A Selective “Modern” History of the Boltzmann and Related Equations

Reinhard Illner, Victoria

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Synopsis

1. I have an ambivalent relation to surveys!
2. Key Words, Tools, People
3. Powerful Tools, I: Potentials for Interaction
4. An entertaining digression: The Digits of Π
5. Powerful Tools, II: Velocity Averaging
6. Powerful Tools, III: Functionals
7. % Powerful Tools, IV: Metrics on measures
8. Rest of the Digression
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- velocity averaging
- functionals
- metrics on measures, with applications.

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Cabannes (2), Toscani (2,6), Bobylev (1,2,4,6), DiPerna, Lions (1), Golse, Perthame, Degond, Wennberg (1,2,4,5,6) Desvillettes, Villani, Carrillo (1,5,6) Levermore (1,4,5), Gamba (3,4,5,6), St. Raymond (4). Morimoto, Ukai, Yang (7).

If I have not listed (forgotten) you or one of your friends, forgive me...
A List of Tools

- BBGKY & Boltzmann hierarchies (Bogolyubov, Cercignani, Lanford)
- Perturbation Series as solutions (control of the hierarchies)
- Free Flow domination for rare clouds (I, Shinbrot)
- Velocity Averaging & renormalization (DiPerna, Lions)
- Potentials for Interaction (Varadhan, Bony, Beale for DVMs)
- Regularization by the collision operator (Yang, Morimoto, Ukai)
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A Selective “Modern” History of the Boltzmann and Related Equations
An Example: Discrete Velocity Models in 1 Dimension

Equations:

\[ u_i, t + c_i u_i, x = \sum_{j,k} A_{jk} i u_j u_k =: F_i \]

Potential for interaction gives uniform global control of

\[ \int_0^t \int u_i u_j \, dx \, dt. \]

This, combined with some other (older) tricks, produces global uniform boundedness and the existence of wave operators (in the absence of boundaries).

All we need is

\[ \sum F_i = 0 = \sum c_i F_i \]

(mass and momentum conservation). Then the following fantastic calculation works:
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All we need is \( \sum F_i = 0 = \sum c_i F_i \) (mass and momentum conservation). Then the following fantastic calculation works:
Assume $c_i \neq c_j$ if $i \neq j$.
Let
\[
I(t) = \sum_{i,j} \int_y \int_{x < y} (c_i - c_j) u_i(x) u_j(y) \, dx \, dy.
\]
Assume $c_i \neq c_j$ if $i \neq j$.

Let

$$I(t) = \sum_{i,j} \int_y \int_{x < y} (c_i - c_j)u_i(x)u_j(y) \, dx \, dy.$$ 

Note: $I(t)$ is bounded by mass conservation! One computes

$$\frac{dl}{dt} = \sum_{i,j} \int_y \int_{x < y} (c_i - c_j)[F_i(y)u_j(x) + u_i(y)F_j(x)] \, dx \, dy$$

sum to 0, by conservations

$$+ \int \int_{-\infty}^{y} (c_i - c_j)(-c_iu_{i,x})u_j(y) \, dx \, dy$$

$$+ \int \int_{x}^{\infty} (c_i - c_j)u_i(x)(-c_ju_{j,y}) \, dy \, dx$$

Do the inner integrals, collect terms....
So,

\[ I(t) = \sum_{i,j} \int_y \int_{x < y} (c_i - c_j) u_i(x) u_j(y) \, dx \, dy \]

gives

\[ \frac{dl}{dt} = -\sum_{ij} \int (c_i - c_j)^2 u_i(x) u_j(x) \, dx, \]
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gives

\[ \frac{dI}{dt} = -\sum_{ij} \int (c_i - c_j)^2 u_i(x) u_j(x) \, dx, \]

or

\[ I(t) - I(0) = -\int_0^t \int \sum_{ij} (c_i - c_j)^2 u_i(x) u_j(x) \, dx \, dt, \]

and \( |I(t)| \leq C (\text{mass})^2 \),

so \( \int_0^t \int u_i u_j \, dx \, dt \leq C m^2 \).
These estimates were then used by Bony and Beale to prove global boundedness of solutions (using a trick pioneered by Crandall and Tartar 40 years ago).
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masses $m_i > 0$, radii $d_i > 0$, $i = 1 \ldots N$
Positions $x_i(t) \in \mathbb{R}^3$, velocities $v_i(t) \in \mathbb{R}^3$. 
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**ingoing** collision configuration

\[
x_j = x_i + (d_i + d_j)n,
\]

where \( n \in S^2 \) is such that

\[
n \cdot (v_i - v_j) > 0 \\
= 0 \text{ (grazing)} \\
< 0 \text{ (outgoing)}
\]
The post-collisional velocities $v'_i, v'_j$ are computed from

a) momentum transfer in direction $n$

\[
\begin{align*}
  m_i v'_i + m_j v'_j &= m_i v_i + m_j v_j \quad \text{(momentum conservation)} \\
  m_i (v'_i)^2 + m_j (v'_j)^2 &= m_i v_i^2 + m_j v_j^2 \quad \text{(energy conservation)}
\end{align*}
\]

This defines the collision transformation $J$: $(v_i, v_j) \rightarrow (v'_i, v'_j)$.
The post-collisional velocities $v_i', v_j'$ are computed from:

a) momentum transfer in direction $n$:

$$b) \quad m_i v_i' + m_j v_j' = m_i v_i + m_j v_j$$

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The post-collisional velocities $v'_i, v'_j$ are computed from
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\[ c) \quad m_i (v_i')^2 + m_j (v_j')^2 = m_i v_i^2 + m_j v_j^2 \]
(energy conservation) \implies

\[
\begin{align*}
  v_i' &= v_i - \frac{2m_j}{m_i+m_j} (n \cdot (v_i - v_j)) n \\
  v_j' &= v_j + \frac{2m_i}{m_i+m_j} (n \cdot (v_i - v_j)) n
\end{align*}
\]

This defines the collision transformation $J : (v_i, v_j) \rightarrow (v_i', v_j')$. 
Abbreviate $x = (x_1, \ldots, x_N) \in \mathbb{R}^{3N}$, $v = (v_1, \ldots, v_N) \in \mathbb{R}^{3N}$.

Define, in $\mathbb{R}^{3N}$,

$$\langle x, y \rangle_m = \sum_{i=1}^{N} m_i \langle x_i, y_i \rangle.$$ 

This is a useful inner product, for example, we have

$$\langle v(t), v(t) \rangle_m = \langle v(0), v(0) \rangle_m.$$ 

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If \( t \) is a collision instant, write \( v^-(t) \) (ingoing) and \( v^+(t) \) (outgoing). We will also write \( x^0(t) = x(0) + tv(0) \) (free flow).
Assume \( v(0) \neq 0 \). Define

\[
u(t) := \frac{v(t)}{\|v(t)\|_m}, \quad e(t) := \frac{x(t)}{\|x(t)\|_m} \in S^{3N-1}.
\]

**Theorem.** There is \( e \in S^{3N-1} : \lim_{t \to \infty} e(t) = e = \lim_{t \to \infty} u(t) \). The product \( \langle u(t), e(t) \rangle_m \) is monotonically increasing to 1 (a potential for interaction; when it is equal to 1, there can be no more collisions).
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Proof.
STEP 1. By explicit calculation, if there are no collisions in \([t_1, t_2),\]
we have for \(t\) in that interval
\[
\langle e(t), u(t) \rangle_m = \langle e(t), u(t_1) \rangle_m \leq \langle e(t_1), u(t_1) \rangle_m. \]
Geometric meaning... picture:
A family of nested cone sections

STEP 2. Let $C(e(t)) := \{ u \in S^{3N-1}; \langle u, e(t) \rangle_m \geq \langle u(t), e(t) \rangle_m \}$. This is a cone section.

**Lemma.** If $t_2 \geq t_1$ then $C(e(t_2)) \subset C(e(t_1))$. 

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A family of nested cone sections

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**Lemma.** If $t_2 \geq t_1$ then $C(e(t_2)) \subset C(e(t_1))$.

Proof: If there are no collisions between $t_1$ and $t_2$ then this follows from the calculation in STEP 1. Revisit the picture! If there is a collision at a time $t_2$, one computes (this is where the *ingoing* configuration ($n \cdot (v_i^- - v_j^-) > 0$) property enters!)

$$\langle u(t_2)^+, e(t_2) \rangle_m \geq \langle u(t_2)^-, e(t_2) \rangle_m.$$ 

This means that the cone $C$ collapses around its axis $e(t): C^+(e(t_2)) \subset C^-(e(t_2))$. 

$\implies$ **The product** $\langle u(t), e(t) \rangle_m$ **is a potential for interaction!**
An entertaining digression (Godunov, Sultanghazin, Galperin)

Consider:

\[
\begin{align*}
\text{A: Mass } & 1 \\
\text{B: Mass } & m
\end{align*}
\]
The collision transformation takes the form

\[ u'_0 = u_0 - \frac{2m}{m+1}(u_0 - v_0) \quad (1) \]

\[ v'_0 = v_0 + \frac{2}{m+1}(u_0 - v_0) \quad (2) \]
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\[ v_0' = v_0 + \frac{2}{m+1}(u_0 - v_0) \]  

(2)

momentum, energy are conserved:

\[ u_0' + mv_0' = u_0 + mv_0 \]

\[ (u_0')^2 + m(v_0')^2 = (u_0)^2 + m(v_0)^2 \]
Ball A will bounce off the wall and head back right; it will collide again with ball B, but if ball B is heavier than ball A, this will not be the last collision:

**Figure**: Many collisions in spacetime
Let $u_0, u_1, u_2, \ldots$ denote the velocities of A initially, after the first wall bounce, then after the second wall bounce, etc.
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Then
\[
\begin{align*}
u_1 &= -u'_0, \\
v_1 &= v'_0,
\end{align*}
\]

or
\[
\begin{align*}
u_1 &= m_{-1}u_0 - 2m_{+1}v_0 \quad (3) \\
v_1 &= 2m_{+1}u_0 + m_{-1}m_{+1}v_0 \quad (4)
\end{align*}
\]

The two particles were originally in a collision configuration because 
\( v_0 - u_0 = -1 < 0; \) if \( v_1 - u_1 < 0, \) they will collide again. 
We can then compute \((u_2, v_2), (u_3, v_3)\) etc., until we find a number \( k \) such that, for the first time, 
\( v_k - u_k > 0. \)
A can then not catch up with B, and there will be no more collisions.
Terminology

Let \( u_0, u_1, u_2, \ldots \) denote the velocities of A initially, after the first wall bounce, then after the second wall bounce, etc.
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\] (3) (4)
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Find the number $k$ with little effort. The following table shows $k$ as a function of $m$, the mass of particle B. Following Galperin’s idea, we have taken $m = 100^n$, where $n = 0, 1, 2, 3, \ldots$.

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Table: Number of collisions: THE DIGITS OF $\pi$!

$N$ and $M$ are the numbers of total collisions and wall collisions, respectively. Remember: particle A is initially at rest, and particle B moves initially at $v_0 = -1$. 
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Explanation? ... is another talk!
Observed around 1987 (?) by Sentis, Golse, Lions, Perthame. DiPerna and Lions figured out how to use this for BE.

**The Result** For $f = f(x, v, t)$, let $Tf := (\partial_t + v \cdot \nabla_x)f$.

**Lemma.** (velocity averaging) Assume that $f \in L^2(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R})$, has compact support, and is such that $Tf \in L^2(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R})$. Then

$$\int f \, dv \in H^{1/2}(\mathbb{R}^3 \times \mathbb{R}).$$

(meaning $\int (\tau^2 + |z|^2)^{1/2} \left| \int \hat{f}(z, v, \tau) \, dv \right|^2 dz \, d\tau < \infty$.)
By entropy theorems (to be revisited later) can construct weakly approximating sequence \( \{f_n\} \) by, say, modifying the BE. A limit exists! : \( f_n \rightharpoonup_w f \).
How this is used:

By entropy theorems (to be revisited later) can construct weakly approximating sequence \( \{f_n\} \) by, say, modifying the BE. A limit exists! : \( f_n \to_w f \). But nonlinear functionals are in general not weakly continuous (ask me for an example if you wish), so we need better than weak convergence!

Fortunately, the loss term of BE, 
\[
Q(f_n, f) = f_R(f_n) = \int \nu(v - w) f_n(x, w, t) \, dw
\]

The velocity averaging lemma and compact embeddings can be used, with intermediate steps, to prove Lemma. For a subsequence
\[
\int f_n \, dv \to \int f \, dv \quad \text{strongly in } L^1
\]
\[
R_n(f_n) \to R(f) \quad \text{strongly in } L^1
\]

... convergence of the gain term... requires much hard work.
By entropy theorems (to be revisited later) can construct weakly approximating sequence \( \{f_n\} \) by, say, modifying the BE. A limit exists! : \( f_n \to_w f \). But nonlinear functionals are in general not weakly continuous (ask me for an example if you wish), so we need better than weak convergence! Fortunately, the loss term of BE, \( Q^-(f, f) = fR(f) \) where \( R(f) = \int \nu(v - w)f(x, w, t)dw \).
How this is used:

By entropy theorems (to be revisited later) can construct weakly approximating sequence \( \{ f_n \} \) by, say, modifying the BE. A limit exists! \( f_n \rightarrow_w f \). But nonlinear functionals are in general not weakly continuous (ask me for an example if you wish), so we need better than weak convergence! Fortunately, the loss term of BE, \( Q(f, f) = fR(f) \) where \( R(f) = \int \nu(v - w)f(x, w, t)dw \). The velocity averaging lemma and compact embeddings can be used, with intermediate steps, to prove

**Lemma.** For a subsequence

i) \( \int f_n dv \rightarrow \int fdv \) strongly in \( L^1 \)

ii) \( R_n(f_n) \rightarrow R(f) \) strongly in \( L^1 \)

iii) ... convergence of the gain term... requires much hard work.
A Case Study: Kinetic Granular Media Model (Benedetto, Caglioti, Pulvirenti, in 1 D, 1997-1999).

Equation:
\[ \frac{\partial}{\partial t} f + v \cdot \nabla_x f = \lambda \text{div}_v[(\nabla W \ast_v f)f] \]

(think \( W(v) = \frac{1}{3}|v|^3 \).) General \( W \) such that \( W(-v) = W(v) \).
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Formal properties: Mass and momentum conservation. Kinetic energy decrease:

\[ K(t) := \frac{1}{2} \int \int |\mathbf{v}|^2 f(x, \mathbf{v}, t) dx \, dv \leq K(0) \]

(in general strict decrease).
Consider a particle system

\[
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= \epsilon \sum_{j=1}^{N} \eta_\alpha(x_i - x_j) \nabla W(v_j - v_i) \\
&= \epsilon N \frac{1}{N} \sum \ldots
\end{align*}
\]

Define a measure \( \mu_t^N = \frac{1}{N} \sum \delta_{(x_j, v_j)} \), \( F_\alpha(x, v) = \eta_\alpha(x) \nabla W(v) \).

Then

\[
(F_\alpha * \mu_t^N)(x, v) = -\frac{1}{N} \sum \eta_\alpha(x - x_j) \nabla W(v_j - v).
\]
Formally, one takes the limit $N\epsilon \to \lambda$, in which

$$\mu_t^N(x, v) \to f(x, v, t)$$

and the system becomes

$$\dot{x} = v, \quad \dot{v} = -\lambda F_\alpha \ast f.$$  

Then one sends $\alpha$ to zero, and the model equation appears.
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Issues.

- Validation! (the order of limits is a subtle point).
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- Qualitative behaviour.
1. Entropy: Let $U : [0, \infty) \rightarrow \mathbb{R}$, $U(0) = 0$, convex, and set $P_U(r) = rU'(r) - U(r) \geq 0$. Examples are $r^p$, $p > 1$, and $r \ln r$. Then, if $f$ solves the model equation,

$$
\frac{d}{dt} \int \int U(f) = \ldots = \lambda \int \int \int \Delta W(v - u) P_U(f)(x, v) f(x, u) \, du \, dv \, dx
$$

r.h.s. is $\geq 0$ because $W$ is convex, so $\Delta W \geq 0$. For $U = r \ln r$ one computes $P_U(f) = f$, and the r.h.s. is $\lambda \int \int \int \Delta W(v - u) f(x, v) f(x, u) \, du \, dv \, dx$. 

Reinhard Illner, Victoria
1. Entropy: Let $U : [0, \infty) \to \mathbb{R}$, $U(0) = 0$, convex, and set $P_U(r) = rU'(r) - U(r) \geq 0$. Examples are $r^p$, $p > 1$, and $r \ln r$. Then, if $f$ solves the model equation,

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Let \( J(f)(t) := \int \int (x - tv)^2 f(x, v, t) dv \, dx \), then

\[
\frac{d}{dt} J = \int \int 2(x - tv)(-v)f + \int \int (x - tv) \partial_t f \\
= \int \int \{-2xv + 2tv^2 + 2(x - tv)v\} f \\
+ \lambda \int \int (x - tv)^2 div_v[(\nabla W * v f) f] dv \, dx \\
= -2\lambda t^2 \int \int \int (\nu - u) \nabla W(\nu - u)f(x, \nu)f(x, u) \, du \, dv \, dx.
\]
So,

\[ J(f)(t) = J(f)(0) - \lambda \int_0^t s^2 \int \int \int (v-u) \nabla W(v-u) f \otimes f \, du \, dv \, dx \, ds \]

Compare with.

\[ H(f)(t) = H(f)(0) + \lambda \int_0^t \int \int \int \Delta W(v-u) f \otimes f \, du \, dv \, dx \, ds \]

Note: in, say, one dimension, for \( W(v) = \frac{1}{3} v^3 \), we have

\[ W''(v) = 2|v|, \quad \text{and} \quad vW'(v) = |v|^3. \]

This is the fundamental difference of the terms on the right. The production term on the right hand side in the second identity is uniformly bounded; however, this does not entail bounded entropy production, because of the different powers of \(|v-u|\).
3. In 1 D: Can use potential for interaction:
Let \( I(f)(t) = \int_v \int_u \int_{x<y} (v - u)f(x, v)f(y, u) \, dx \, dy \, du \, dv \). Then, repeating the calculation done much earlier for DVMs, using only momentum and mass conservation,

\[
\frac{d}{dt} J = - \int \int \int (v - u)^2 f(x, v)f(x, u) \, dx \, dv \, du.
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*Almost* the same r.h.s. emerges from completely different functionals!
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Thank you
Revisit the Digression...

The explanation is hidden in the properties of the transformation (3,4). Things become simpler if one rescales the speeds $v_0, v_1, v_2$, etc. of ball B:

\[ w_0 := \sqrt{m}v_0, \quad w_1 := \sqrt{m}v_1, \]

etc.

Energy conservation then becomes the simpler equation

\[ (u'_0)^2 + (w'_0)^2 = (u_0)^2 + (w_0)^2 \]  (5)

and the collision transformation (4) becomes

\[ u_1 = \frac{m-1}{m+1}u_0 - \frac{2\sqrt{m}}{m+1}w_0 \]  (6)

\[ w_1 = \frac{2\sqrt{m}}{m+1}u_0 + \frac{m-1}{m+1}w_0 \]  (7)
In this new coordinate system, the equations (6,7) are where the circle is hiding: set

\[ \alpha = \frac{m - 1}{m + 1}, \quad \beta = \frac{2\sqrt{m}}{m + 1} \quad \Rightarrow \]

\[ \alpha^2 + \beta^2 = 1 \quad \Rightarrow \]
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there is an angle \( \theta \) such that \( \cos \theta = \alpha, \sin \theta = \beta \). Geometrically this means that in the \( u - w \) plane, (6,7) is a rotation in the counterclockwise sense by the angle \( \theta \);
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\[
\begin{pmatrix}
  u_{j+1} \\
  w_{j+1}
\end{pmatrix} =
\begin{pmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
  u_j \\
  w_j
\end{pmatrix}
\]
Energy conservation as stated in (5) is the key ingredient in this: the collision transformation must conserve the length of the vector \((u_0, w_0)\), and only rotations or reflections do this.

\[
\theta
\]

\[
w = m^{1/2} u
\]

\[
\theta
\]

\[
\theta
\]

\[
-\frac{1}{2} m
\]

\[
\theta
\]

---

**Figure**: Collisions are rotations!
Almost there!

No more collisions after the first $k$ for which $v_k > u_k$, or, equivalently, $w_k > \sqrt{mu_k}$.
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No more collisions after the first $k$ for which $v_k > u_k$, or, equivalently, $w_k > \sqrt{mu_k}$. $\implies$ have to find out for which $k$ the sum of the angles will have crossed the line with slope $\sqrt{m}$. 
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For a large $m$: $\tan^{-1}\sqrt{m} \approx \frac{\pi}{2}$ (there have been enough collisions to go almost through a half-circle, meaning $k\theta \approx \pi$).

We can also approximate $\theta$ in terms of $m$ by observing that $\alpha = \cos\theta \approx 1 - \frac{\theta^2}{2}$, hence $\theta \approx 2\sqrt{m} + 1$. Together: $k \approx \pi \sqrt{m} + 1$, and this is an approximation of the expected number of wall touches. For example, for $m = 10^4$, we find $2k \approx 100\pi \approx 314$. 

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"Selective "Modern” History of the Boltzmann and Related E
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