Geometric rigidity in normed spaces[1]

Derek Kitson

Department of Mathematics and Statistics
Lancaster University

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Let X be a finite dimensional real vector space.

The dual space X^* is the vector space of linear functionals $f : X \to \mathbb{R}$.

Let $\|\cdot\|$ be a norm on X and let $S = \{x \in X : \|x\| = 1\}$ be the unit sphere.

A support functional for a point $x_0 \in S$ is a linear functional $f \in X^*$ such that $|f(x)| \leq 1$ for all $x \in S$ and $f(x_0) = 1$.

The norm is smooth at a point $x_0 \in S$ if there exists a unique support functional for x_0.

Let G be a finite simple graph.

A bar-joint framework is a pair (G, p) with $p = (p_v)_{v \in V} \in X^{|V|}$ a placement of the vertices of G in X.

An edge vw is well-positioned if the norm on X is smooth at the point $\frac{p_v - p_w}{\|p_v - p_w\|} \in S$.

The unique support functional at $\frac{p_v - p_w}{\|p_v - p_w\|}$ is denoted $\varphi_{v,w}$.

The bar-joint framework (G, p) is well-positioned if every edge is well-positioned.

Let (G, p) be a well-positioned bar-joint framework.

The rigidity matrix $R(G, p)$ is an $|E| \times |V|$ matrix with entries in the dual space X^* given by,

$$a_{(e,v)} = \begin{cases} \varphi_{v,w} & \text{if } e = vw \text{ for some vertex } w, \\ 0 & \text{otherwise} \end{cases}$$

for all $(e, v) \in E \times V$.

The rigidity map for G is,

$$f_G : X^{|V|} \to \mathbb{R}^{|E|}, \qquad x = (x_v)_{v\in V} \mapsto (\|x_v - x_w\|)_{vw\in E}$$

The configuration space for (G, p) is,

$$\mathcal{C}(G, p) = \{x \in X^{|V|} : f_G(x) = f_G(p)\}$$

An infinitesimal flex of (G, p) is a vector $u \in X^{|V|}$ such that,

$$D_u f_G(p) = 0.$$

A rigid motion of $(X, \|\cdot\|)$ is a collection of continuous paths

$$\gamma_x : [-1,1] \to X, \qquad x \in X$$

such that for all $x, y \in X$

1. $\gamma_x(0) = x$,
2. γ_x is differentiable at 0,
3. $\|\gamma_x(t) - \gamma_y(t)\| = \|x - y\|$ for all t.

An infinitesimal flex u is trivial if it is derived from a rigid motion,

$$u_v = \gamma'_{p_v}(0) \qquad \text{for all } v \in V.$$

A bar-joint framework (G, p) in $(X, \|\cdot\|)$ is

- infinitesimally rigid if every infinitesimal flex of (G, p) is trivial.
- isostatic if
  1. it is infinitesimally rigid, and,
  2. removing any bar will result in a framework which is not infinitesimally rigid.

A bar-joint framework (G, p) is Γ-symmetric if there exists

(i) a group action $\theta : \Gamma \to \mathrm{Aut}(G)$, and,

(ii) an isometry-valued representation $\tau : \Gamma \to \mathrm{GL}(X)$ such that $\tau(\gamma)(p_v) = p_{\theta(\gamma)v}$ for all $\gamma \in \Gamma$ and all $v \in V$.

Define external and internal permutation representations,

$$P_V : \Gamma \to \mathrm{GL}(\mathbb{R}^{|V|}), \qquad P_V(\gamma)(a_v)_{v\in V} = (a_{\theta(\gamma)^{-1}v})_{v\in V}$$

$$P_E : \Gamma \to \mathrm{GL}(\mathbb{R}^{|E|}), \qquad P_E(\gamma)(a_e)_{e\in E} = (a_{\theta(\gamma)^{-1}e})_{e\in E}$$

The sets of vertices and edges which are fixed by $\gamma \in \Gamma$ are denoted by V_γ and E_γ, respectively.

Theorem (DK - B Schulze, 2014)

If (G, p) *is well-positioned and* Γ*-symmetric then,*

(i) $df_G(p) \in \mathrm{Hom}_\Gamma(\tau \otimes P_V, P_E)$.

(ii) $\mathcal{T}(G, p)$ *is* $\tau \otimes P_V$*-invariant.*

If (G, p) *is also isostatic then,*

(i) $\chi(P_E) = \chi(\tau \otimes P_V) - \chi((\tau \otimes P_V)^{(\mathcal{T})})$.

(ii) $|E_\gamma| = \mathrm{tr}(\tau(\gamma))\,|V_\gamma| - \mathrm{tr}((\tau \otimes P_V)^{(\mathcal{T})}(\gamma))$.

If the group of linear isometries of $(X, \|\cdot\|)$ *is finite then,*

(i) $|E| = \dim(X)\,(|V| - 1)$, *and,*

(ii) $|E_\gamma| = \mathrm{tr}(\tau(\gamma))\,(|V_\gamma| - 1)$ *for each* $\gamma \in \Gamma$.

A symmetry operation $\gamma \in \Gamma$ is called

1. an inversion if $\tau(\gamma) = -I$.
2. a reflection if $\tau(\gamma) = I - 2P$ where P is a rank one projection on X.
3. an n-fold rotation if there exists a two-dimensional subspace Y of X with a complementary space Z such that $\tau(\gamma) = S \oplus I_Z$ where $S : Y \to Y$ has matrix representation

$$\begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$$

with respect to some basis for Y.

Corollary

Suppose (G,p) is well-positioned, isostatic, Γ-symmetric and $\gamma \in \Gamma$ is a reflection.

(i) $|E_\gamma| = (\dim(X) - 2)\,(|V_\gamma| - 1)$.

(ii) *If* $\dim(X) = 2$ *then* $|E_\gamma| = 0$.

(iii) *If* $\dim(X) \geq 3$ *then the following two conditions hold,*

  (a) $|V_\gamma| \geq 1$, *and,*

  (b) $|V_\gamma| = 1$ *if and only if* $|E_\gamma| = 0$.

Corollary

Suppose (G, p) is well-positioned, isostatic, Γ-symmetric and $\gamma \in \Gamma$ is a half-turn rotation.

(i) $|E_\gamma| = (\dim(X) - 4)\ (|V_\gamma| - 1)$.

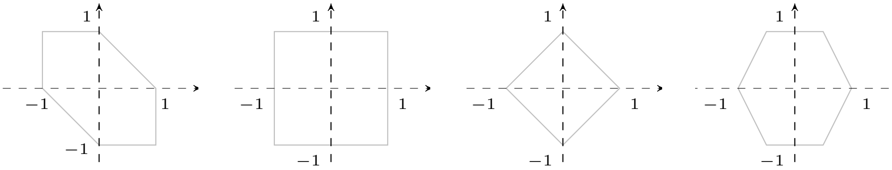
(ii) *If* $\dim(X) = 2$, *then one of the following conditions holds,*
   (a) $|V_\gamma| = 0$, *and,* $|E_\gamma| = 2$.
   (b) $|V_\gamma| = 1$, *and,* $|E_\gamma| = 0$.

(iii) *If* $\dim(X) = 3$, *then one of the following conditions holds,*
   (a) $|V_\gamma| = 0$, *and,* $|E_\gamma| = 1$.
   (b) $|V_\gamma| = 1$, *and,* $|E_\gamma| = 0$.

(iv) *If* $\dim(X) = 4$, *then* $|E_\gamma| = 0$.

(v) *If* $\dim(X) \geq 5$, *then the following conditions hold,*
   (a) $|V_\gamma| \geq 1$, *and,*
   (b) $|V_\gamma| = 1$ *if and only if* $|E_\gamma| = 0$.

Polyhedral norms



Let $\mathcal{P}$ be a convex symmetric d-dimensional polytope in $\mathbb{R}^d$.

The Minkowski functional for $\mathcal{P}$ defines a norm on $\mathbb{R}^d$,

$$\|x\|_{\mathcal{P}} = \inf\{\lambda \geq 0 : x \in \lambda\mathcal{P}\}$$

and is characterised by,

$$\|x\|_{\mathcal{P}} = \|x\|_{\mathcal{P}}^{**} = \|x\|_{\mathcal{P}^\triangle}^{*} = \max_{y \in \mathcal{P}^\triangle} x \cdot y = \max_{y \in ext(\mathcal{P}^\triangle)} x \cdot y$$

Let $\tau : \Gamma \to \mathrm{GL}(\mathbb{R}^d)$ be an isometry-valued group rep.

A group element $\gamma \in \Gamma$ preserves the facets of $\mathcal{P}$ if $\tau(\gamma)F \in \{F, -F\}$ for each facet F of $\mathcal{P}$.

Theorem (DK, 2013)

If (G, p) is well-positioned in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ then the following statements are equivalent.

(i) *(G, p) is continuously rigid.*
(ii) *(G, p) is infinitesimally rigid.*

Theorem (DK, 2013)

Let G be a finite simple graph and let $\|\cdot\|_{\mathcal{P}}$ be a polyhedral norm on $\mathbb{R}^2$. The following statements are equivalent.

(i) *There exists p such that (G, p) is well-positioned and isostatic in $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$.*
(ii) *G is $(2, 2)$-tight.*

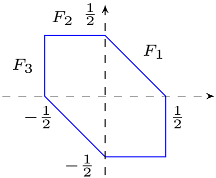
For each edge $vw \in E(G)$ define the set of framework colours,

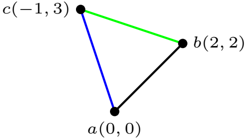
$$\Phi(vw) = \{\{F, -F\} : p_v - p_w \in \text{cone}(F) \cup \text{cone}(-F)\}$$

Let G_F denote the monochrome subgraph of G spanned by edges with framework colour $[F] = \{F, -F\}$.

Denote by $\Phi(G, p)$ the set of all framework colours of (G, p),

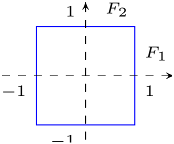
$$\Phi(G, p) = \bigcup_{vw \in E(G)} \Phi(vw)$$

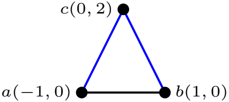




Lemma

Let (G, p) be well-positioned and Γ-symmetric in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$. If each symmetry operation $\gamma \in \Gamma$ preserves the facets of $\mathcal{P}$ then the monochrome subgraphs of G are Γ-symmetric.





$$\begin{array}{c} \\ ab \\ bc \\ ac \end{array} \begin{array}{c} \begin{array}{cccccc} a_x & a_y & b_x & b_y & c_x & c_y \end{array} \\ \left(\begin{array}{cc|cc|cc} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{array}\right) \end{array}$$

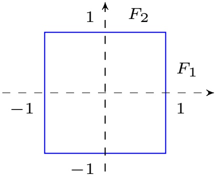
Polyhedral norms

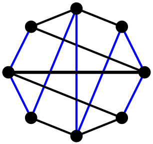
Theorem (DK - S Power, BLMS 2014)

If (G,p) is well-positioned in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ and $|\Phi(G,p)| = d$ then the following statements are equivalent.

(i) *(G,p) is isostatic.*

(ii) *G_F is a spanning tree in G for each $[F] \in \Phi(G,p)$.*





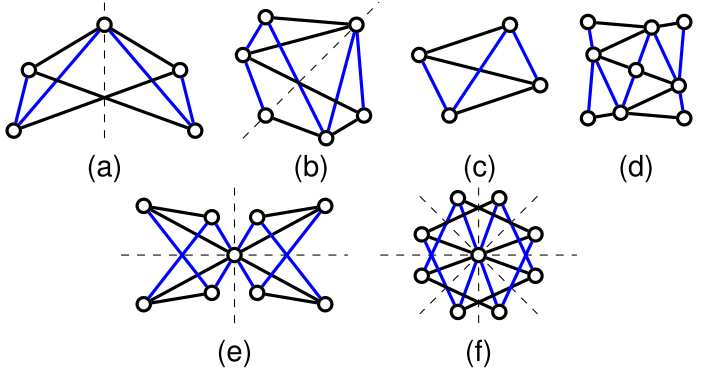


Figure: Symmetric isostatic bar-joint frameworks in $(\mathbb{R}^2, \|\cdot\|_\infty)$: (a), (b) $\mathcal{C}_s$-symmetry; (c) $\mathcal{C}_2$-symmetry; (d) $\mathcal{C}_4$-symmetry; (e) $\mathcal{C}_{2v}$-symmetry; (f) $\mathcal{C}_{4v}$-symmetry.

Symmetric tree packings

Theorem (DK - B Schulze, 2014)

Let $\|\cdot\|_{\mathcal{P}}$ be a polyhedral norm on $\mathbb{R}^2$ for which the unit ball $\mathcal{P}$ is a quadrilateral and let G be a finite simple graph with a group action $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(G)$ where $\mathbb{Z}_2 = \langle s \rangle$.
The following statements are equivalent.

(i) *There exists a representation $\tau : \mathbb{Z}_2 \to \mathrm{GL}(\mathbb{R}^2)$ and a point p such that the bar-joint framework (G, p) is well-positioned and isostatic in $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$ and $\mathcal{C}_s$-symmetric with respect to θ and τ, where the symmetry operation s is a reflection which preserves the facets of $\mathcal{P}$.*

(ii) *G is expressible as a union of two edge-disjoint spanning trees, both of which are $\mathbb{Z}_2$-symmetric with respect to θ, and no edge of G is fixed by s.*

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(ii) *G is expressible as a union of two edge-disjoint spanning trees, both of which are $\mathbb{Z}_2$-symmetric with respect to θ, and either no edge or two edges of G are fixed by s.*

Let $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(G)$ be an action of $\mathbb{Z}_2 = \langle s \rangle$ on G.

The pair (G, θ) is admissible if,

(i) no edge of G is fixed by s, and,

(ii) G is expressible as a union of two edge-disjoint spanning trees, both of which are $\mathbb{Z}_2$-symmetric with respect to θ.

Lemma

If (G, θ) is an admissible pair then,

(i) *there exists exactly one vertex v_0 in G which is fixed by s.*

(ii) *the unique fixed vertex v_0 has even degree and degree at least 4.*

Theorem (DK - B Schulze, 2014)

If (G,θ) is admissible then there exists a construction chain,

$$W_5 = G^1 \to G^2 \to \cdots \to G^n = G,$$

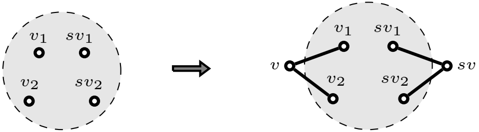
such that for each $k = 1, 2, \ldots, n-1$,

(i) *(G^k, θ) is admissible, and,*

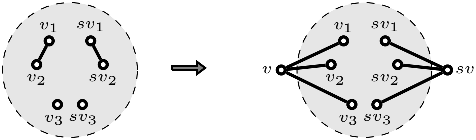
(ii) *$G^k \to G^{k+1}$ is one of six allowable $(\mathbb{Z}_2, \theta)$ graph extensions.*

Allowable graph extensions

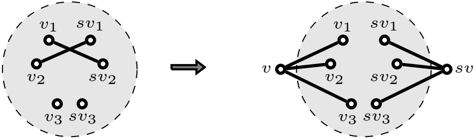
1. 1-extension.



2. 2-extension.

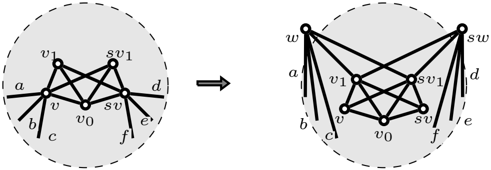


3. Modified 2-extension.

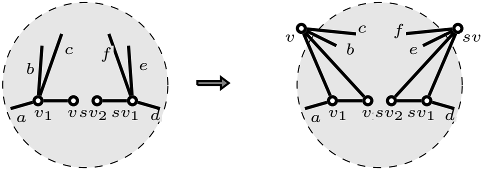


Allowable graph extensions

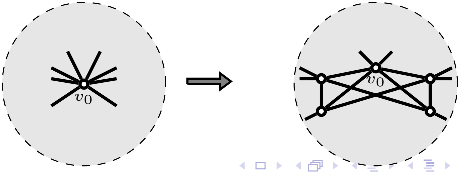
4. Wheel extension.



5. Vertex split.



6. Fixed-vertex-to-W_5.



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Thank you