ex-ante perspective
  - cannot occur in the “1-d supermodular framework,” where the matching of cost types must be positively assortative in any equilibrium
Contributions (II)

Two kinds of inefficiency

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Main contributions

- new sufficient condition for ruling out inefficiency of joint investments: “absence of technological multiplicity”
- analysis of mismatch in multi-dimensional environments without technological multiplicity
  - examples, require some insights from optimal transport
- new insights about the role of ex-ante heterogeneity for ruling out inefficiencies in environments with technological multiplicity
Related literature

Investments and matching
- Acemoglu (1996); Mailath, Postlewaite and Samuelson (2013)
- Peters and Siow (2002); Bhaskar and Hopkins (2013); Gall, Legros and Newman (2013)
- Chiappori, Iyigun and Weiss (2009); McCann, Shi, Siow and Wolthoff (2013)
- Cole, Mailath and Postlewaite (2001a,b); Felli and Roberts (2001)
- Nöldeke and Samuelson (2014)

Assignment games, optimal transport and hedonic pricing
- Shapley and Shubik (1971); Becker (1973); Gretzky, Ostroy and Zame (1992, 1999)
- Villani (2009)
- Rosen (1974); Ekeland (2005, 2010); Chiappori, McCann and Nesheim (2010)
There is a continuum of buyers and sellers with quasi-linear utility functions
Timing

There is a continuum of buyers and sellers with quasi-linear utility functions

- at stage 1, all agents simultaneously and non-cooperatively choose investments
- at stage 2, agents compete for partners
Match surplus, costs and ex-ante heterogeneity
choosing an investment means choosing an attribute (deterministic investment technology)

- the sets of possible attribute choices are $X$ (for buyers) and $Y$ (for sellers)
- generic elements are denoted $x$ and $y$
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**gross match surplus** $\nu(x, y)$
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**gross match surplus** \( v(x, y) \)

agents are ex-ante heterogeneous
  - characterized by **cost types** \( b \in B \) and \( s \in S \)
  - cost functions \( c_B(x, b) \) and \( c_S(y, s) \)
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$B, S, X$ and $Y$ are compact metric spaces

$v : X \times Y \rightarrow \mathbb{R}_+, c_B : X \times B \rightarrow \mathbb{R}_+$ and $c_S : Y \times S \rightarrow \mathbb{R}_+$ are continuous

for simplicity: unmatched agents create zero surplus
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- for simplicity: unmatched agents create zero surplus
- the heterogeneous ex-ante populations of buyers and sellers are described by probability measures $\mu_B$ on $B$ and $\mu_S$ on $S$
- $\mu_B$, $\mu_S$, $\nu$, $c_B$ and $c_S$ are common knowledge at stage 1
Stage 2: The matching market
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- the match surplus function \( v \)
- the distributions of buyer and seller attributes that result from agents’ sunk investments: \( \mu_X \) on \( X \) and \( \mu_Y \) on \( Y \)
Stage 2: The matching market

Matching is frictionless, utility is transferable: the matching market is a continuum assignment game, described by

- the match surplus function $\nu$
- the distributions of buyer and seller attributes that result from agents’ sunk investments: $\mu_X$ on $X$ and $\mu_Y$ on $Y$

The possible matchings of $\mu_X$ and $\mu_Y$ are the measures $\pi_2$ on $X \times Y$ with marginal measures $\mu_X$ and $\mu_Y$: $\pi_2 \in \Pi(\mu_X, \mu_Y)$
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- an efficient, surplus-maximizing matching $\pi_2^*$
- $\pi_2^* \in \Pi(\mu_X, \mu_Y)$ attains $\sup_{\pi_2 \in \Pi(\mu_X, \mu_Y)} \int \nu \, d\pi_2$
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  \[\pi_2^* \in \Pi(\mu_X, \mu_Y) \text{ attains } \sup_{\pi_2 \in \Pi(\mu_X, \mu_Y)} \int \nu \, d\pi_2\]

- core payoff functions \(\psi_X^*: \text{Supp}(\mu_X) \to \mathbb{R}\) and \(\psi_Y^*: \text{Supp}(\mu_Y) \to \mathbb{R}\)
  
  \[\psi_Y^*(y) + \psi_X^*(x) = \nu(x, y) \text{ on } \text{Supp}(\pi_2^*)\]

  for all \((x, y) \in \text{Supp}(\mu_X) \times \text{Supp}(\mu_Y)\): \(\psi_Y^*(y) + \psi_X^*(x) \geq \nu(x, y)\)
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- core payoff functions \(\psi^*_X : \text{Supp}(\mu_X) \to \mathbb{R}\) and \(\psi^*_Y : \text{Supp}(\mu_Y) \to \mathbb{R}\)
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Stable outcomes exist (Gretzky, Ostroy and Zame 1992; Villani 2009)
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Stage 2: Stable outcomes

Competition without frictions results in a stable outcome: a **stable outcome** of \((\mu_X, \mu_Y, v)\) consists of

- an efficient, surplus-maximizing matching \(\pi^*_2\)
  \[\pi^*_2 \in \Pi(\mu_X, \mu_Y) \text{ attains } \sup_{\pi_2 \in \Pi(\mu_X, \mu_Y)} \int v \, d\pi_2\]
- core payoff functions \(\psi^*_X : \text{Supp}(\mu_X) \to \mathbb{R}\) and \(\psi^*_Y : \text{Supp}(\mu_Y) \to \mathbb{R}\)
  \[\psi^*_Y(y) + \psi^*_X(x) = v(x, y) \text{ on Supp}(\pi^*_2)\]
  \[\text{for all } (x, y) \in \text{Supp}(\mu_X) \times \text{Supp}(\mu_Y): \psi^*_Y(y) + \psi^*_X(x) \geq v(x, y)\]

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- \(\psi^*_X\) and \(\psi^*_Y\) are continuous
Stage 1: Best replies
Stage 1: Best replies

In **ex-post contracting equilibrium**, agents’ attribute choices must “best-reply” to the correctly anticipated trading possibilities and the equilibrium outcome \((\pi^*_2, \psi^*_X, \psi^*_Y)\) of the endogenous market \((\mu_X, \mu_Y, \nu)\) that results from others’ sunk investments. In particular,

- if \(x \in \text{Supp}(\mu_X)\) is an equilibrium investment of type \(b\), then \(x\) must satisfy
  \[
  \psi^*_X(x) - c_B(x, b) = \max_{x' \in X, y \in \text{Supp}(\mu_Y)} (\nu(x', y) - \psi^*_Y(y) - c_B(x', b))
  \]

- if \(y \in \text{Supp}(\mu_Y)\) is an equilibrium investment of type \(s\), then \(y\) must satisfy
  \[
  \psi^*_Y(y) - c_S(y, s) = \max_{y' \in Y, x \in \text{Supp}(\mu_X)} (\nu(x, y') - \psi^*_X(x) - c_S(y', s))
  \]
Ex-post contracting equilibrium

Formal definition
**Ex-post contracting equilibrium**

**Formal definition**

**Definition**

An *ex-post contracting equilibrium* is a tuple \(((\beta, \sigma, \pi_1), (\pi_2^*, \psi_X^*, \psi_Y^*))\), in which \((\beta, \sigma, \pi_1)\) is a regular investment profile and \((\pi_2^*, \psi_X^*, \psi_Y^*)\) is a stable and feasible bargaining outcome for \((\mu_X, \mu_Y, \nu)\), such that for all \((b, s) \in \text{Supp}(\pi_1)\) it holds:

\[
\begin{align*}
\psi_X^*(\beta(b, s)) - c_B(\beta(b, s), b) &= \max_{x' \in X, y \in \text{Supp}(\mu_Y)} (\nu(x', y) - \psi_Y^*(y) - c_B(x', b)) =: r_B(b), \\
\psi_Y^*(\sigma(b, s)) - c_S(\sigma(b, s), s) &= \max_{y' \in Y, x \in \text{Supp}(\mu_X)} (\nu(x, y') - \psi_X^*(x) - c_S(y', s)) =: r_S(s).
\end{align*}
\]
The efficiency benchmark
The maximal net surplus that a pair \((b, s)\) can generate is

\[
w(b, s) = \max_{x \in X, y \in Y} v(x, y) - c_B(x, b) - c_S(y, s)
\]

- **jointly optimal** attributes \((x^*(b, s), y^*(b, s))\) exist for all \((b, s)\)
- \(w\) is continuous
The efficiency benchmark

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The stable outcomes \((\pi_1^*, \psi_B^*, \psi_S^*)\) of the assignment game \((\mu_B, \mu_S, w)\) provide the benchmark of ex-ante efficiency
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The stable outcomes \((\pi^*_1, \psi^*_B, \psi^*_S)\) of the assignment game \((\mu_B, \mu_S, w)\) provide the benchmark of ex-ante efficiency

- they describe how agents would match and divide net surplus if buyers and sellers could bargain in a frictionless market and write complete contracts before they invest, so that partners choose jointly optimal attributes
Efficient equilibria
Efficient equilibria

Result

Every stable outcome \((\pi_1^*, \psi_B^*, \psi_S^*)\) of \((\mu_B, \mu_S, w)\) can be supported by an ex-post contracting equilibrium. In particular, an efficient equilibrium exists.
Two manifestations of inefficiency
Two manifestations of inefficiency

Buyers and sellers may be mismatched from an ex-ante perspective:
- The matching of cost types that is associated with the equilibrium investment behavior and the matching of attributes is not efficient for the benchmark assignment game \((\mu_B, \mu_S, w)\).
Two manifestations of inefficiency

Buyers and sellers may be **mismatched** from an ex-ante perspective

- the matching of cost types that is associated with the equilibrium investment behavior and the matching of attributes is not efficient for the benchmark assignment game \((\mu_B, \mu_S, w)\)

There may be **inefficiency of joint investments**

- agents’ attributes are not jointly optimal in a strictly positive mass of matches that arise in equilibrium
Full appropriation games
Consider the following complete information, “full appropriation” (FA) game between a buyer of type $b$ and a seller of type $s$

- strategy spaces are $X$ and $Y$
- payoffs are $v(x, y) - c_B(x, b)$ and $v(x, y) - c_S(y, s)$
Consider the following complete information, “full appropriation” (FA) game between a buyer of type $b$ and a seller of type $s$

- strategy spaces are $X$ and $Y$
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**Lemma**

The attributes of a buyer of type $b$ and a seller of type $s$ who are matched in equilibrium must be a Nash equilibrium (NE) of the FA game between them.
Technological multiplicity
Proposition

Assume that for all $b \in \text{Supp}(\mu_B)$ and $s \in \text{Supp}(\mu_S)$, the FA game between $b$ and $s$ has a unique NE. Then ex-post contracting equilibria cannot feature inefficiency of joint investments.
Technological multiplicity

Proposition

Assume that for all $b \in \text{Supp}(\mu_B)$ and $s \in \text{Supp}(\mu_S)$, the FA game between $b$ and $s$ has a unique NE. Then ex-post contracting equilibria cannot feature inefficiency of joint investments.

Note

- jointly optimal attributes $x^*(b,s)$ and $y^*(b,s)$ are always a NE of the FA game between $b$ and $s$, as they maximize $v(x,y) - c_B(x,b) - c_S(y,s)$.
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Proposition

Assume that for all \( b \in \text{Supp}(\mu_B) \) and \( s \in \text{Supp}(\mu_S) \), the FA game between \( b \) and \( s \) has a unique NE. Then ex-post contracting equilibria cannot feature inefficiency of joint investments.

Note

- jointly optimal attributes \( x^*(b, s) \) and \( y^*(b, s) \) are always a NE of the FA game between \( b \) and \( s \), as they maximize \( v(x, y) - c_B(x, b) - c_S(y, s) \)

Definition

An environment displays technological multiplicity if FA games have more than one pure strategy NE for some \((b, s) \in \text{Supp}(\mu_B) \times \text{Supp}(\mu_S)\).
The 1-d supermodular framework
Condition (1dS)

Let $X \setminus \{x_{\emptyset}\}, Y \setminus \{y_{\emptyset}\}, B \setminus \{b_{\emptyset}\}, S \setminus \{s_{\emptyset}\} \subset \mathbb{R}_+$. Assume that $\nu$ is strictly supermodular in $(x, y)$, $c_B$ is strictly submodular in $(x, b)$, and $c_S$ is strictly submodular in $(y, s)$. 

The 1-d supermodular framework
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**Condition (1dS)**

Let $X \setminus \{x_{\emptyset}\}$, $Y \setminus \{y_{\emptyset}\}$, $B \setminus \{b_{\emptyset}\}$, $S \setminus \{s_{\emptyset}\} \subset \mathbb{R}_+$. Assume that $v$ is strictly supermodular in $(x, y)$, $c_B$ is strictly submodular in $(x, b)$, and $c_S$ is strictly submodular in $(y, s)$.

**Lemma**

Let Condition 1dS hold. Then the induced matching of buyer and seller cost types is positively assortative in every ex-post contracting equilibrium. Mismatch is impossible.
The 1-d supermodular framework

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Let $X \setminus \{x_\emptyset\}, Y \setminus \{y_\emptyset\}, B \setminus \{b_\emptyset\}, S \setminus \{s_\emptyset\} \subset \mathbb{R}_+$. Assume that $v$ is strictly supermodular in $(x, y)$, $c_B$ is strictly submodular in $(x, b)$, and $c_S$ is strictly submodular in $(y, s)$.

**Lemma**

Let Condition 1dS hold. Then the induced matching of buyer and seller cost types is positively assortative in every ex-post contracting equilibrium. Mismatch is impossible.

- equilibrium attribute choices are increasing in type
  - an equilibrium attribute $x$ of type $b$ must belong to $\arg\max_{x' \in X} \left( \max_{y \in \text{Supp}(\mu_Y)} (v(x', y) - \psi^*_Y(y)) - c_B(x', b) \right)$
- the matching of attributes is positively assortative
Mismatch without technological multiplicity
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Problem: beyond the 1-d supermodular framework, it is a priori unclear which matchings of buyers and sellers can occur in equilibrium
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An intuition for cases without technological multiplicity
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An intuition for cases without technological multiplicity

- equilibrium partners have jointly optimal attributes, and \((x^*(b, s), y^*(b, s))\) is continuous on \(\text{Supp}(\mu_B) \times \text{Supp}(\mu_S)\)
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An intuition for cases without technological multiplicity:

- equilibrium partners have jointly optimal attributes, and \((x^*(b, s), y^*(b, s))\) is **continuous** on \(\text{Supp}(\mu_B) \times \text{Supp}(\mu_S)\).
- any attribute choice displays the preparation for the intended match, but it also strongly reflects the agent’s own type.
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- any attribute choice displays the preparation for the intended match, but it also strongly reflects the agent’s own type

- marketed attributes \(x^*(b, s)\) are attractive targets for deviations by agents \(s'\) not too different from \(s\), similarly for \(y^*(b, s)\) and buyers \(b'\)
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- profitable deviations at stage 1 must be ruled out by sufficiently high net equilibrium payoffs
- these requirements constrain mismatch if there is some differentiation of agents ex-ante
Mismatch and its constraints in a 2-d bilinear model (I)
Mismatch and its constraints in a 2-d bilinear model (1)

The standard bilinear model

Let \( \text{Supp}(\mu_B) \setminus \{b_\emptyset\} \subset \mathbb{R}_2^+ \setminus \{0\} \), \( \text{Supp}(\mu_S) \setminus \{s_\emptyset\} \subset \mathbb{R}_2^+ \setminus \{0\} \) and \( X \setminus \{x_\emptyset\} = Y \setminus \{y_\emptyset\} = \mathbb{R}_2^+ \). Surplus and costs are given by

\[
\nu(x, y) = x \cdot y = x_1 y_1 + x_2 y_2, \quad c_B(x, b) = \frac{x_1^4}{b_1^4} + \frac{x_2^4}{b_2^4} \quad \text{and} \quad c_S(y, s) = \frac{y_1^4}{s_1^4} + \frac{y_2^4}{s_2^4}.
\]
Mismatch and its constraints in a 2-d bilinear model (I)

The standard bilinear model

Let $\text{Supp}(\mu_B) \setminus \{b_\emptyset\} \subset \mathbb{R}_+^2 \setminus \{0\}$, $\text{Supp}(\mu_S) \setminus \{s_\emptyset\} \subset \mathbb{R}_+^2 \setminus \{0\}$ and $X \setminus \{x_\emptyset\} = Y \setminus \{y_\emptyset\} = \mathbb{R}_+^2$. Surplus and costs are given by

$$v(x, y) = x \cdot y = x_1 y_1 + x_2 y_2, \quad c_B(x, b) = \frac{x_1^4}{b_1^4} + \frac{x_2^4}{b_2^4} \quad \text{and} \quad c_S(y, s) = \frac{y_1^4}{s_1^4} + \frac{y_2^4}{s_2^4}.$$  

FA games have unique non-trivial NE, given by

$$(x^*(b, s), y^*(b, s)) = \frac{1}{2} \left( \left( b_1^{\frac{3}{4}} s_1^{\frac{1}{4}}, b_2^{\frac{3}{4}} s_2^{\frac{1}{4}} \right), \left( b_1^{\frac{1}{4}} s_1^{\frac{3}{4}}, b_2^{\frac{1}{4}} s_2^{\frac{3}{4}} \right) \right)$$
Mismatch and its constraints in a 2-d bilinear model (I)

The standard bilinear model

Let \( \text{Supp}(\mu_B) \setminus \{b_\emptyset\} \subset \mathbb{R}_+^2 \setminus \{0\} \), \( \text{Supp}(\mu_S) \setminus \{s_\emptyset\} \subset \mathbb{R}_+^2 \setminus \{0\} \) and \( X \setminus \{x_\emptyset\} = Y \setminus \{y_\emptyset\} = \mathbb{R}_+^2 \). Surplus and costs are given by

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\]

- FA games have unique non-trivial NE, given by
  \[
  (x^*(b, s), y^*(b, s)) = \frac{1}{2} \left( \left( \frac{b_1^3}{s_1^4}, \frac{b_2^3}{s_2^4} \right), \left( \frac{b_1^3}{s_1^4}, \frac{b_2^3}{s_2^4} \right) \right)
  \]

- \( w(b, s) = \frac{1}{8}(b_1 s_1 + b_2 s_2) \)
Example 1

Let $\mu_S = a_H \delta(s_H,s_H) + (1 - a_H) \delta(s_L,s_L)$, where $0 < s_L < s_H$ and $0 < a_H < 1$. Moreover, $\mu_B = a_1 \delta(b'_1,0) + a_2 \delta(0,b'_2) + (1 - a_1 - a_2) \delta_{b\emptyset}$, where $0 < a_1, a_2, b'_1, b'_2$ and $a_1 + a_2 < 1$. Finally, let $b'_1 > b'_2$ and $a_H < a_1 + a_2$. 
Example 1

Let $\mu_S = a_H \delta_{(s_H,s_H)} + (1 - a_H) \delta_{(s_L,s_L)}$, where $0 < s_L < s_H$ and $0 < a_H < 1$. Moreover, $\mu_B = a_1 \delta_{(b'_1,0)} + a_2 \delta_{(0,b'_2)} + (1 - a_1 - a_2) \delta_{b_\emptyset}$, where $0 < a_1, a_2, b'_1, b'_2$ and $a_1 + a_2 < 1$. Finally, let $b'_1 > b'_2$ and $a_H < a_1 + a_2$.

- $w((b_1, b_2), (s_1, s_1)) = \frac{1}{8} (b_1 + b_2) s_1$
- the ex-ante efficient matching is positively assortative in $s_1$ and $b_1 + b_2$
Example 1

Let \( \mu_S = a_H \delta_{(s_H, s_H)} + (1 - a_H) \delta_{(s_L, s_L)} \), where \( 0 < s_L < s_H \) and \( 0 < a_H < 1 \).
Moreover, \( \mu_B = a_1 \delta_{(b_1', 0)} + a_2 \delta_{(0, b_2')} + (1 - a_1 - a_2) \delta_{b_\varnothing} \), where \( 0 < a_1, a_2, b_1', b_2' \) and \( a_1 + a_2 < 1 \). Finally, let \( b_1' > b_2' \) and \( a_H < a_1 + a_2 \).

- \( w((b_1, b_2), (s_1, s_1)) = \frac{1}{8}(b_1 + b_2)s_1 \)
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- \( w((b_1, b_2), (s_1, s_1)) = \frac{1}{8}(b_1 + b_2)s_1 \)
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- depicted case: \( a_H < a_1 \)
Mismatch and its constraints in a 2-d bilinear model (II)

Example 1

Let \( \mu_S = a_H \delta(s_H, s_H) + (1 - a_H) \delta(s_L, s_L) \), where \( 0 < s_L < s_H \) and \( 0 < a_H < 1 \).

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- \( w((b_1, b_2), (s_1, s_1)) = \frac{1}{8}(b_1 + b_2)s_1 \)
- the ex-ante efficient matching is positively assortative in \( s_1 \) and \( b_1 + b_2 \)
- depicted case: \( a_H < a_1 \)
- attributes in the endogenous market:
  \[ x^*((b'_1, 0), (s_H, s_H)), x^*((b'_1, 0), (s_L, s_L)), x^*((0, b'_2), (s_L, s_L)), y^*((b'_1, 0), (s_H, s_H)), y^*((b'_1, 0), (s_L, s_L)), y^*((0, b'_2), (s_L, s_L)) \]
- e.g. \( x^*((b'_1, 0), (s_H, s_H)) = \left( \frac{1}{2} b'_1^3 \frac{1}{4} s_H^\frac{3}{4}, 0 \right) \), \( y^*((b'_1, 0), (s_H, s_H)) = \left( \frac{1}{2} b'_1^4 s_H^\frac{3}{4}, 0 \right) \)
Consider the environment of Example 1. If $a_H < a_2$, then there is exactly one additional, mismatch inefficient equilibrium if and only if

$$\frac{2}{3} \frac{b_2'}{b_1'} \geq \left( \frac{s_H}{s_L} \right)^{\frac{2}{3}} - 1.$$

Otherwise, only the ex-ante efficient equilibrium exists.
Claim

Consider the environment of Example 1. If $a_H < a_2$, then there is exactly one additional, mismatch inefficient equilibrium if and only if

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Claim

Consider the environment of Example 1. If $a_H < a_2$, then there is exactly one additional, mismatch inefficient equilibrium if and only if

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\]

Otherwise, only the ex-ante efficient equilibrium exists.

attributes in the endogenous market:
\[
x^*((b_1', 0), (s_L, s_L)), \quad x^*((0, b_2'), (s_H, s_H)), \quad x^*((0, b_2'), (s_L, s_L)), \quad y^*((b_1', 0), (s_L, s_L)), \quad y^*((0, b_2'), (s_H, s_H)), \quad y^*((0, b_2'), (s_L, s_L))
\]
Claim

Consider the environment of Example 1. If $a_H < a_2$, then there is exactly one additional, mismatch inefficient equilibrium if and only if

$$\frac{2}{3} \frac{b'_2}{b'_1} \geq \left( \frac{s_H}{s_L} \right)^{\frac{2}{3}} - 1$$

Otherwise, only the ex-ante efficient equilibrium exists.

- attributes in the endogenous market:
  - $x^*((b'_1, 0), (s_L, s_L))$, $x^*((0, b'_2), (s_H, s_H))$
  - $x^*((0, b'_2), (s_L, s_L))$, $y^*((b'_1, 0), (s_L, s_L))$
  - $y^*((0, b'_2), (s_H, s_H))$, $y^*((0, b'_2), (s_L, s_L))$

- $(s_H, s_H)$-sellers have no incentive to deviate by investing optimally for a match with $x^*((b'_1, 0), (s_L, s_L))$ if and only if the condition of the Claim holds
Example 2

\[ \text{Supp}(\mu_S) = \{(s_1, s_1)| s_L \leq s_1 \leq s_H\}, \text{ for some } s_L < s_H. \ \mu_B \text{ is compactly supported in the union of } (\mathbb{R}_+ \setminus \{0\}) \times \{0\}, \{0\} \times (\mathbb{R}_+ \setminus \{0\}) \text{ and } \{b_{\emptyset}\}. \text{ The restrictions of } \mu_B \text{ to } (\mathbb{R}_+ \setminus \{0\}) \times \{0\} \text{ and } \{0\} \times (\mathbb{R}_+ \setminus \{0\}) \text{ have interval support.} \]
Example 2

\[ \text{Supp}(\mu_S) = \{(s_1, s_1) | s_L \leq s_1 \leq s_H\}, \text{ for some } s_L < s_H. \]
\[ \mu_B \text{ is compactly supported in the union of } (\mathbb{R}_+ \setminus \{0\}) \times \{0\}, \{0\} \times (\mathbb{R}_+ \setminus \{0\}) \text{ and } \{b_\emptyset\}. \]
\[ \text{The restrictions of } \mu_B \text{ to } (\mathbb{R}_+ \setminus \{0\}) \times \{0\} \text{ and } \{0\} \times (\mathbb{R}_+ \setminus \{0\}) \text{ have interval support.} \]

- result: the only ex-post contracting equilibrium is the ex-ante efficient one
- cost types are matched positively assortatively in \( s_1 \) and \( b_1 + b_2 \)
Mismatch and its constraints in a 2-d bilinear model (V)
Remarks
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- in Examples 1 and 2, results from the theory of assortative matching can be used to identify the efficient matching and to evaluate whether inefficient equilibria exist
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- this is not feasible in more complex environments

Characterization from optimal transport

- a matching $\pi_1 \in \Pi(\mu_B, \mu_S)$ is efficient if and only if it is concentrated on a $w$-cyclically monotone set

**Definition**

A set $A \subset B \times S$ is called **$w$-cyclically monotone** if for all $K \in \mathbb{N}$, $(b_1, s_1), \ldots, (b_K, s_K) \in A$ and $s_{K+1} = s_1$, the following inequality is satisfied.

$$\sum_{i=1}^{K} w(b_i, s_i) \geq \sum_{i=1}^{K} w(b_i, s_{i+1}).$$
Mismatch and its constraints in a 2-d bilinear model (VI)
Theorem (Villani)

Let $\text{Supp}(\mu_B), \text{Supp}(\mu_S) \subset (\mathbb{R}_+ \setminus \{0\})^2$ be closures of bounded, open and uniformly convex sets with smooth boundaries. Assume that $\mu_B$ and $\mu_S$ admit smooth, strictly positive densities on $\text{Supp}(\mu_B)$ and $\text{Supp}(\mu_S)$. Then, the stable outcomes $(\pi_1^*, \psi_B^*, \psi_S^*)$ of $(\mu_B, \mu_S, w)$ satisfy:

- $\psi_B^*$ and $\psi_S^*$ are smooth, and unique up to an additive constant,
- $\pi_1^*$ is unique. It is given by a smooth bijection $T^* : \text{Supp}(\mu_B) \rightarrow \text{Supp}(\mu_S)$ satisfying $\frac{1}{8} T^*(b) = \nabla \psi_B^*(b)$.

\[ b_2 \quad b_1 \quad s_2 \quad s_1 \]

\[ \text{Supp}(\mu_B) \quad \text{Supp}(\mu_S) \]
Let $\text{Supp}(\mu_B), \text{Supp}(\mu_S) \subset (\mathbb{R}_+ \setminus \{0\})^2$ be closures of bounded, open and uniformly convex sets with smooth boundaries. Assume that $\mu_B$ and $\mu_S$ admit smooth, strictly positive densities on $\text{Supp}(\mu_B)$ and $\text{Supp}(\mu_S)$. Then, the stable outcomes $(\pi_1^*, \psi_B^*, \psi_S^*)$ of $(\mu_B, \mu_S, w)$ satisfy:

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Theorem (Villani)
Mismatch and its constraints in a 2-d bilinear model (VII)
Consider the environment of Theorem (Villani), and assume in addition that 
\[ \left( \frac{s_1 b_2}{b_1 s_2} + \frac{s_2 b_1}{b_2 s_1} \right) < 32 \] for all \( b \in \text{Supp}(\mu_B), s \in \text{Supp}(\mu_S) \). If \( T : \text{Supp}(\mu_B) \to \text{Supp}(\mu_S) \) is a smooth matching of buyer and seller types that is compatible with an ex-post contracting equilibrium, then \( T \) is ex-ante efficient.
Theorem

Consider the environment of Theorem (Villani), and assume in addition that

\[ \left( \frac{s_1 b_2}{b_1 s_2} + \frac{s_2 b_1}{b_2 s_1} \right) < 32 \]

for all \( b \in \text{Supp}(\mu_B), \ s \in \text{Supp}(\mu_S) \). If \( T : \text{Supp}(\mu_B) \rightarrow \text{Supp}(\mu_S) \) is a smooth matching of buyer and seller types that is compatible with an ex-post contracting equilibrium, then \( T \) is ex-ante efficient.

Sketch of proof

- show that \( \nabla r_B(b) = \frac{1}{8} T(b) \), where \( r_B \) is the buyer net payoff in the ex-post contracting equilibrium.
Theorem
Consider the environment of Theorem (Villani), and assume in addition that
\[
\left( \frac{s_1 b_2}{b_1 s_2} + \frac{s_2 b_1}{b_2 s_1} \right) < 32 \text{ for all } b \in \text{Supp}(\mu_B), s \in \text{Supp}(\mu_S). \]
If \( T : \text{Supp}(\mu_B) \to \text{Supp}(\mu_S) \) is a smooth matching of buyer and seller types that is compatible with an ex-post contracting equilibrium, then \( T \) is ex-ante efficient.

Sketch of proof
- show that \( \nabla r_B(b) = \frac{1}{8} T(b) \), where \( r_B \) is the buyer net payoff in the ex-post contracting equilibrium
- use the equilibrium conditions to show that \( r_B \) must be convex
Consider the environment of Theorem (Villani), and assume in addition that
\[ \left( \frac{s_1 b_2}{b_1} + \frac{s_2 b_1}{b_2} \right) < 32 \text{ for all } b \in \text{Supp}(\mu_B), s \in \text{Supp}(\mu_S). \]
If \( T : \text{Supp}(\mu_B) \rightarrow \text{Supp}(\mu_S) \) is a smooth matching of buyer and seller types that is compatible with an ex-post contracting equilibrium, then \( T \) is ex-ante efficient.

Sketch of proof

- show that \( \nabla r_B(b) = \frac{1}{8} T(b) \), where \( r_B \) is the buyer net payoff in the ex-post contracting equilibrium
- use the equilibrium conditions to show that \( r_B \) must be convex
- hence, the matching \( T \) of buyer and seller types associated with the equilibrium is concentrated on the subdifferential of a convex function
Mismatch and its constraints in a 2-d bilinear model (VII)

**Theorem**

Consider the environment of Theorem (Villani), and assume in addition that
\[
\left( \frac{s_1 b_2}{b_1 s_2} + \frac{s_2 b_1}{b_2 s_1} \right) < 32 \text{ for all } b \in \text{Supp}(\mu_B), s \in \text{Supp}(\mu_S). \]

If \( T : \text{Supp}(\mu_B) \to \text{Supp}(\mu_S) \) is a smooth matching of buyer and seller types that is compatible with an ex-post contracting equilibrium, then \( T \) is ex-ante efficient.

**Sketch of proof**

- show that \( \nabla r_B(b) = \frac{1}{8} T(b) \), where \( r_B \) is the buyer net payoff in the ex-post contracting equilibrium
- use the equilibrium conditions to show that \( r_B \) must be **convex**
- hence, the matching \( T \) of buyer and seller types associated with the equilibrium is concentrated on the subdifferential of a convex function
- for bilinear \( w \), this is a \( w \)-cyclically monotone set \( \Rightarrow T \) is efficient
Environments with technological multiplicity

An under-investment example à la (CMP) (I)
Environments with technological multiplicity
An under-investment example à la (CMP) (I)

Example 3

Let \( v(x, y) = \max \left( xy, \frac{1}{2} x^\frac{3}{2} y^\frac{3}{2} \right) \), \( c_B(x, b) = \frac{x^4}{b^2} \) and \( c_S(y, s) = \frac{y^4}{s^2} \). \( \mu_B \) and \( \mu_S \) have interval support. For simplicity, \( \mu_B = \mu_S \).
Environments with technological multiplicity
An under-investment example à la (CMP) (I)

Example 3

Let \( v(x, y) = \max \left( xy, \frac{1}{2} x^{\frac{3}{2}} y^{\frac{3}{2}} \right) \), \( c_B(x, b) = \frac{x^4}{b^2} \) and \( c_S(y, s) = \frac{y^4}{s^2} \). \( \mu_B \) and \( \mu_S \) have interval support. For simplicity, \( \mu_B = \mu_S \).

- \( v \) has two regimes of complementarity
- \( x^*(b, s = b) \) and \( y^*(b, s = b) \) jump from \( \frac{b}{2} \) to \( \frac{3b^2}{16} \) at \( b = b_{12} \)

The efficient equilibrium
Environments with technological multiplicity
An under-investment example à la (CMP) (I)

Example 3

Let $v(x, y) = \max \left( xy, \frac{1}{2}x^{3/2}y^{3/2} \right)$, $c_B(x, b) = \frac{x^4}{b^2}$ and $c_S(y, s) = \frac{y^4}{s^2}$. $\mu_B$ and $\mu_S$ have interval support. For simplicity, $\mu_B = \mu_S$.

- $v$ has two regimes of complementarity
- $x^*(b, s = b)$ and $y^*(b, s = b)$ jump from $\frac{b}{2}$ to $\frac{3b^2}{16}$ at $b = b_{12}$
- however, attributes $\left( \frac{b}{2}, \frac{b}{2} \right)$ remain a NE of the FA game between $b$ and $s = b$ for $b > b_{12}$
Environments with technological multiplicity
An under-investment example à la (CMP) (II)
Example 3

Let $v(x, y) = \max \left( xy, \frac{1}{2} x^\frac{3}{2} y^\frac{3}{2} \right)$, $c_B(x, b) = \frac{x^4}{b^2}$ and $c_S(y, s) = \frac{y^4}{s^2}$. $\mu_B$ and $\mu_S$ have interval support. For simplicity, $\mu_B = \mu_S$.

- this enables an equilibrium in which types with lower costs than the “indifference type” $b_{12}$ under-invest, unless...
Environments with technological multiplicity
An under-investment example à la (CMP) (II)

Example 3

Let \( v(x, y) = \max \left( xy, \frac{1}{2}x^{\frac{3}{2}}y^{\frac{3}{2}} \right) \), \( c_B(x, b) = \frac{x^4}{b^2} \) and \( c_S(y, s) = \frac{y^4}{s^2} \). \( \mu_B \) and \( \mu_S \) have interval support. For simplicity, \( \mu_B = \mu_S \).

- this enables an equilibrium in which types with lower costs than the “indifference type” \( b_{12} \) under-invest, unless...

- ... populations are so heterogeneous that \( \left( \frac{b}{2}, \frac{b}{2} \right) \) is not a NE of the FA game between the pair of highest types.
Example 3

Let \( v(x, y) = \max \left( xy, \frac{1}{2}x^\frac{3}{2}y^\frac{3}{2} \right) \), \( c_B(x, b) = \frac{x^4}{b^2} \) and \( c_S(y, s) = \frac{y^4}{s^2} \). \( \mu_B \) and \( \mu_S \) have interval support. For simplicity, \( \mu_B = \mu_S \).

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Environments with technological multiplicity
Simultaneous under- and over-investment: the case of missing middle sectors (I)
Environments with technological multiplicity
Simultaneous under- and over-investment: the case of missing middle sectors (I)

Example 4

Let \( v(x, y) = \max \left( x^{\frac{1}{10}} y^{\frac{1}{10}}, \frac{3}{2} x^{\frac{3}{5}} y^{\frac{3}{5}}, x^{\frac{8}{5}} y^{\frac{8}{5}} \right) \), \( c_B(x, b) = \frac{x^4}{b^2} \) and \( c_S(y, s) = \frac{y^4}{s^2} \). \( \mu_B \) and \( \mu_S \) have interval support. For simplicity, \( \mu_B = \mu_S \).
Environments with technological multiplicity
Simultaneous under- and over-investment: the case of missing middle sectors (I)

Example 4

Let \( v(x, y) = \max \left( x^{10} y^{10}, \frac{3}{2} x^{\frac{3}{5}} y^{\frac{3}{5}}, x^{\frac{8}{5}} y^{\frac{8}{5}} \right) \), \( c_B(x, b) = \frac{x^4}{b^2} \) and \( c_S(y, s) = \frac{y^4}{s^2} \). \( \mu_B \) and \( \mu_S \) have interval support. For simplicity, \( \mu_B = \mu_S \).

\[ v \] has three regimes of complementarity

The efficient equilibrium

b
\[ b_{12} \]
\[ b_{23} \]
Environments with technological multiplicity
Simultaneous under- and over-investment: the case of missing middle sectors (II)
Example 4

Let \( v(x, y) = \max \left( x^{\frac{1}{10}} y^{\frac{1}{10}}, \frac{3}{2} x^{\frac{3}{5}} y^{\frac{3}{5}}, x^{\frac{8}{5}} y^{\frac{8}{5}} \right) \), \( c_B(x, b) = \frac{x^4}{b^2} \) and \( c_S(y, s) = \frac{y^4}{s^2} \). \( \mu_B \) and \( \mu_S \) have interval support. For simplicity, \( \mu_B = \mu_S \).
Example 4

Let $v(x, y) = \max \left( x^{1 \over 10} y^{1 \over 10}, {3 \over 2} x^{3 \over 5} y^{3 \over 5}, x^{8 \over 5} y^{8 \over 5} \right)$, $c_B(x, b) = {x^4 \over b^2}$ and $c_S(y, s) = {y^4 \over s^2}$. $\mu_B$ and $\mu_S$ have interval support. For simplicity, $\mu_B = \mu_S$. 

An inefficient equilibrium
Example 4

Let \( v(x, y) = \max \left( x^{\frac{1}{10}} y^{\frac{1}{10}}, \frac{3}{2} x^3 y^5, x^8 y^8 \right) \), \( c_B(x, b) = \frac{x^4}{b^2} \) and \( c_S(y, s) = \frac{y^4}{s^2} \). \( \mu_B \) and \( \mu_S \) have interval support. For simplicity, \( \mu_B = \mu_S \).

Even extreme exogenous heterogeneity does not rule out the inefficient equilibrium.
Conclusion

Take home messages

- technological multiplicity is the key source of potential inefficiencies
- even extreme ex-ante heterogeneity may be insufficient for ruling out inefficient equilibria