The Generic Rigidity of Point-line Frameworks

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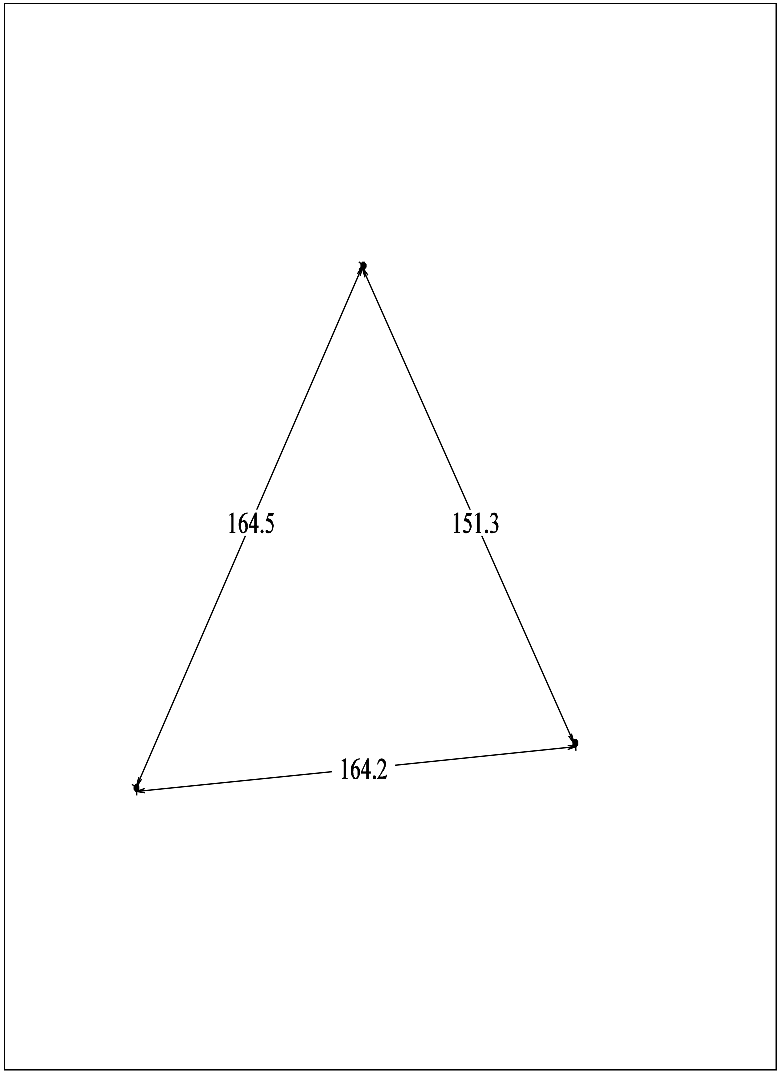
Point-line graphs and frameworks

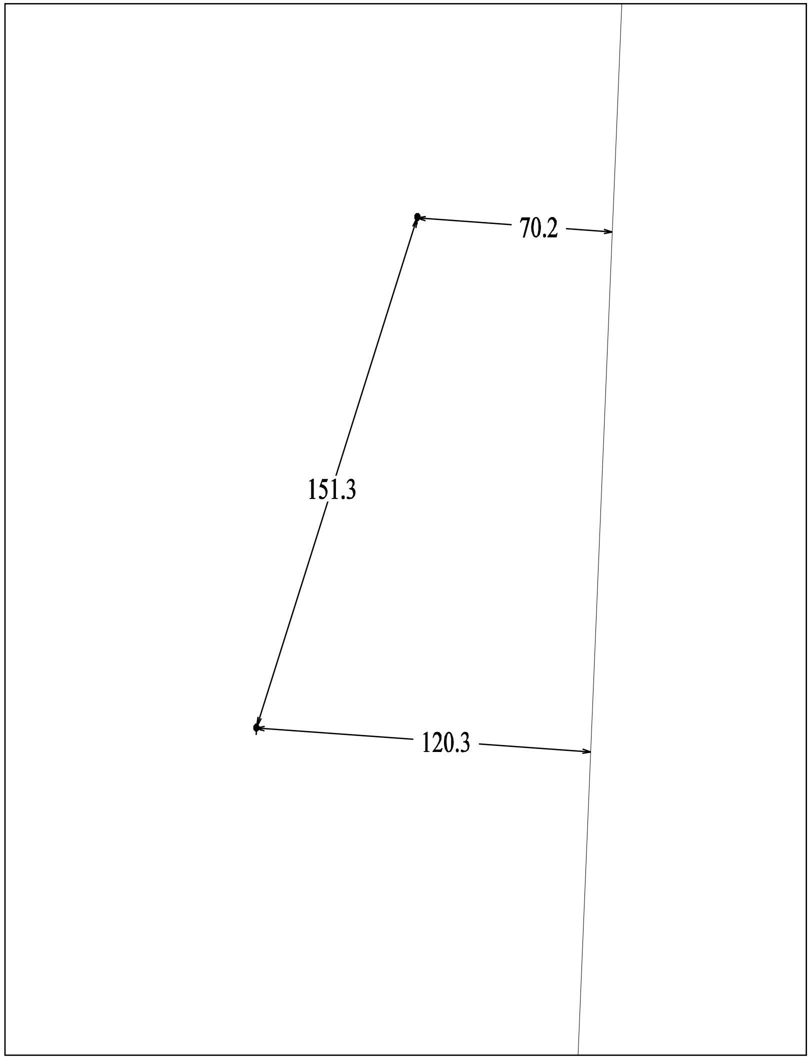
A point-line graph $G=(V,E)$ is a graph where the vertices in V are labelled as either point vertices or line vertices. $V = V_P \cup V_L$, $E=E_{PP} \cup E_{PL} \cup E_{LL}$.

A point-line framework (G,p) is a point-line graph G and an assignment p of two real numbers to each vertex. For each point-vertex these represent its two position coordinates and for each line-vertex these represent a position and a direction coordinate. (G,p) is generic if p is algebraically independent.

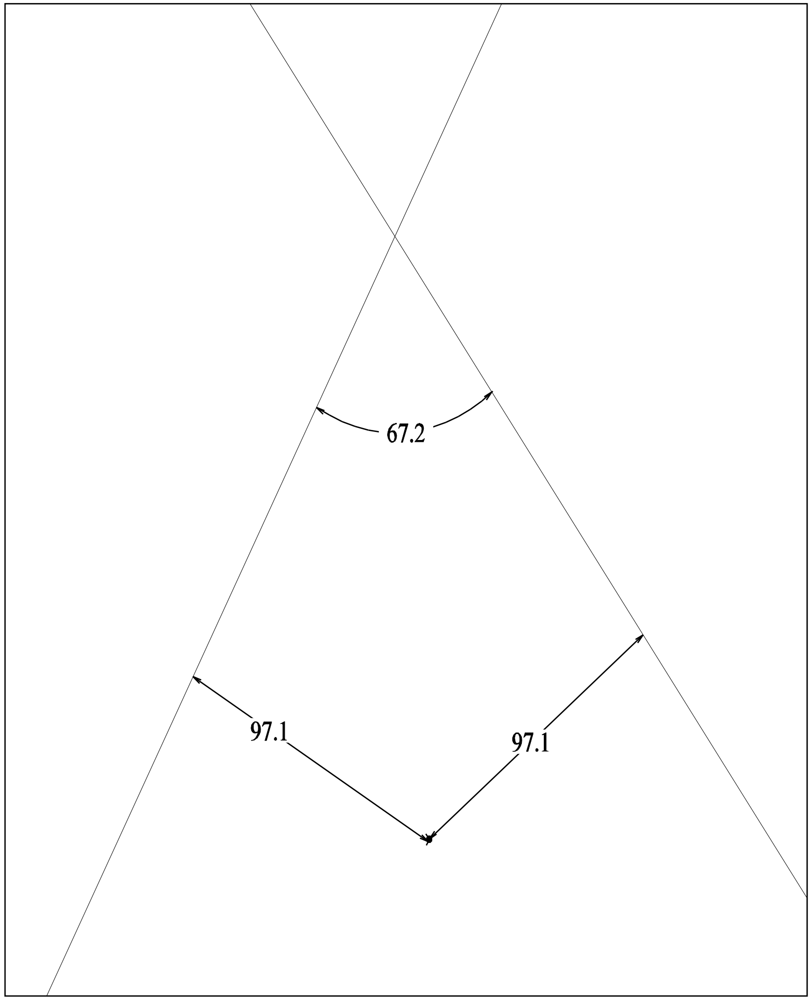
An edge $u_1u_2 \in E_{PP}$ preserves the distance between u_1 and u_2,
An edge $u_1v_2 \in E_{PL}$ preserves the distance between u_1 and v_2
An edge $v_1v_2 \in E_{LL}$ preserves the angle between v_1 and v_2.

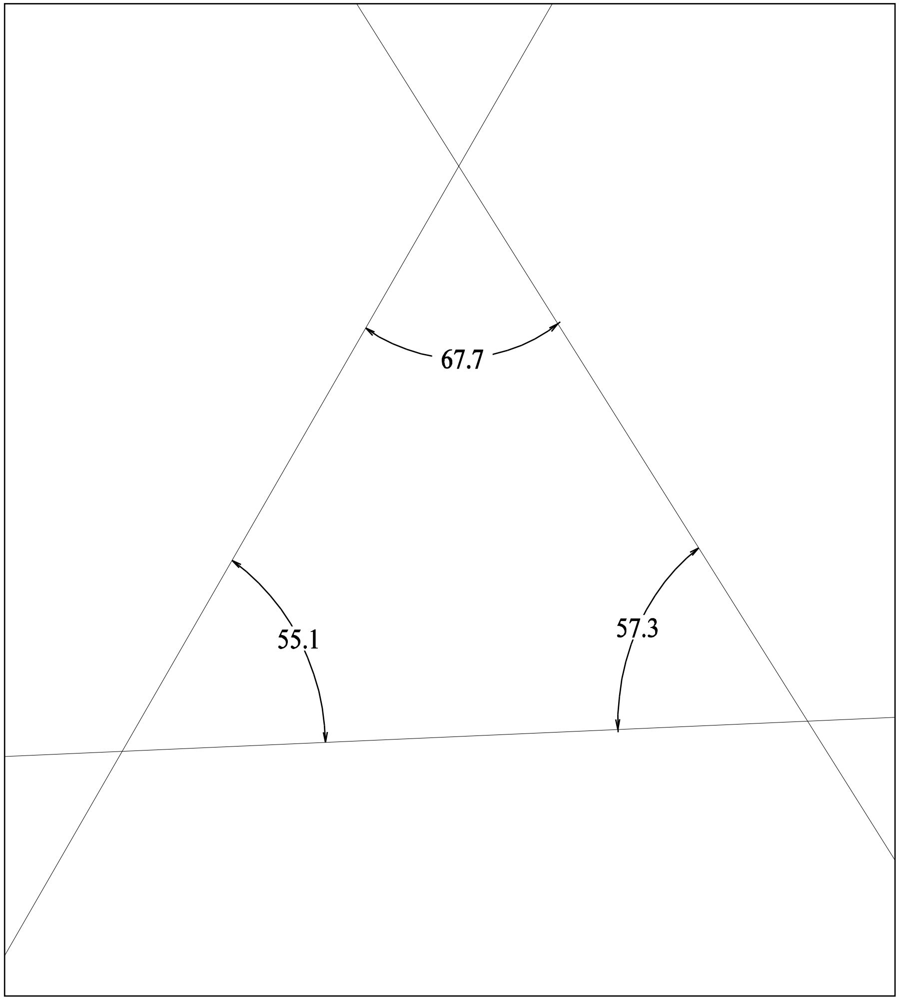
Drawing point-line frameworks



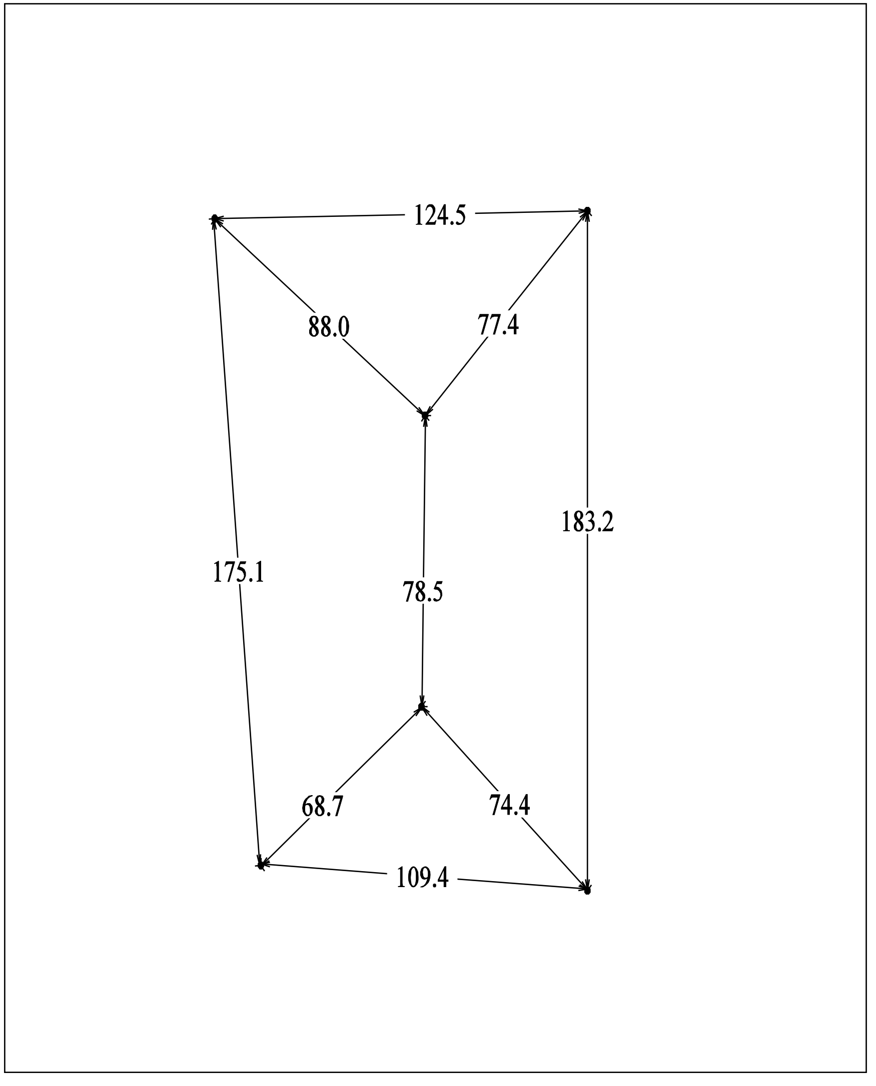


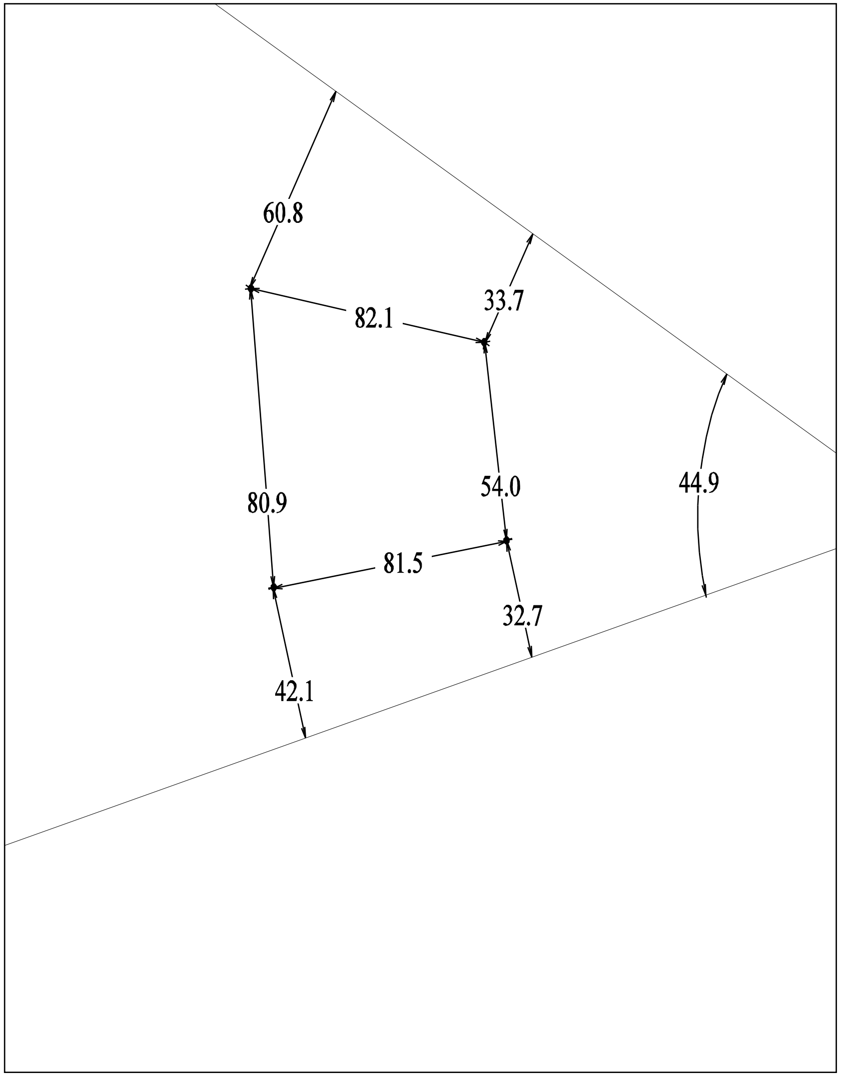
Drawing Point-line frameworks





Drawing Point-line frameworks





Independent Point-line graphs

A point-line framework (G, p) determines a rigidity matrix R(G, p). There are two types of (pairs of) columns representing point vertices and line vertices. There are three types of rows representing edges in E_{PP}, E_{PL} and E_{LL}.

A point-line graph G=(V,E) is independent if rank(R(G,p))=|E| for generic p.

A point-line graph G=(V,E) is rigid if rank(R(G, p))=2|V|-3 for generic p.

Rigidity matroid for point-line graphs

The rigidity matroid $M_{PL}(G)$ of G is the row matroid of R(G,p) for generic p.

A point-line graph G=(V,E) is dependent if, for any $H \subseteq G$ with $|E(H)|>0$,

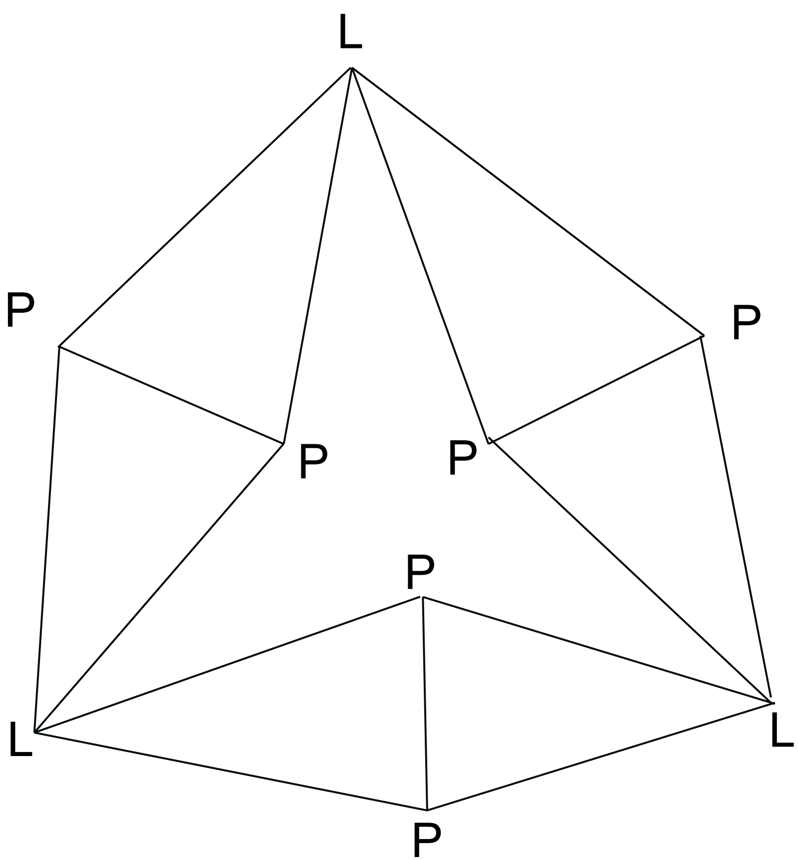
(1) $|E(H)| > 2|V(H)|-3$

(2) $|E(H)| > |V(H)|-1$ when $|V_P(H)|=0$ (or when $E(H)=E_{LL}(H)$)

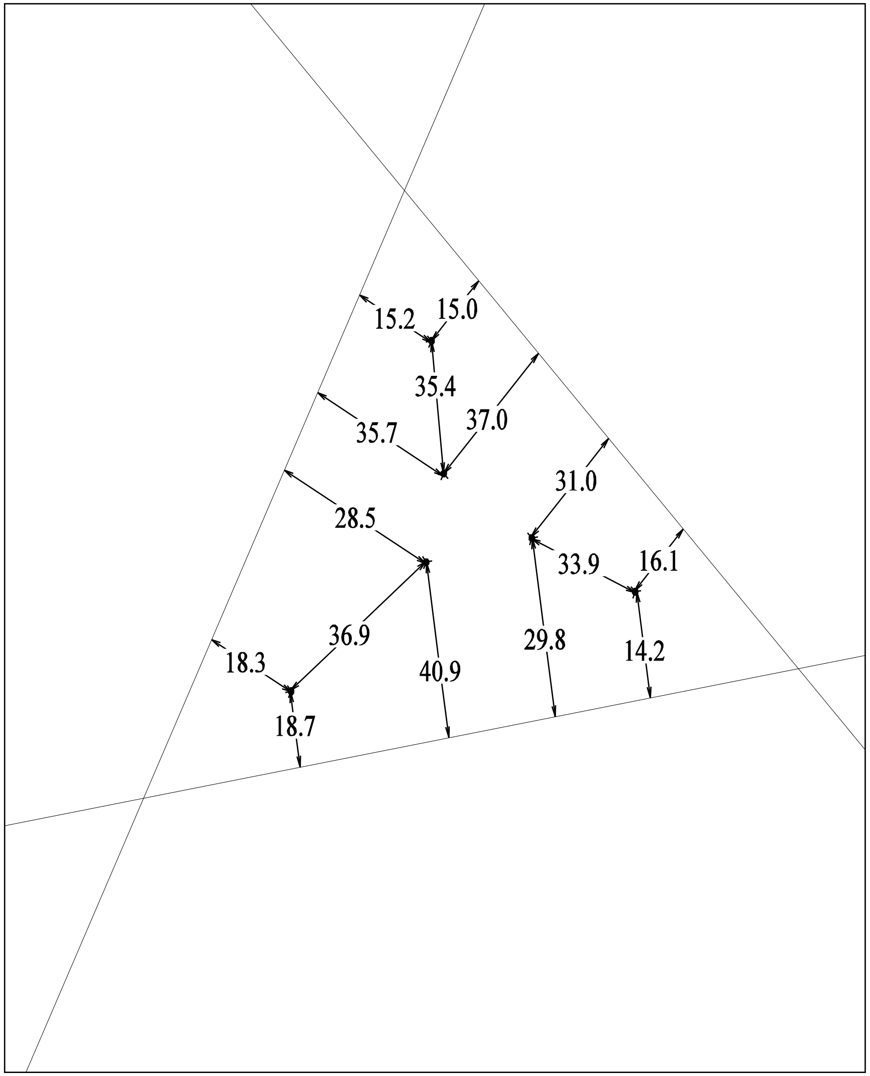
Subsets E(H) which satisfy these conditions <u>do not</u> determine the dependent sets of a matroid.

We will determine a maximal matroid $M_\#(G)$ which contains these as dependent sets, then show $M_{PL}(G)=M_\#(G)$.

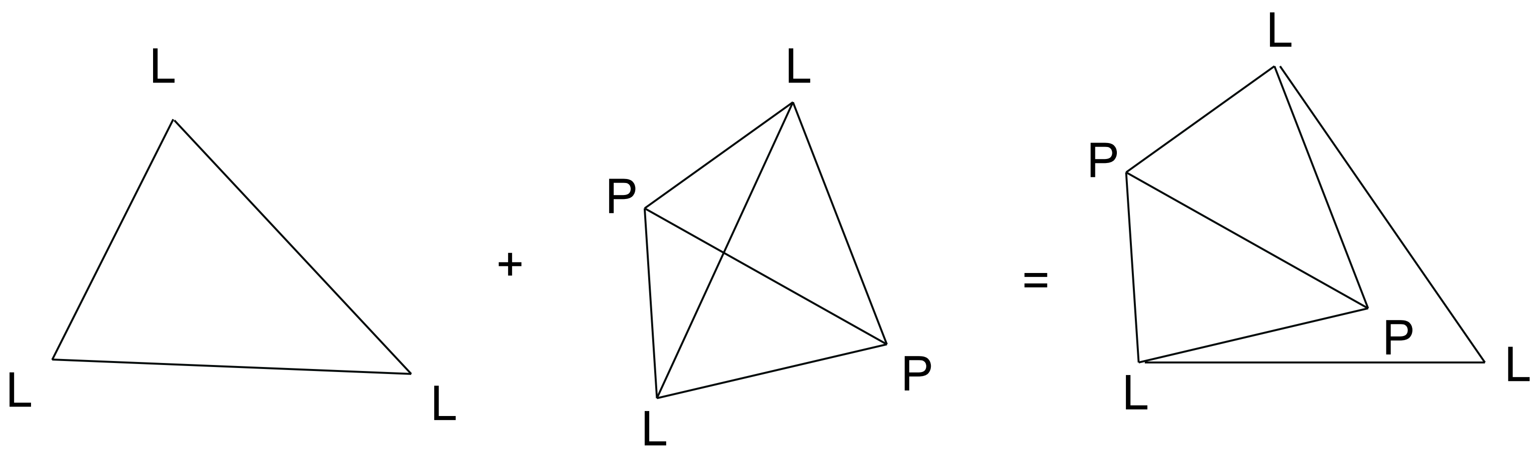
A point-line framework – is it dependent?



$|E| = 2|V|-3$ and $|E_{LL}| = 0$



it is dependent by the “circuit exchange” property



is a circuit

is a circuit

contains (is) a circuit

$M_{\#}$ definition

$H \subseteq G$ is dependent if $|E(H)| > \rho(E(H))+|V_L(H)|-1$ (1)

where $\rho(A)=\min\{\Sigma_i(2|V_P(A_i)|+|V_L(A_i)|-2\}$
and the minimum is over all partitions $\{A_1,\ldots,A_s\}$ of A.

If $|V_P(H)|=0$ then $\rho(E(H))=0$ and $|E(H)| > |V(H)|-1$.
If $|V_L(H)|=0$ then $\rho(E(H))=2|V_P(E(H))|+|V_L(E(H))|-2$ and $|E(H)| >2|V(E(H))|-3$.

Subsets satisfying (1) are the dependent sets of a matroid.

The set function $f(A) = \rho(A)+|V_L(A)|-1$ is
(a) Increasing: $f(A) \geq f(A-e)$ for e in A
(b) Positive: $f(e) > 0$
(c) Intersecting submodular:

Properties of $M_{\#}$

A count matroid M(f) has dependent sets defined by $|S| > f(S)$

Thus $M_{\#} = M(\rho+|V_L|-1)$

1. $M(\rho+|V_L|-1)$ is a Dilworth truncation of $M(\rho+|V_L|)$.
   The following are equivalent:
   (a) G is $M(\rho+|V_L|-1)$-independent.
   (b) $G+w_1w_2$ is $M(\rho+|V_L|)$-independent for all $w_1,w_2 \in E$.
   (c) $G+w_1w_2$ is $M(\rho+|V_L|)$-independent for all $w_1w_2 \in E$.

2. $M(\rho+|V_L|) = M(\rho) \vee M(|V_L|) = M(2|V_P|+|V_L|-2) \vee M(|V_L|)$.

3. $M(2|V_P|+|V_L|-2)$ is a Dilworth truncation of $M(2|V_P|+|V_L|-1)$.

4. $M(2|V_P|+|V_L|-1) = M(|V|-1) \vee M(|V_P|)$.

Thus: $M_{\#} = \mathrm{Dil}(M(2|V_P|+|V_L|-2) \vee M(|V_L|)) = \mathrm{Dil}(M(|V_L|) \vee \mathrm{Dil}(M(|V|-1) \vee |V_P|))$

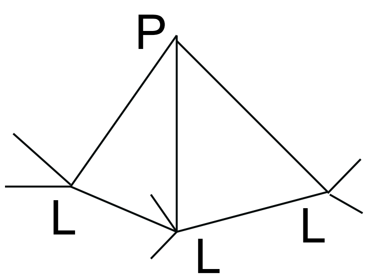
$M_{PL} = M_{\#}$

We have shown that if G is M_{PL}-independent then G is $M_{\#}$-independent

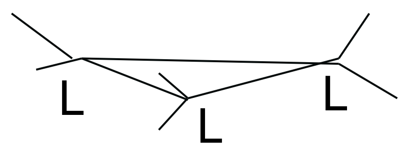
It remains to show that if G is $M_{\#}$-independent then G is M_{PL}-independent

If $|V_L(G)|=0$ Laman used a recursive construction of all independent graphs to do this.

When $|V_L(G)|>0$: there is a problem with some Henneberg 2 moves



may be independent



dependent

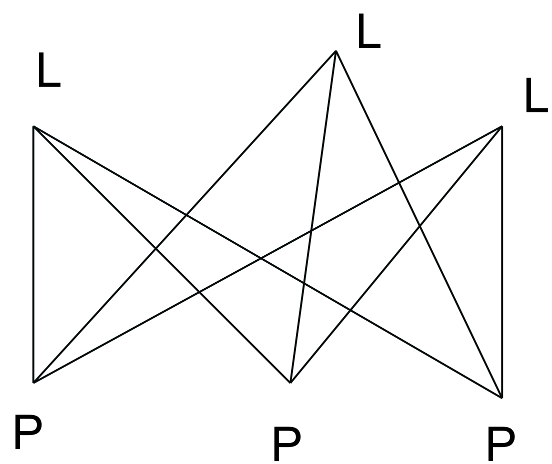
Open problem: We have not found a recursive construction of all independent graphs

Instead: For any G which is $M_{\#}$-independent we will find a (non-generic) realisation such that R(G,p) is independent.

Simplify point-line frameworks

A point-line graph is *naturally bipartite* if $E = E_{PL}$
so $|E_{PP}|=|E_{LL}|=0$.

Then R(G,p) has only one type of row.



A generic framework on naturally bipartite K(3,3) is rigid.

We can convert any point-line graph to a naturally bipartite graph by replacing edges in E_{PP} and E_{LL} by copies of naturally bipartite K(3,3)

Coordinates for points and lines

$p : V \to \mathbb{R}^{2|V|}$

$p(u_i) = (x_i, y_i)$ for $u_i \in V_P$

$p(v_j) = (a_j, b_j)$ for $v_j \in V_L$ where line is $x = -a_j y + b_j$

$f_e(p) = (x_i + y_i a_j - b_j)(1 + a_j^2)^{-1/2}$ for $e = u_i v_j \in E_{PL}$

R(G,p) has columns	u_i in V_P		v_j in V_L		
	x_i	y_i	b_j	a_j	
R(G,p) has entries	1	a_j	-1	$x_i a_j - y_i$	for all $e = u_i v_j$

All other entries in R(G,p) are zero.

Frames

following Walter Whiteley "Union of Matroids and the rigidity of Frameworks" (1988)

Let G be a naturally bipartite point-line graph

Recall $p(u_i)=(x_i,y_i)$, u_i in V_P; $p(v_j)=(a_j,b_j)$, v_j in V_L

Also define map $c : E \to \mathbb{R}^{|E|+1}$

Show $M_{PL} = Dil(M(|V_L|) \vee Dil(M(|V|-1) \vee M(|V_P|))) = M_\#$

	Matrix	columns				row matroid
		u_i in V_P		v_j in V_L		
		x_i	y_i	b_j	a_j	
	R(G,x,y,a)	1	a_j	-1	$x_ia_j-y_i$	$M_{PL}=Dil(M_A)$
	A(G,a,c)	1	a_j	-1	c_e	$M_A=M_C \vee M(\|V_L\|)$
row $e=u_iv_j$	C(G,a)	1	a_j	-1		$M_C=Dil(M_B)$
	B(G,c)	1	c_e	-1		$M_B=M(\|V\|-1) \vee M(\|V_P\|)$

all other entries zero

Algorithms for independence and rigidity

G is again any point-line graph (not necessarily bipartite)

We have shown that the following are equivalent

(a) G is M_{PL}-independent.

(b) $G+w_1w_2$ is $M(2|V_P|+|V_L|-2) \vee M(|V_L|)$-independent for all w_1w_2 □ E.

This implies: Suppose $I \subseteq E$ is independent in M_{PL}. Then I+e is independent in M_{PL} if and only if I+2e is independent in $M(2|V_P|+|V_L|-2) \vee M(|V_L|)$.

This means we can test for independence in M_{PL} by testing for independence in $M(2|V_P|+|V_L|-2) \vee M(|V_L|)$.

Matroid Union

I is independent in $M(2|V_P|+|V_L|-2) \vee M(|V_L|)$ if and only if $I=T \cup R$ and T is independent in $M(2|V_P|+|V_L|-2)$ and R is independent in $M(|V_L|)$.

If $e \in E_{PP} \cap I$ then $e \in T$

If $e \in E_{LL} \cap I$ then $e \in R$

If $e \in E_{PL} \cap I$ then $e \in T$ or $e \in R$

Matroid algorithms

We need to:

1. Partition $E(G)$ into sets T and R such that T is independent in $M(2|V_P|+|V_L|-2)$ and R is independent in $M(|V_L|)$. We use the *augmenting path* algorithm of Edmonds to generate partitions.

2. Test independence and find circuits in $M(2|V_P|+|V_L|-2)$ and $M(|V_L|)$. We use *directed graph* algorithms of Berg and Jordan.

3. Our implementation of a combined algorithm can be described as a two-colour pebble game using two types of pebbles – t-pebbles and r-pebbles.

See B. Jackson and J. C. Owen "A characterisation of the generic rigidity of 2-dimensional point-line frameworks" (2014) for details and references.

Two-colour Pebble game

For any point-line graph $G=(V,E)$ we represent partitions of $E = T \cup R$ into two sets T and R together with orientations of $G(T)$ and $G(R)$ as follows:

1. Each point vertex gets two t-pebbles and each line vertex gets one t-pebble and one r-pebble.
2. Every edge in G is assigned one pebble from one or other of its end vertices subject to the restrictions:

    an edge in E_{LL} must be assigned an r-pebble

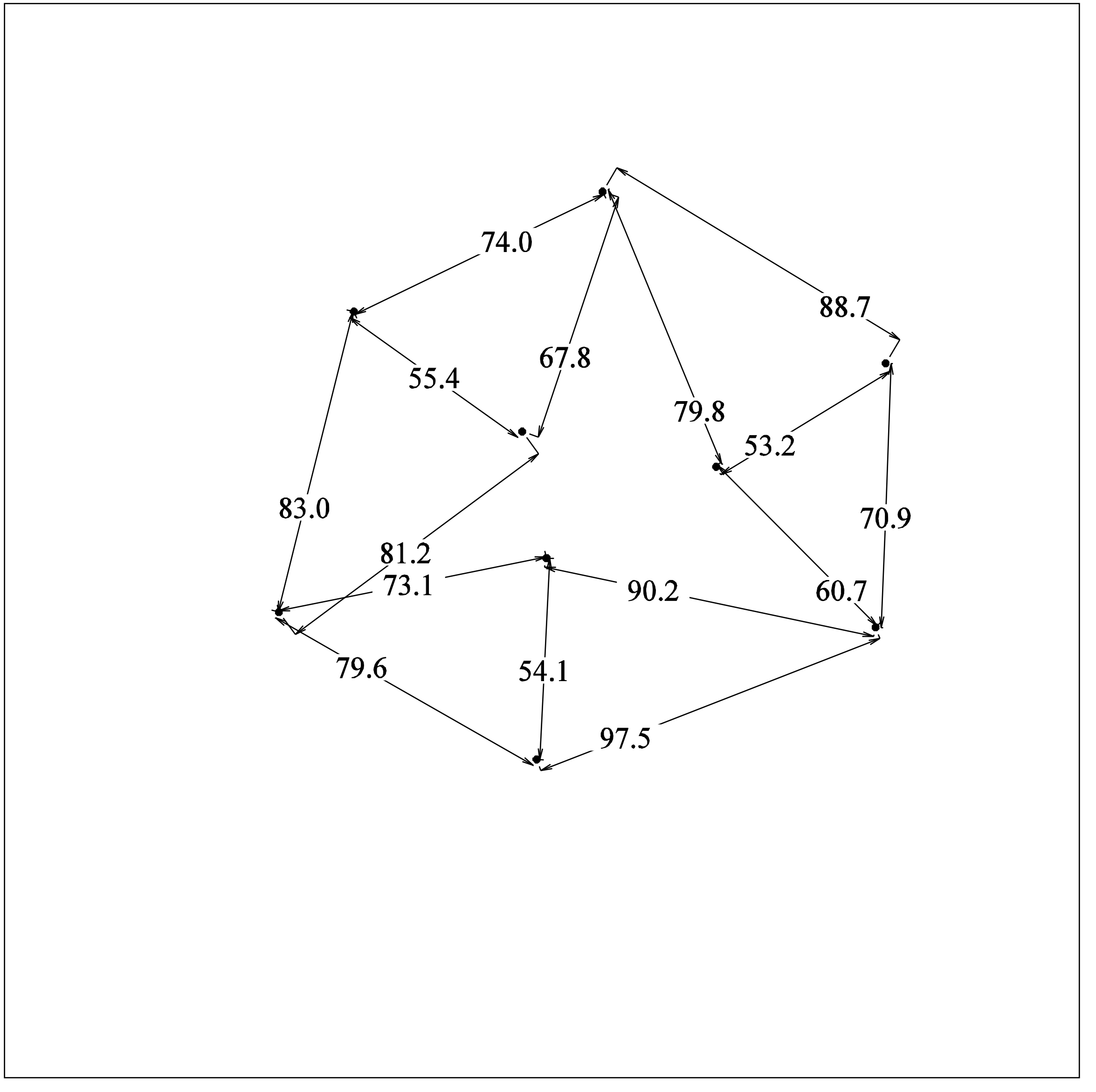
    an edge in E_{PP} must be assigned a t-pebble

    an edge in E_{PL} may be assigned either type of pebble.

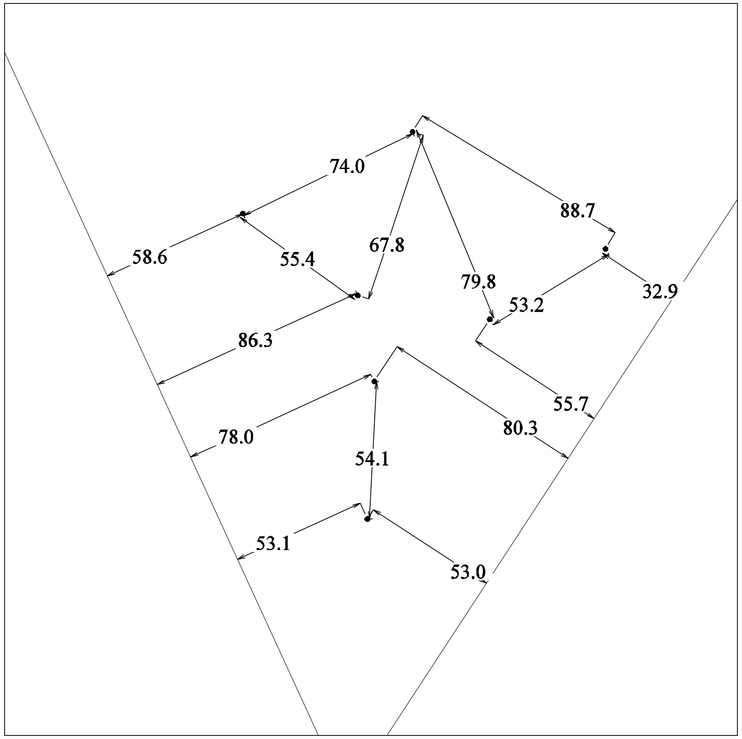
If $I = T \cup R \subseteq E$ is independent then $I+e$ is independent if $G(I+e+e')$ has an assignment of pebbles which can be derived from $G(T)$ and $G(R)$ leaving 3 free pebbles.

Demonstration

Using demo6, dem7, demo8

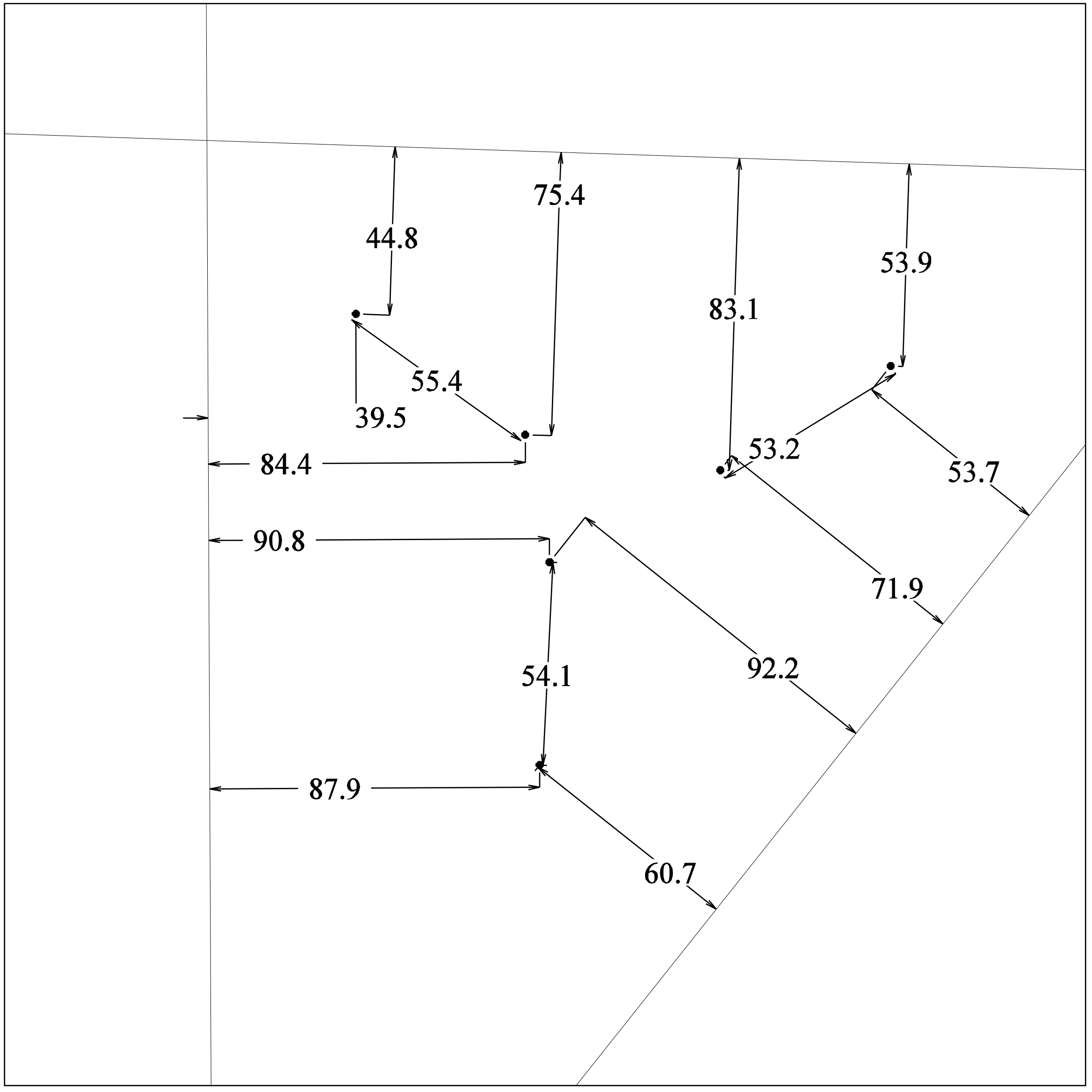


demo6



demo7.eps

demo7



demo8

Open Problems

Find a recursive construction for independent point-line graphs

Which point-line frameworks are bounded? Is this a generic property? Is continuous boundedness equivalent to boundedness?

Characterise global rigidity of point-line frameworks - is it a generic property?