Rigidity Theory through a Geometric Lens

Walter Whiteley

York University, Toronto, Canada

August 5, 2014

Outline

Thanks to the Attendees

I am looking forward to a range in stimulating discussions and sharing in this gathering of friends, collaborators and people with shared interests.

Special thanks to the organizers - who made the proposal, revised it, and very occasionally consulted with me:

Stewart Craven, Wendy Finbow-Singh, Ami Mamolo,

Elissa Ross, Bernd Schulze, Adnan Sljoka.

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- Founding of Structural Topology by Henry Crapo - who did much of the writing, editing, translation, type setting, printing ....
- Much of the early mathematical work had a projective background and representation - a *geometric lens*.

Frameworks

Euclidean Metric $\mathbb{E}^d$ $\|(x_1, \ldots, x_d)\| = x_1^2 + \ldots + x_d^2$

- A framework (in $\mathbb{E}^d$) is a pair (G, p), where G is a graph and $p : V(G) \to \mathbb{E}^d$ is a map with $p(u) \neq p(v)$ for all $\{u, v\} \in E(G)$.

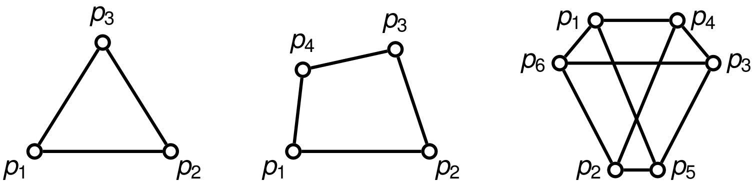

Figure : Some frameworks in $\mathbb{E}^2$

- finite motion of a framework (G, p), is an assignment continuous function $p(t)$, $0 \leq t < 1$ to the vertices such that: $|p_i(t) - p_j(t)| = |p_i(0) - p_j(0)|$, for all $(i, j) \in E$;
- non-trivial if $|p_h(t) - p_k(t)| \neq |p_h(0) - p_k(0)|$ for some $(h, k) \notin E$ and all $0 < t < 1$.

Infinitesimal motion

An infinitesimal motion of a framework (G, p) in $\mathbb{E}^d$ with $V(G) = \{1, \ldots, n\}$ is a function $u : V(G) \to \mathbb{R}^d$ such that

$$(p_i - p_j) \cdot (u_i - u_j) = 0 \quad \text{for all } \{i, j\} \in E(G), \tag{1}$$

where u_i denotes the vector $u(i)$ for each i.

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The rigidity matrix of (G, p) is the $|E(G)| \times dn$ matrix $\mathbf{R}(G, p)$

$$\begin{array}{c} \\ \{i,j\} \end{array} \begin{pmatrix} & & & i & & & & j & & & \\ & & & & & \vdots & & & & & \\ 0 & \ldots & 0 & (p_i - p_j) & 0 & \ldots & 0 & (p_j - p_i) & 0 & \ldots & 0 \\ & & & & & \vdots & & & & & \end{pmatrix}$$

kernel is space of infinitesimal motions.

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row dependencies are self-stresses

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Appears Euclidean - but actually projectively invariant.

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6. Baracs Conjecture / Maxwell-Cremona theory: the connection of stresses in planar graphs with projections of plane-faced spherical polyhedra;

Development of Baracs Conjecture

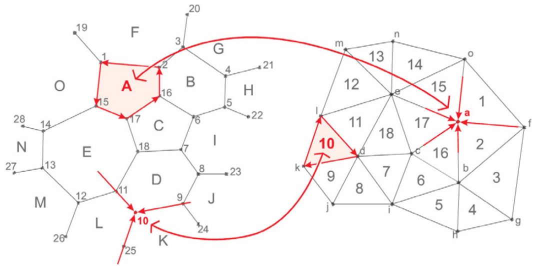

Figure : from http://complexitys.com

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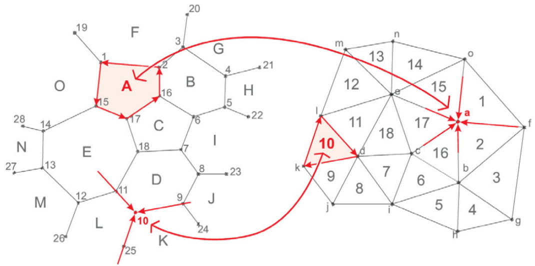

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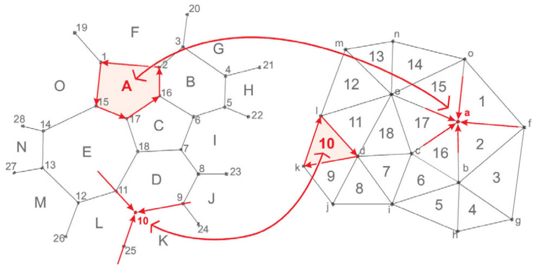

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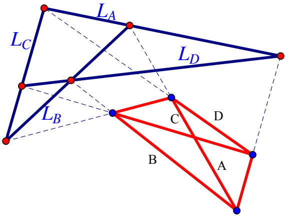
an original area of ‘diagrammatic reasoning’ (Wittgenstein, draftsmen);

Reciprocal Diagrams



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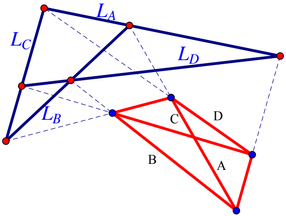

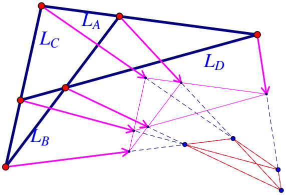

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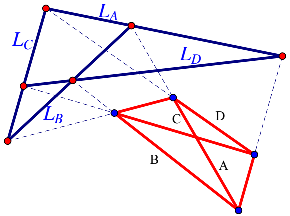

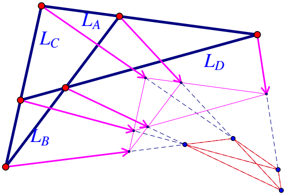

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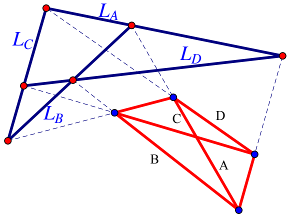

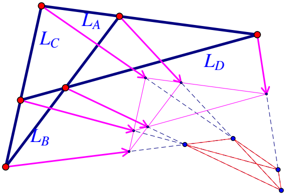

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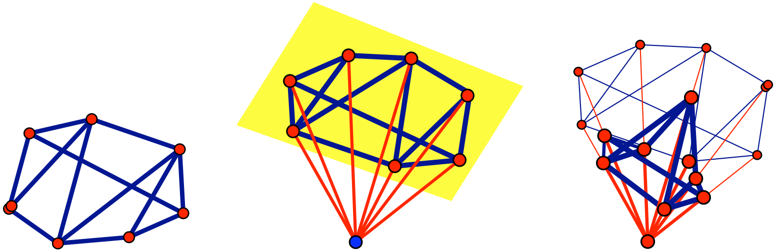
- extensions to toroidal, periodic (Voronoi , Crapo-Whiteley);
- still an active area of work in Engineering;
- 'projection and section' are basic areas in research and teaching on spatial reasoning

Further Development of a Projective Foundation

Coning as a key construction for lifting up a dimension (old folklore learned from Janos Baracs), proved in various forms.



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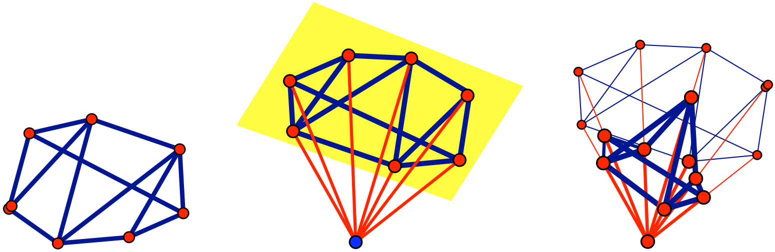

1. preserves stresses and infinitesimal motions
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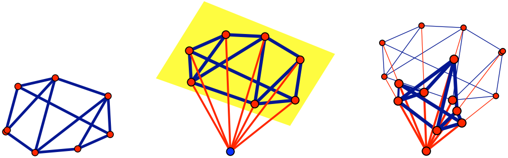
3. transfer of infinitesimal rigidity (stresses) from Euclidean space to spherical space;

More Coning



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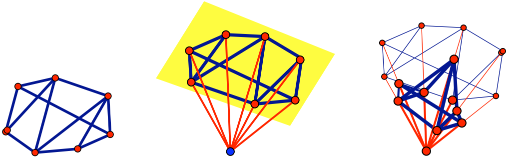

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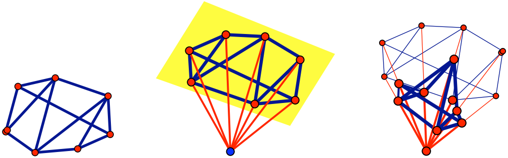

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- metric transfer extended to universal rigidity by Gortler and Thurston.

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- how does coning extend to CAD?

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Observations / partial development of many people

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Project points on the equator of the sphere?

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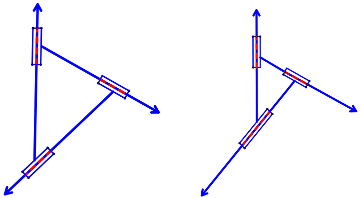

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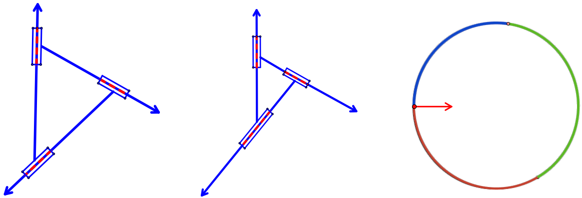

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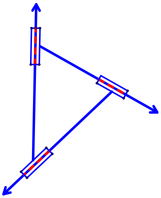
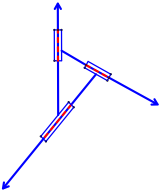
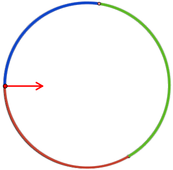
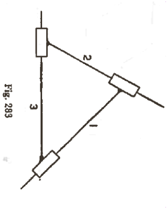

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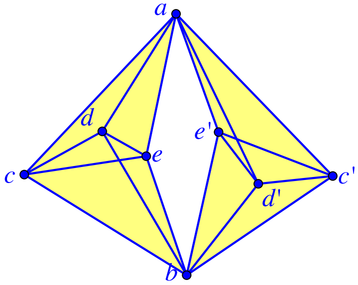
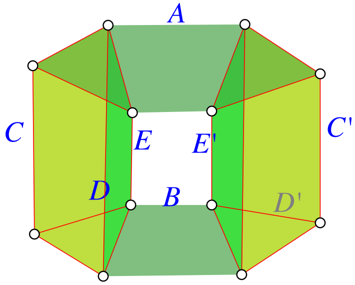
- sliders appear regularly in mechanical engineering:

Polarity

Basic projective transformation:





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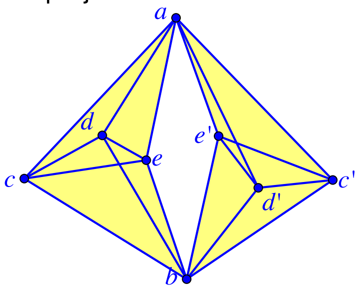

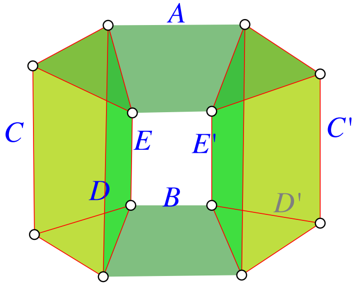

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- line joining two points to line of intersection of two planes
- ‘faces’ not really there - ‘vertices’ not really there in polar;
- point becomes sheet : plane-rigid framework on its edge segments

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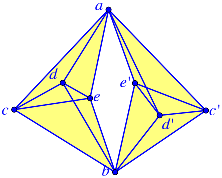

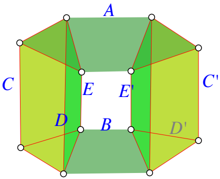

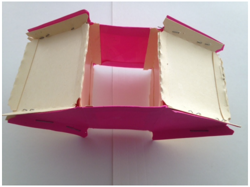

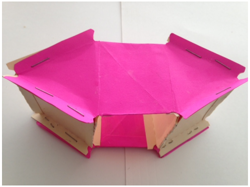

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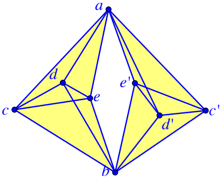

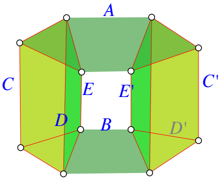

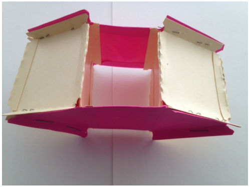

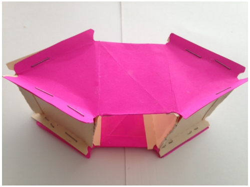

- Has rigidity transfer: polar of infinitesimal flex is infinitesimal flex
- polar of double banana clearly has (infinitesimal) motion

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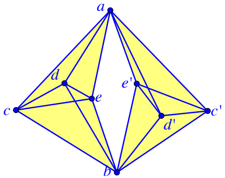

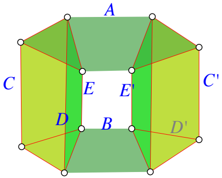

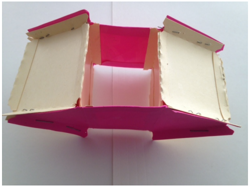

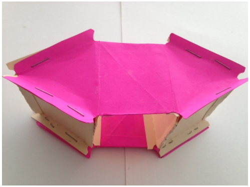

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- polar of double banana clearly has (infinitesimal) motion
- has rigidity transfer in 3-D;
- same counts, same theorems, $|E| = 3|F| - 6$ etc. ;

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Polar of self-stress is self-stress of sheet structure.

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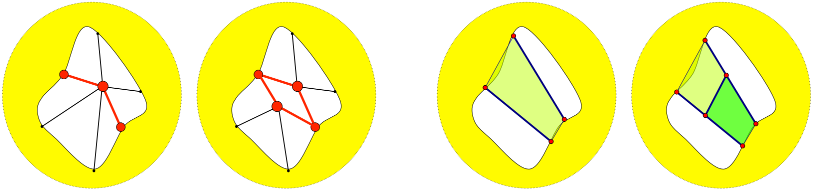

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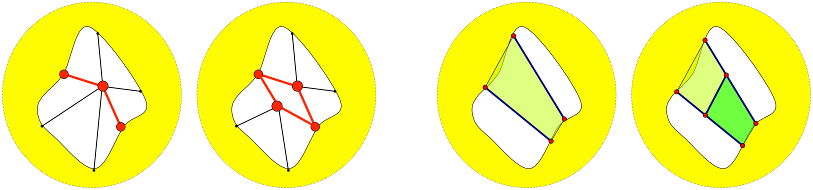

- Cauchy for triangulated convex polyhedra goes to Alexandrov (triangulated faces) for simple polyhedra

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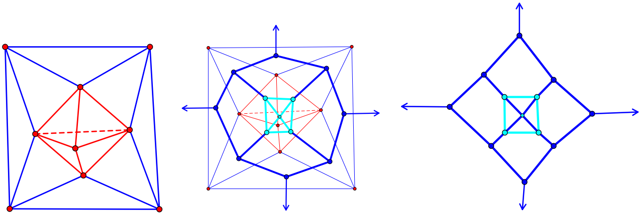

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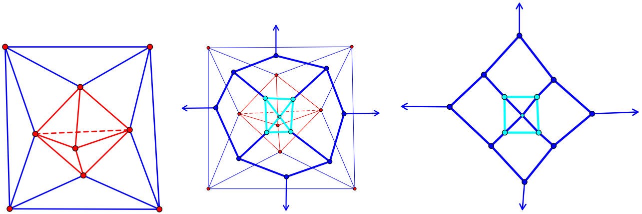
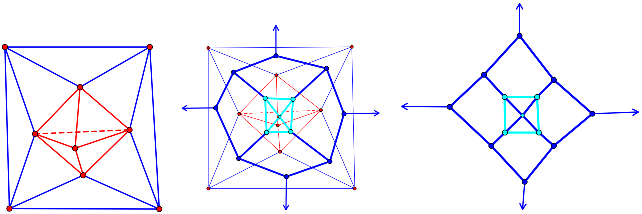


- Polar of block and hole polyhedra is block and hole sheet structure:
- block goes to block, hole to hole
- triangles go to sheets with simple vertices

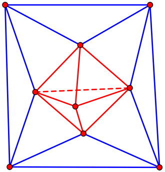
Polarity

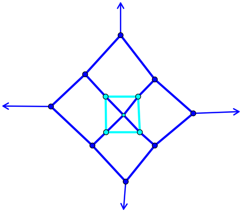
Has rigidity transfer: polar of infinitesimal flex is infinitesimal flex



- Polar of block and hole polyhedra is block and hole sheet structure:
- block goes to block, hole to hole
- triangles go to sheets with simple vertices
- how about mix of sheets (not all simple) created by further face splits?

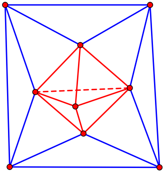
Polarity

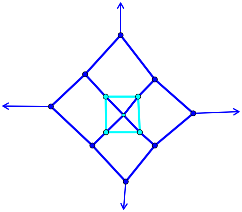




- Transfers across metrics
- extends to combined self-polar theory of sheets and points.

Polarity





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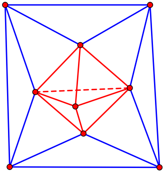

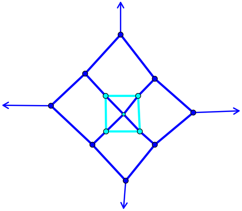

- Transfers across metrics
- extends to combined self-polar theory of sheets and points.
- Deserves to be explored beyond bar-bones publication on statics.
- There is a second polarity for sphere and hyperbolic (distances go to angles, ...) in all dimensions;
- other connections of polarity to be explored and exploited.

Concluding Remarks

- other connections of polarity to be explored and exploited.
- Tensegrity and sheet structures:

  plane theory of woven sticks

  3-D theory of slotted sheet structures

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- Geometry is one of the roots of the theory of rigidity.
- Applied geometry is a key support and application for spatial reasoning.

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  plane theory of woven sticks

  3-D theory of slotted sheet structures
- Many more geometric possibilities, insights
- Geometry is one of the roots of the theory of rigidity.
- Applied geometry is a key support and application for spatial reasoning.
- More questions, conjectures for problem session!

Thanks

Thanks

Questions?

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