A microscopic approach to Souslin trees constructions

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This is joint work with Ari M. Brodsky, and still *in progress*..
Some conventions

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- $\text{CH}_\lambda$ asserts that $2^\lambda = \lambda^+$
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- \( \text{succ}_\sigma(C) = \{ \alpha \in C \mid \text{otp}(C \cap \alpha) = j + 1 \text{ for some } j < \sigma \} \)
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E.g., $\text{succ}_3(\omega_1 \setminus \omega) = \{\omega + 1, \omega + 2, \omega + 3\}$. 
\( \kappa \)-trees

Definition

- A tree is a poset \( \mathcal{T} = \langle T, \triangleleft \rangle \) in which \( x_\downarrow := \{ y \in T \mid y \triangleleft x \} \) is well-ordered for all \( x \in T \);
κ-trees

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- A tree is a poset $\mathcal{T} = \langle T, \triangleleft \rangle$ in which $x_\downarrow := \{ y \in T \mid y \triangleleft x \}$ is well-ordered for all $x \in T$;
- $\mathcal{T}_\delta = \{ x \in T \mid \operatorname{otp}(x_\downarrow, \triangleleft) = \delta \}$ is the $\delta^{th}$-level of $\mathcal{T}$;
κ-trees

Definition

- A tree is a poset \( T = \langle T, \triangleleft \rangle \) in which \( x_\downarrow := \{ y \in T \mid y \triangleleft x \} \) is well-ordered for all \( x \in T \);
- \( T_\delta = \{ x \in T \mid \text{otp}(x_\downarrow, \triangleleft) = \delta \} \) is the \( \delta^{th} \)-level of \( T \);
- The height of \( T \) is \( \min\{ \delta \in \text{Ord} \mid T_\delta = \emptyset \} \);
- A \( \kappa \)-Aronszajn tree is a \( \kappa \)-tree having no branches of size \( \kappa \);
- A \( \kappa \)-Souslin tree is a \( \kappa \)-Aronszajn tree having no antichains of size \( \kappa \).
$\kappa$-trees

Definition

- A tree is a poset $\mathcal{T} = \langle T, \prec \rangle$ in which $x_\downarrow := \{y \in T \mid y \prec x\}$ is well-ordered for all $x \in T$;
- $\mathcal{T}_\delta = \{x \in T \mid \otp(x_\downarrow, \prec) = \delta\}$ is the $\delta^{th}$-level of $\mathcal{T}$;
- The height of $\mathcal{T}$ is $\min\{\delta \in \text{Ord} \mid T_\delta = \emptyset\}$;
- $\mathcal{T}$ is $\chi$–complete if any $\prec$-increasing sequence of length $< \chi$ admits a bound;
$\kappa$-trees

Definition

- A tree is a poset $\mathcal{T} = \langle T, \preccurlyeq \rangle$ in which $x_\downarrow := \{y \in T \mid y \prec x\}$ is well-ordered for all $x \in T$;
- $\mathcal{T}_\delta = \{x \in T \mid \text{otp}(x_\downarrow, \preccurlyeq) = \delta\}$ is the $\delta^{th}$-level of $\mathcal{T}$;
- The height of $\mathcal{T}$ is $\min\{\delta \in \text{Ord} \mid \mathcal{T}_\delta = \emptyset\}$;
- $\mathcal{T}$ is $\chi$–complete if any $\prec$-increasing sequence of length $< \chi$ admits a bound;
- $\mathcal{T}$ is $\chi$–slim if $|\mathcal{T}_\alpha| = |\alpha|$ whenever $\alpha \geq \chi$.
κ-trees

Definition

- A tree is a poset $\mathcal{T} = \langle T, \triangleleft \rangle$ in which $x^\downarrow := \{y \in T \mid y \triangleleft x\}$ is well-ordered for all $x \in T$;
- $\mathcal{T}_\delta = \{x \in T \mid \text{otp}(x^\downarrow, \triangleleft) = \delta\}$ is the $\delta^{th}$-level of $\mathcal{T}$;
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- $\mathcal{T}$ is $\chi$–complete if any $\triangleleft$-increasing sequence of length $< \chi$ admits a bound;
- $\mathcal{T}$ is $\chi$–slim if $|\mathcal{T}_\alpha| = |\alpha|$ whenever $\alpha \geq \chi$.

Definition

- A $\kappa$-tree is a tree of height $\kappa$ whose levels are of size $< \kappa$;
- A $\kappa$-Aronszajn tree is a $\kappa$-tree having no branches of size $\kappa$;
- A $\kappa$-Souslin tree is a $\kappa$-Aronszajn tree having no antichains of size $\kappa$. 
Aronszajn and Souslin trees are useful objects that give rise to rich counterexamples in mathematics. The literature concerning these trees splits roughly into two:

- Papers that deal with the construction of Aronszajn/Souslin trees with some additional features.
- Papers that deal with the construction of the trees from weaker and weaker hypotheses, or consistency results concerning non-existence.
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- Papers that deal with the construction of Aronszajn/Souslin trees with some additional features.
- Papers that deal with the construction of the trees from weaker and weaker hypotheses, or consistency results concerning non-existence.

We shall now dedicate a few minutes to review some known results, highlighting that the behavior of $\kappa$-Aronszajn and $\kappa$-Souslin trees depends heavily on the identity of $\kappa$. 

The role of $\kappa$
Theorem (König, 1927)

There exists no $\aleph_0$-Aronszajn tree.
$\kappa$-Aronszajn trees

Theorem (König, 1927)

There exists no $\aleph_0$-Aronszajn tree.

Theorem (Aronszajn, 1935)

There exists an $\aleph_1$-Aronszajn tree.
\(\kappa\)-Aronszajn trees

Theorem (König, 1927)

There exists no \(\aleph_0\)-Aronszajn tree.

Theorem (Specker, 1949. \(\lambda = \omega\) is due to Aronszajn, 1935)

If \(\lambda\) is regular and \(\lambda^{<\lambda} = \lambda\), then there exists a \(\lambda^+\)-Aronszajn tree.
$\kappa$-Aronszajn trees

Theorem (König, 1927)
There exists no $\aleph_0$-Aronszajn tree.

Theorem (Specker, 1949. $\lambda = \omega$ is due to Aronszajn, 1935)
If $\lambda$ is regular and $\lambda^< \leq \lambda$, then there exists a $\lambda^+$-Aronszajn tree.

Theorem (Magidor-Shelah, 1996)
Modulo large cardinals, it is consistent with GCH, that for some singular cardinal $\lambda$, there exists no $\lambda^+$-Aronszajn tree.
\(\kappa\)-Aronszajn trees

Theorem (König, 1927)

There exists no \(\aleph_0\)-Aronszajn tree.

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If \(\lambda\) is regular and \(\lambda^{<\lambda} = \lambda\), then there exists a \(\lambda^+\)-Aronszajn tree.

Theorem (Magidor-Shelah, 1996)

Modulo large cardinals, it is consistent with GCH, that for some singular cardinal \(\lambda\), there exists no \(\lambda^+\)-Aronszajn tree.

Theorem (Erdős-Taski, 1943)

If \(\kappa\) is weakly compact, then there exists no \(\kappa\)-Aronszajn tree.
\( \lambda^+ \)-Souslin trees

**Definition (Jensen, 1972)**

For \( S \subseteq \kappa \), \( \diamond (S) \) asserts the existence of a sequence \( \langle A_\alpha \mid \alpha \in S \rangle \) such that \( \{ \alpha \in S \mid A \cap \alpha = A_\alpha \} \) is stationary for all \( A \subseteq \kappa \).
\(\lambda^+\)-Souslin trees

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**Theorem (Jensen, 1972)**

*If* \(\lambda^{<\lambda} = \lambda\) *and* \(\diamond(E_\lambda^{\lambda^+})\) *holds, then there exists a \(\lambda\)-complete \(\lambda^+\)-Souslin tree.*
**$\lambda^+$-Souslin trees**

**Definition (Jensen, 1972)**

For $S \subseteq \kappa$, $\Diamond (S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S \rangle$ such that $\{ \alpha \in S \mid A \cap \alpha = A_\alpha \}$ is stationary for all $A \subseteq \kappa$.

**Theorem (Jensen, 1972)**

If $\lambda^+ = \lambda$ and $\Diamond (E^\lambda_\lambda^+)$ holds, then there exists a $\lambda$-complete $\lambda^+$-Souslin tree.

This gives a method to construct Souslin tree at the level of successor of regulars. How to handle successor of singulars?
λ⁺-Souslin trees

Definition (Jensen, 1972)
For \( S \subseteq \kappa \), \( \diamond (S) \) asserts the existence of a sequence \( \langle A_\alpha \mid \alpha \in S \rangle \) such that \( \{ \alpha \in S \mid A \cap \alpha = A_\alpha \} \) is stationary for all \( A \subseteq \kappa \).

Theorem (Jensen, 1972)
If \( \lambda^\lt^\lambda = \lambda \) and \( \diamond (E_\lambda^{\lambda^+}) \) holds, then there exists a \( \lambda \)-complete \( \lambda^+ \)-Souslin tree.

Definition (Jensen, 1972)
\( \Box_\lambda (S) \) asserts the existence of a sequence \( \langle C_\delta \mid \delta < \lambda^+ \rangle \) such that for all limit \( \delta < \lambda^+ \):
- \( C_\delta \) is a club in \( \delta \) of order-type \( \leq \lambda \);
- if \( \beta \in \text{acc}(C_\delta) \), then \( \beta \notin S \) and \( C_\delta \cap \beta = C_\beta \).

Write \( \Box_\lambda \) for \( \Box_\lambda (\emptyset) \).
$\lambda^+$-Souslin trees

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For $S \subseteq \kappa$, $\diamond (S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S \rangle$ such that $\{ \alpha \in S \mid A \cap \alpha = A_\alpha \}$ is stationary for all $A \subseteq \kappa$.

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If $\lambda^{<\lambda} = \lambda$ and $\diamond (E_\lambda^{\lambda^+})$ holds, then there exists a $\lambda$-complete $\lambda^+$-Souslin tree.

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$\Box_\lambda (S)$ asserts the existence of a sequence $\langle C_\delta \mid \delta < \lambda^+ \rangle$ such that for all limit $\delta < \lambda^+$:

- $C_\delta$ is a club in $\delta$ of order-type $\leq \lambda$;
- if $\beta \in \text{acc}(C_\delta)$, then $\beta \notin S$ and $C_\delta \cap \beta = C_\beta$.

Theorem (Jensen, 1972)

If there exists $S \subseteq \lambda^+$ for which $\Box_\lambda (S) + \diamond (S)$ holds, then there exists a $\lambda^+$-Souslin tree.
Special and specializable $\lambda^+$-trees

**Definition**

A $\lambda^+$-tree is *special* if it is the union of $\lambda$ many antichains.
Special and specializable $\lambda^+$-trees

Definition
A $\lambda^+$-tree is special if it is the union of $\lambda$ many antichains.

Note
- A special $\lambda^+$-tree is $\lambda^+$-Aronszajn;
- A $\lambda^+$-Souslin tree is non-special.
Special and specializable $\lambda^+$-trees

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A $\lambda^+$-tree is special if it is the union of $\lambda$ many antichains.

Note
- A special $\lambda^+$-tree is $\lambda^+$-Aronszajn;
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Remark
Aronszajn’s and Specker’s constructions from $\lambda^{<\lambda} = \lambda$ may be steered to yield a special $\lambda^+$-tree.
Special and specializable $\lambda^+$-trees

**Definition**
A $\lambda^+$-tree is **special** if it is the union of $\lambda$ many antichains.

**Definition**
A $\lambda^+$-tree is **specializable** if it is special in some extended universe of ZFC with the same cardinal structure.
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A $\lambda^+$-tree is **specializable** if it is special in some extended universe of ZFC with the same cardinal structure.

Theorem (Baumgartner-Mailtz-Reinhardt, 1970)
An $\aleph_1$-tree is Aronszajn iff it is specializable.
Special and specializable $\lambda^+$-trees

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A $\lambda^+$-tree is **special** if it is the union of $\lambda$ many antichains.

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A $\lambda^+$-tree is **specializable** if it is special in some extended universe of ZFC with the same cardinal structure.

Theorem (Baumgartner-Mailtz-Reinhardt, 1970)
An $\aleph_1$-tree is Aronszajn iff it is specializable.

Theorem (implicit in David, 1990)
If $V = L$, then for every **regular** $\lambda$, the canonical $\lambda$-complete $\lambda^+$-Souslin tree constructed using fine structure, is specializable.
Non-specializable $\lambda^+$-Souslin trees

Theorem (Baumgartner, 1970’s, building on Laver)

$\square_{\aleph_1}$ entails a non-specializable $\aleph_2$-Aronszajn tree.
Non-specializable $\lambda^+$-Souslin trees

Theorem (Baumgartner, 1980’s, improving Devlin)

$GCH + \Box_{\aleph_1}$ entails a non-specializable $\aleph_2$-Souslin tree.

Theorem (Cummings, 1997)

If $\lambda$ is a singular cardinal of countable cofinality,

$\Box \lambda + CH \lambda$ and $\mu \aleph_1 < \lambda$ for all $\mu < \lambda$,

then there exists a non-specializable $\lambda^+$-Souslin tree.

Theorem (Cummings, 1997)

If $\lambda$ is a singular cardinal of uncountable cofinality,

$\Box \lambda + CH \lambda$ and $\mu \aleph_0 < \lambda$ for all $\mu < \lambda$,

then there exists a non-specializable $\lambda^+$-Souslin tree.
Non-specializable $\lambda^+$-Souslin trees

Theorem (Baumgartner, 1980’s, improving Devlin)

$GCH + \square_{\aleph_1}$ entails a non-specializable $\aleph_2$-Souslin tree.

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$\aleph_1 \leq \lambda < \lambda = \lambda + \Box \lambda$ entails a non-specializable $\lambda$-complete $\lambda^+$-Souslin tree.
Non-specializable $\lambda^+$-Souslin trees

Theorem (Baumgartner, 1980’s, improving Devlin)

$GCH + \Box_{\aleph_1}$ entails a non-specializable $\aleph_2$-Souslin tree.

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$\aleph_1 \leq \lambda^{<\lambda} = \lambda + \Box_{\lambda}$ entails a non-specializable $\lambda$-complete $\lambda^+$-Souslin tree.

Theorem (Cummings, 1997)

If $\lambda$ is a singular cardinal of countable cofinality, $\Box_{\lambda} + \text{CH}_{\lambda}$ and $\mu^{\aleph_1} < \lambda$ for all $\mu < \lambda$, then there exists a non-specializable $\lambda^+$-Souslin tree.
Non-specializable $\lambda^+-$Souslin trees

Theorem (Baumgartner, 1980’s, improving Devlin)

$\text{GCH} + \Box_{\aleph_1}$ entails a non-specializable $\aleph_2$-Souslin tree.

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Theorem (Cummings, 1997)

If $\lambda$ is a singular cardinal of countable cofinality, $\Box_{\lambda} + \text{CH}_{\lambda}$ and $\mu^{\aleph_1} < \lambda$ for all $\mu < \lambda$, then there exists a non-specializable $\lambda^+$-Souslin tree.

Theorem (Cummings, 1997)

If $\lambda$ is a singular cardinal of uncountable cofinality, $\Box_{\lambda} + \text{CH}_{\lambda}$ and $\mu^{\aleph_0} < \lambda$ for all $\mu < \lambda$, then there exists a non-specializable $\lambda^+$-Souslin tree.
To sum up

The construction of $\lambda^+$-Souslin trees often makes an explicit distinction between the case that $\lambda$ is a regular cardinal and the case that $\lambda$ is singular.
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Question
Do one really have to come up with such a long list of variations each time that a fundamental construction is discovered?
To sum up

The construction of $\lambda^+$-Souslin trees often makes an explicit distinction between the case that $\lambda$ is a regular cardinal and the case that $\lambda$ is singular. Some of them also depend on whether or not $\lambda$ is of countable cofinality.

Question
Do one really have to come up with such a long list of variations each time that a fundamental construction is discovered? Isn’t there any automatic translation between the different cardinals?
An idea

Find a proxy!

1. Introduce a family of combinatorial principles from which the constructions can be carried out uniformly;
2. Prove that this operational principle is a consequence of the “usual” hypotheses.
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Find a proxy!

1. Introduce a family of combinatorial principles from which the constructions can be carried out uniformly;

2. Prove that this operational principle is a consequence of the “usual” hypotheses. This part is done only once, and then will be utilized each time that a new construction is discovered.
The proxy principle

Goal
The proxy principle will allow to translate constructions from one cardinal to another, to calibrate the hypotheses needed to carry a construction, and will capture all known ♦-based constructions.
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**Definition**

\[ P(\kappa, \mu, R, \theta, S, \nu, \sigma, \varpi) \] asserts that ♦(κ) holds, and so is the corresponding \( P^-(\kappa, \mu, R, \theta, S, \nu, \sigma, \varpi) \).
The proxy principle

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$P(\kappa, \mu, \mathcal{R}, \theta, S, \nu, \sigma, \varpi)$ asserts that $\Diamond(\kappa)$ holds, and so is the corresponding $P^-(\kappa, \mu, \mathcal{R}, \theta, S, \nu, \sigma, \varpi)$. 
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The proxy principle

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$P^{-}(\kappa, \mu, \mathcal{R}, \theta, S, \nu, \sigma, \varpi)$ asserts the existence of $\langle C_{\delta} \mid \delta < \kappa \rangle$ s.t.:

- for every limit $\delta < \kappa$, $C_{\delta}$ is a collection of club subsets of $\delta$;
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- for every limit $\delta < \kappa$, $C_\delta$ is a collection of club subsets of $\delta$;
- $0 < |C_\delta| < \mu$ for all $\delta < \kappa$;
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\( P^{-}(\kappa, \mu, \mathcal{R}, \theta, S, \nu, \sigma, \varpi) \) asserts the existence of \( \langle C_\delta \mid \delta < \kappa \rangle \) s.t.:

- for every limit \( \delta < \kappa \), \( C_\delta \) is a collection of club subsets of \( \delta \);
- \( 0 < |C_\delta| < \mu \) for all \( \delta < \kappa \);
- \( \mathcal{R} \) is a binary relation over \( \mathcal{P}(\kappa) \);
The proxy principle

Definition

\( P^-(\kappa, \mu, \mathcal{R}, \theta, S, \nu, \sigma, \omega) \) asserts the existence of \( \langle C_\delta \mid \delta < \kappa \rangle \) s.t.:

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The proxy principle

Definition

$P^- (\kappa, \mu, R, \theta, S, \nu, \sigma, \varpi)$ asserts the existence of $\langle C_\delta \mid \delta < \kappa \rangle$ s.t.:

- for every limit $\delta < \kappa$, $C_\delta$ is a collection of club subsets of $\delta$;
- $0 < |C_\delta| < \mu$ for all $\delta < \kappa$;
- $R$ is a binary relation over $\mathcal{P}(\kappa)$;
- if $C \in C_\delta$ and $\beta \in \text{acc}(C)$, then $\exists D \in C_\beta$ with $(D, C) \in R$;
The proxy principle

Example of a binary relation $\mathcal{R}$
$\sqsubseteq$, where $D \sqsubseteq C$ iff $\exists \beta$ such that $D = C \cap \beta$.

Definition

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- for every limit $\delta < \kappa$, $C_\delta$ is a collection of club subsets of $\delta$;
- $0 < |C_\delta| < \mu$ for all $\delta < \kappa$;
- if $C \in C_\delta$ and $\beta \in \text{acc}(C)$, then $\exists D \in C_\beta$ with $(D, C) \in \mathcal{R}$;
The proxy principle

Example of a binary relation $\mathcal{R}$

$\sqsubseteq_\chi$, where $D \sqsubseteq_\chi C$ iff $D \sqsubseteq C$ or $\text{otp}(C) < \chi$.

Definition

$P^-(\kappa, \mu, \mathcal{R}, \theta, S, \nu, \sigma, \omega)$ asserts the existence of $\langle C_\delta \mid \delta < \kappa \rangle$ s.t.:

- for every limit $\delta < \kappa$, $C_\delta$ is a collection of club subsets of $\delta$;
- $0 < |C_\delta| < \mu$ for all $\delta < \kappa$;
- if $C \in C_\delta$ and $\beta \in \text{acc}(C)$, then $\exists D \in C_\beta$ with $(D, C) \in \mathcal{R}$;
The proxy principle

Example of a binary relation $\mathcal{R}$
$\sqsubseteq^*$, where $D \sqsubseteq^* C$ iff $\exists \alpha < \sup(D)$ with $D \setminus \alpha \sqsubseteq C \setminus \alpha$.

Definition
$P^-(\kappa, \mu, \mathcal{R}, \theta, S, \nu, \sigma, \varpi)$ asserts the existence of $\langle C_\delta \mid \delta < \kappa \rangle$ s.t.:
- for every limit $\delta < \kappa$, $C_\delta$ is a collection of club subsets of $\delta$;
- $0 < |C_\delta| < \mu$ for all $\delta < \kappa$;
- if $C \in C_\delta$ and $\beta \in \text{acc}(C)$, then $\exists D \in C_\beta$ with $(D, C) \in \mathcal{R}$;
The proxy principle

Example of a binary relation $\mathcal{R}$

$\chi \sqsubseteq^*$, where $D \chi \sqsubseteq^* C$ iff $\text{cf}(\text{sup}(D)) < \chi$ or $D \sqsubseteq^* C$.

Definition

$P^-(\kappa, \mu, \mathcal{R}, \theta, S, \nu, \sigma, \varpi)$ asserts the existence of $\langle C_\delta \mid \delta < \kappa \rangle$ s.t.:

- for every limit $\delta < \kappa$, $C_\delta$ is a collection of club subsets of $\delta$;
- $0 < |C_\delta| < \mu$ for all $\delta < \kappa$;
- if $C \in C_\delta$ and $\beta \in \text{acc}(C)$, then $\exists D \in C_\beta$ with $(D, C) \in \mathcal{R}$;
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\( P^- (\kappa, \mu, R, \theta, S, \nu, \sigma, \varpi) \) asserts the existence of \( \langle C_\delta \mid \delta < \kappa \rangle \) s.t.:

- for every limit \( \delta < \kappa \), \( C_\delta \) is a collection of club subsets of \( \delta \);
- \( 0 < |C_\delta| < \mu \) for all \( \delta < \kappa \);
- if \( C \in C_\delta \) and \( \beta \in \text{acc}(C) \), then \( \exists D \in C_\beta \) with \((D, C) \in R\);
- for every sequence \( \langle A_i \mid i < \theta \rangle \) of cofinal subsets of \( \kappa \).
The proxy principle

Definition

$P^-(\kappa, \mu, \mathcal{R}, \theta, S, \nu, \sigma, \omega)$ asserts the existence of $\langle C_\delta \mid \delta < \kappa \rangle$ s.t.:

- for every limit $\delta < \kappa$, $C_\delta$ is a collection of club subsets of $\delta$;
- $0 < \lvert C_\delta \rvert < \mu$ for all $\delta < \kappa$;
- if $C \in C_\delta$ and $\beta \in \text{acc}(C)$, then $\exists D \in C_\beta$ with $(D, C) \in \mathcal{R}$;
- for every sequence $\langle A_i \mid i < \theta \rangle$ of cofinal subsets of $\kappa$, and every $S \in S$. 
The proxy principle

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\(P^{-}(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \varpi)\) asserts the existence of \(\langle C_\delta \mid \delta < \kappa \rangle\) s.t.: 

- for every limit \(\delta < \kappa\), \(C_\delta\) is a collection of club subsets of \(\delta\);
- \(0 < |C_\delta| < \mu\) for all \(\delta < \kappa\);
- if \(C \in C_\delta\) and \(\beta \in \text{acc}(C)\), then \(\exists D \in C_\beta\) with \((D, C) \in \mathcal{R}\);
- for every sequence \(\langle A_i \mid i < \theta \rangle\) of cofinal subsets of \(\kappa\), and every \(S \in \mathcal{S}\), there exists \(\delta \in S\).
Definition

$P^-(\kappa, \mu, \mathcal{R}, \theta, S, \nu, \sigma, \omega)$ asserts the existence of $\langle C_\delta \mid \delta < \kappa \rangle$ s.t.:

- for every limit $\delta < \kappa$, $C_\delta$ is a collection of club subsets of $\delta$;
- $0 < |C_\delta| < \mu$ for all $\delta < \kappa$;
- if $C \in C_\delta$ and $\beta \in \text{acc}(C)$, then $\exists D \in C_\beta$ with $(D, C) \in \mathcal{R}$;
- for every sequence $\langle A_i \mid i < \theta \rangle$ of cofinal subsets of $\kappa$, and every $S \in S$, there exists $\delta \in S$ with $|C_\delta| < \nu$ such that:
The proxy principle

Recall

\[ \text{succ}_\sigma(C) = \{ \alpha \in C \mid \text{otp}(C \cap \alpha) = j + 1 \text{ for some } j < \sigma \}. \]

Definition

\( P^-(\kappa, \mu, R, \theta, S, \nu, \sigma, \varpi) \) asserts the existence of \( \langle C_\delta \mid \delta < \kappa \rangle \) s.t.:

- for every limit \( \delta < \kappa \), \( C_\delta \) is a collection of club subsets of \( \delta \);
- \( 0 < |C_\delta| < \mu \) for all \( \delta < \kappa \);
- if \( C \in C_\delta \) and \( \beta \in \text{acc}(C) \), then \( \exists D \in C_\beta \) with \( (D, C) \in R \);
- for every sequence \( \langle A_i \mid i < \theta \rangle \) of cofinal subsets of \( \kappa \), and every \( S \in S \), there exists \( \delta \in S \) with \( |C_\delta| < \nu \) such that:
  - \( \forall i < \min\{\delta, \theta\} \forall C \in C_\delta \sup\{\beta \in C \mid \text{succ}_\sigma(C \setminus \beta) \subseteq A_i\} = \delta; \)
The proxy principle

Recall
\[ \text{succ}_{\omega}(C) = \{ \alpha \in C \mid \text{otp}(C \cap \alpha) = j + 1 \text{ for some } j < \omega \}. \]

Definition
\[ P^-(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \omega) \] asserts the existence of \( \langle C_\delta \mid \delta < \kappa \rangle \) s.t.:
- for every limit \( \delta < \kappa \), \( C_\delta \) is a collection of club subsets of \( \delta \);
- \( 0 < |C_\delta| < \mu \) for all \( \delta < \kappa \);
- if \( C \in C_\delta \) and \( \beta \in \text{acc}(C) \), then \( \exists D \in C_\beta \) with \( (D, C) \in \mathcal{R} \);
- for every sequence \( \langle A_i \mid i < \theta \rangle \) of cofinal subsets of \( \kappa \), and every \( S \in \mathcal{S} \), there exists \( \delta \in S \) with \( |C_\delta| < \nu \) such that:
  - \( \forall i < \min\{\delta, \theta\} \forall C \in C_\delta \sup\{\beta \in C \mid \text{succ}_{\sigma}(C \setminus \beta) \subseteq A_i\} = \delta \);
  - \( \forall i < \min\{\delta, \theta\} \sup \bigcap_{C \in C_\delta} \{\beta \in C \mid \text{succ}_{\omega}(C \setminus \beta) \subseteq A_i\} = \delta \), unless \( \omega = 0 \).
Default values

Don’t worry, we have some default values!
Whenever omitted, let $\theta = 1, S = \{\kappa\}, \nu = 2, \sigma = 1, \varpi = 0.$
Default values

Don’t worry, we have some default values!
Whenever omitted, let \( \theta = 1, S = \{ \kappa \}, \nu = 2, \sigma = 1, \varpi = 0 \).

Definition

\( P^- (\kappa, \mu, \mathcal{R}, \theta, S, \nu, \sigma, \varpi) \) asserts the existence of \( \langle C_\delta \mid \delta < \kappa \rangle \) s.t.:

- for every limit \( \delta < \kappa \), \( C_\delta \) is a collection of club subset of \( \delta \);
- \( 0 < |C_\delta| < \mu \) for all \( \delta < \kappa \);
- if \( C \in C_\delta \) and \( \beta \in \text{acc}(C) \), then \( \exists D \in C_\beta \) with \( (D, C) \in \mathcal{R} \);
- for every sequence \( \langle A_i \mid i < \theta \rangle \) of cofinal subsets of \( \kappa \), and every \( S \in S \), there exists \( \delta \in S \) with \( |C_\delta| < \nu \) such that:
  - \( \forall i < \min \{ \delta, \theta \} \forall C \in C_\delta \sup \{ \beta \in C \mid \text{succ}_\sigma (C \setminus \beta) \subseteq A_i \} = \delta \);
  - \( \forall i < \min \{ \delta, \theta \} \sup \bigcap_{C \in C_\delta} \{ \beta \in C \mid \text{succ}_\varpi (C \setminus \beta) \subseteq A_i \} = \delta \),
    unless \( \varpi = 0 \).
Default values

Don’t worry, we have some default values!
Whenever omitted, let $\theta = 1, S = \{\kappa\}, \nu = 2, \sigma = 1, \omega = 0.$

Special case

$P^-(\kappa, \mu, R)$ asserts the existence of $\langle C_\delta \mid \delta < \kappa \rangle$ such that:

- for every limit $\delta < \kappa$, $C_\delta$ is a collection of club subset of $\delta$;
- $0 < |C_\delta| < \mu$ for all $\delta < \kappa$;
- if $C \in C_\delta$ and $\beta \in \text{acc}(C)$, then $\exists D \in C_\beta$ with $(D, C) \in R$;
Default values

Don’t worry, we have some default values!
Whenever omitted, let $\theta = 1, S = \{\kappa\}, \nu = 2, \sigma = 1, \varpi = 0$.

Special case

$P^-(\kappa, \mu, \mathcal{R})$ asserts the existence of $\langle C_\delta \mid \delta < \kappa \rangle$ such that:

- for every limit $\delta < \kappa$, $C_\delta$ is a collection of club subset of $\delta$;
- $0 < |C_\delta| < \mu$ for all $\delta < \kappa$;
- if $C \in C_\delta$ and $\beta \in \text{acc}(C)$, then $\exists D \in C_\beta$ with $(D, C) \in \mathcal{R}$;
- for every cofinal $A \subseteq \kappa$, there exists $\delta < \kappa$ with $C_\delta = \{C_\delta\}$, such that $\sup(\text{nacc}(C_\delta) \cap A) = \delta$. 

A Souslin tree from the weakest principle

Let $\kappa$ denote a regular uncountable cardinal.

**Proposition**

$P(\kappa, \kappa, \sqsubseteq^*, 1, \{\kappa\}, \kappa)$ entails a $\kappa$-Souslin tree.
A Souslin tree from the weakest principle

Let $\kappa$ denote a regular uncountable cardinal.

Proposition

$P(\kappa, \kappa, \sqsubseteq^*, 1, \{\kappa\}, \kappa)$ entails a $\kappa$-Souslin tree.

Proposition

$P(\kappa, \kappa, \sqsubseteq^*, 1, \{E^\kappa_{\geq \chi}\}, \kappa)$ entails a $\chi$-complete $\kappa$-Souslin tree, provided $|\alpha| < \chi < \kappa$ for all $\alpha < \kappa$. 
Let $\kappa$ denote a regular uncountable cardinal.

**Proposition**

$P(\kappa, \kappa, \sqsupset^*, 1, \{\kappa\}, \kappa)$ entails a $\kappa$-Souslin tree.

**Proposition**

$P(\kappa, \kappa, \chi \sqsupset^*, 1, \{E^{\kappa}_{\geq \chi}\}, \kappa)$ entails a $\chi$-complete $\kappa$-Souslin tree, provided $|\alpha| < \chi < \kappa$ for all $\alpha < \kappa$. 
Sanity check #1

Let $\lambda$ denote an uncountable cardinal.

**Theorem (Jensen, 1972)**

If $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E_\lambda^{\lambda^+})$ holds, then there exists a $\lambda$-complete $\lambda^+$-Souslin tree.
Let $\lambda$ denote an uncountable cardinal.

**Theorem (Jensen, 1972)**

If $\lambda^{<\lambda} = \lambda$ and $\diamondsuit (E_\lambda^{\lambda^+})$ holds, then there exists a $\lambda$-complete $\lambda^+$-Souslin tree.

**Theorem**

$\diamondsuit (E_\lambda^{\lambda^+})$ entails $P(\lambda^+, 2, \lambda \subseteq, \{E_\lambda^{\lambda^+}\})$. 
Let $\lambda$ denote an uncountable cardinal.

**Theorem (Jensen, 1972)**

If $\lambda^\lambda = \lambda$ and $\Diamond (E^{\lambda^+}_\lambda)$ holds, then there exists a $\lambda$-complete $\lambda^+$-Souslin tree.

**Theorem**

$\Diamond (E^{\lambda^+}_\lambda)$ entails $P(\lambda^+, 2, \lambda \subseteq, \{E^{\lambda^+}_\lambda\})$.

**Corollary**

If $\lambda^\lambda = \lambda$ and $\Diamond (E^{\lambda^+}_\lambda)$ holds, then there exists a $\lambda$-complete $\lambda^+$-Souslin tree.
Let $\lambda$ denote an uncountable cardinal.

**Theorem (Jensen, 1972)**

*If there exists $S \subseteq \lambda^+$ for which $\square_{\lambda}(S) + \diamond(S)$ holds, then there exists a $\lambda^+$-Souslin tree.*
Let $\lambda$ denote an uncountable cardinal.

**Theorem (Jensen, 1972)**

*If there exists $S \subseteq \lambda^+$ for which $\Box_\lambda(S) + \Diamond(S)$ holds, then there exists a $\lambda^+$-Souslin tree.*

**Theorem**

$\Box_\lambda + \text{CH}_\lambda$ entails $P(\lambda^+, 2, \sqsubseteq, \{E_{\geq \theta}^{\lambda^+} \mid \theta < \lambda\})$. 
Sanity check #2

Let $\lambda$ denote an uncountable cardinal.

**Theorem (Jensen, 1972)**

*If there exists $S \subseteq \lambda^+$ for which $\Box_\lambda(S) + \Diamond(S)$ holds, then there exists a $\lambda^+$-Souslin tree.*

**Theorem**

$\Box_\lambda + \text{CH}_\lambda$ entails $P(\lambda^+, 2, \subseteq, \{E_{\geq \theta}^{\lambda^+} | \theta < \lambda\}).$

**Corollary**

*If $\Box_\lambda + \text{CH}_\lambda$ holds, then for every $\chi < \lambda$ with $\lambda^{<\chi} = \lambda$, there exists a $\chi$-complete $\lambda^+$-Souslin tree.*
Let $\lambda$ denote an uncountable cardinal.

**Theorem (Gregory, 1976)**

If $\lambda^{<\lambda} = \lambda$, $2^\lambda = \lambda^+$ and exists a nonreflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then there exists a $\lambda^+$-Souslin tree.
Let $\lambda$ denote an uncountable cardinal.

Theorem (Gregory, 1976)

If $\lambda^{<\lambda} = \lambda$, $2^\lambda = \lambda^+$ and exists a nonreflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then there exists a $\lambda^+$-Souslin tree.

Theorem

If $2^\lambda = \lambda^+$ and there exists a nonreflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then $P(\lambda^+, 2, \lambda \sqsubset^*, \{ E^{\lambda^+}_\lambda \})$ holds.
Sanity check #3

Let $\lambda$ denote an uncountable cardinal.

**Theorem (Gregory, 1976)**

If $\lambda^{<\lambda} = \lambda$, $2^\lambda = \lambda^+$ and there exists a nonreflecting stationary subset of $E_{<\lambda}^{\lambda^+}$, then there exists a $\lambda^+$-Souslin tree.

**Theorem**

If $2^\lambda = \lambda^+$ and there exists a nonreflecting stationary subset of $E_{<\lambda}^{\lambda^+}$, then $P(\lambda^+, 2, \lambda \subseteq^*, \{ E_{\lambda}^{\lambda^+} \})$ holds.

**Corollary (Kojman-Shelah, 1993)**

If $\lambda^{<\lambda} = \lambda$, $2^\lambda = \lambda^+$ and there exists a nonreflecting stationary subset of $E_{<\lambda}^{\lambda^+}$, then there exists a $\lambda$-complete $\lambda^+$-Souslin tree.
Let $\lambda$ denote an uncountable cardinal.

**Theorem (Shelah, 1984)**

If $2^{\aleph_0} = \aleph_1$, $NS_{\aleph_1}$ is saturated, then there exists an $\aleph_2$-Souslin tree.
Let $\lambda$ denote an uncountable cardinal.

**Theorem (Shelah, 1984)**

If $2^{\aleph_0} = \aleph_1$, $\text{NS}_{\aleph_1}$ is saturated, then there exists an $\aleph_2$-Souslin tree.

**Theorem**

If $2^{\aleph_1} = \aleph_2$, $\text{NS}_{\aleph_1}$ is saturated, then $P(\aleph_2, 2, \aleph_1 \sqsubseteq \ast, \{ E_{\aleph_1}^{\aleph_2} \})$ holds.
Let $\lambda$ denote an uncountable cardinal.

**Theorem (Shelah, 1984)**

*If $2^{\aleph_0} = \aleph_1$, $\text{NS}_{\aleph_1}$ is saturated, then there exists an $\aleph_2$-Souslin tree.*

**Theorem**

*If $2^{\aleph_1} = \aleph_2$, $\text{NS}_{\aleph_1}$ is saturated, then $P(\aleph_2, 2, \aleph_1 \subseteq^*, \{E^{\aleph_2}_{\aleph_1}\})$ holds.*

**Corollary**

*If $2^{\aleph_0} = \aleph_1$, $\text{NS}_{\aleph_1}$ is saturated, then there exists an $\aleph_1$-complete $\aleph_2$-Souslin tree.*
And so on..
And so on..

Okay, so you seem to found a way to redirect all ◊-based constructions of Souslin trees through a single construction. You haven’t yet shown me anything new!
κ-trees which cohere modulo finite

Definition
A subtree $T$ of $<^\kappa \kappa$ is said to be coherent if for all $\delta < \kappa$:

- if $x, y \in T_\delta$, then $\{ \alpha < \delta \mid x(\alpha) \neq y(\alpha) \}$ is finite;
- if $x, y \in \delta \kappa$, and $\{ \alpha < \delta \mid x(\alpha) \neq y(\alpha) \}$ is finite, then $x \in T_\delta$ iff $y \in T_\delta$. 
\( \kappa \)-trees which cohere modulo finite

**Definition**
A subtree \( T \) of \( <^\kappa \kappa \) is said to be coherent if for all \( \delta < \kappa \):

- if \( x, y \in T_\delta \), then \( \{ \alpha < \delta \mid x(\alpha) \neq y(\alpha) \} \) is finite;
- if \( x, y \in \delta \kappa \), and \( \{ \alpha < \delta \mid x(\alpha) \neq y(\alpha) \} \) is finite,
  then \( x \in T_\delta \) iff \( y \in T_\delta \).

**Theorem (Jensen, 1970’s)**
\( \diamondsuit (\mathcal{N}_1) \) entails a coherent \( \mathcal{N}_1 \)-Souslin tree.
κ-trees which cohere modulo finite

Definition
A subtree $T$ of $<^\kappa \kappa$ is said to be coherent if for all $\delta < \kappa$:

- if $x, y \in T_\delta$, then $\{\alpha < \delta \mid x(\alpha) \neq y(\alpha)\}$ is finite;
- if $x, y \in \delta \kappa$, and $\{\alpha < \delta \mid x(\alpha) \neq y(\alpha)\}$ is finite, then $x \in T_\delta$ iff $y \in T_\delta$.

Theorem (Jensen, 1970’s)
$\Diamond (\aleph_1)$ entails a coherent $\aleph_1$-Souslin tree.

Theorem (Veličković, 1986)
$\Box \aleph_1$ entails a coherent $\aleph_2$-Souslin tree.
κ-trees which cohere modulo finite

In my talk at “Young Set Theory 2011” workshop, I asked about the consistency of a coherent $\lambda^+\text{-Souslin tree}$ for $\lambda$ singular.

Theorem (Jensen, 1970’s)
$\diamondsuit(\aleph_1)$ entails a coherent $\aleph_1\text{-Souslin tree}$.

Theorem (Veličković, 1986)
$\square\aleph_1$ entails a coherent $\aleph_2\text{-Souslin tree}$. 
κ-trees which cohere modulo finite

In my talk at “Young Set Theory 2011” workshop, I asked about the consistency of a coherent \( \lambda^+ \)-Souslin tree for \( \lambda \) singular.

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Theorem (Veličković, 1986)
\[ \Box \mathcal{N}_1 \] entails a coherent \( \mathcal{N}_2 \)-Souslin tree.

Theorem
\[ P(\kappa, 2, \sqsubseteq, \kappa) \] entails a coherent \( \kappa \)-Souslin tree.
\( \kappa \)-trees which cohere modulo finite

Theorem

\begin{itemize}
  \item \( \Diamond (\aleph_1) \) entails \( P(\aleph_1, 2, \subseteq, \aleph_1) \)
\end{itemize}

Theorem (Jensen, 1970’s)

\( \Diamond (\aleph_1) \) entails a coherent \( \aleph_1 \)-Souslin tree.

Theorem (Veličković, 1986)

\( \Box \aleph_1 \) entails a coherent \( \aleph_2 \)-Souslin tree.

Theorem

\( P(\kappa, 2, \subseteq, \kappa) \) entails a coherent \( \kappa \)-Souslin tree.
$\kappa$-trees which cohere modulo finite

Theorem

- $\Diamond (\aleph_1)$ entails $P(\aleph_1, 2, \subseteq, \aleph_1)$
- $\Box \lambda$ entails $P(\lambda^+, 2, \subseteq, \lambda^+)$

Theorem (Veličković, 1986)

$\Diamond \aleph_1$ entails a coherent $\aleph_2$-Souslin tree.

Theorem

$P(\kappa, 2, \subseteq, \kappa)$ entails a coherent $\kappa$-Souslin tree.
κ-trees which cohere modulo finite

Theorem

- ◊(ℵ₁) entails $P(ℵ₁, 2, ⊆, ℵ₁)$
- □λ entails $P(λ^+, 2, ⊆, λ^+)$
- □λ + CHλ entails $P(λ^+, 2, ⊆, λ^+)$ for λ singular

Theorem

$P(κ, 2, ⊆, κ)$ entails a coherent κ-Souslin tree.
$\kappa$-trees which cohere modulo finite

**Theorem**

- $\diamondsuit(\aleph_1)$ entails $P(\aleph_1, 2, \sqsubseteq, \aleph_1)$
- $\Box\lambda$ entails $P(\lambda^+, 2, \sqsubseteq, \lambda^+)$
- $\square\lambda + CH\lambda$ entails $P(\lambda^+, 2, \sqsubseteq, \lambda^+)$ for $\lambda$ singular

**Corollary**

*If $\square\lambda + CH\lambda$ holds for $\lambda$ singular, then there exists a coherent $\lambda^+$-Souslin tree.*

**Theorem**

$P(\kappa, 2, \sqsubseteq, \kappa)$ entails a coherent $\kappa$-Souslin tree.
\( \kappa \)-trees which cohere modulo finite

Theorem

- \( \diamondsuit (\aleph_1) \) entails \( P(\aleph_1, 2, \sqsubseteq, \aleph_1) \)
- \( \Box \lambda \) entails \( P(\lambda^+, 2, \sqsubseteq, \lambda^+) \)
- \( \Box \lambda + \text{CH}_\lambda \) entails \( P(\lambda^+, 2, \sqsubseteq, \lambda^+) \) for \( \lambda \) singular
- \( V = L \) entails \( P(\kappa, 2, \sqsubseteq, \kappa) \) for all regular uncountable \( \kappa \) which is not weakly compact

Corollary

If \( \Box \lambda + \text{CH}_\lambda \) holds for \( \lambda \) singular, then there exists a coherent \( \lambda^+ \)-Souslin tree.

Theorem

\( P(\kappa, 2, \sqsubseteq, \kappa) \) entails a coherent \( \kappa \)-Souslin tree.
κ-trees which cohere modulo finite

Theorem

- ♦(ℵ_1) entails P(ℵ_1, 2, ⊆, ℵ_1)
- □_λ entails P(λ^+, 2, ⊆, λ^+)
- □_λ + CH_λ entails P(λ^+, 2, ⊆, λ^+) for λ singular
- V = L entails P(κ, 2, ⊆, κ) for all regular uncountable κ which is not weakly compact

Corollary

If □_λ + CH_λ holds for λ singular, then there exists a coherent λ^+-Souslin tree.

Corollary

If V = L, then any regular uncountable κ is not weakly compact iff there exists a coherent κ-Souslin tree.
A concept of “being productive” for Souslin trees

Definition
A $\kappa$-Souslin tree $T$ is free, if for every nonzero $n < \omega$ and any sequence of distinct nodes $\langle t_i \mid i < n \rangle$ from a fixed level $\delta < \kappa$, the product tree of the upper cones $\bigotimes_{i < n} t_i^{\uparrow}$ is again $\kappa$-Souslin.
A concept of “being productive” for Souslin trees

Definition
A $\kappa$-Souslin tree $T$ is free, if for every nonzero $n < \omega$ and any sequence of distinct nodes $\langle t_i \mid i < n \rangle$ from a fixed level $\delta < \kappa$, the product tree of the upper cones $\bigotimes_{i < n} t_i^\uparrow$ is again $\kappa$-Souslin.

Theorem (Jensen, 1970’s)
$\Diamond(\aleph_1)$ entails a free $\aleph_1$-Souslin tree.
A concept of “being productive” for Souslin trees

Definition
A \( \kappa \)-Souslin tree \( T \) is free, if for every nonzero \( n < \omega \) and any sequence of distinct nodes \( \langle t_i \mid i < n \rangle \) from a fixed level \( \delta < \kappa \), the product tree of the upper cones \( \prod_{i < n} t_i \uparrow \) is again \( \kappa \)-Souslin.

Theorem (Jensen, 1970’s)
\( \Diamond (\mathbb{N}_1) \) entails a free \( \aleph_1 \)-Souslin tree.

Jensen construct the levels of the tree by recursion, where the nodes of limit level \( \alpha \) are obtained by forcing with finite conditions over some countable elementary submodel that knows about the diamond sequence and the tree constructed so far.
A concept of “being productive” for Souslin trees

Definition
A $\kappa$-Souslin tree $T$ is free, if for every nonzero $n < \omega$ and any sequence of distinct nodes $\langle t_i \mid i < n \rangle$ from a fixed level $\delta < \kappa$, the product tree of the upper cones $\bigotimes_{i < n} t_i \uparrow$ is again $\kappa$-Souslin.

Theorem (Jensen, 1970’s)
$\diamondsuit(\aleph_1)$ entails a free $\aleph_1$-Souslin tree.

Jensen construct the levels of the tree by recursion, where the nodes of limit level $\alpha$ are obtained by forcing with finite conditions over some countable elementary submodel that knows about the diamond sequence and the tree constructed so far. Genericity entails freeness.
A concept of “being productive” for Souslin trees

Definition
A $\kappa$-Souslin tree $T$ is free, if for every nonzero $n < \omega$ and any sequence of distinct nodes $\langle t_i \mid i < n \rangle$ from a fixed level $\delta < \kappa$, the product tree of the upper cones $\bigotimes_{i < n} t_i^{\uparrow}$ is again $\kappa$-Souslin.

Observation
$\text{CH} + \diamondsuit(E_{\aleph_1}^{\aleph_2})$ entails a free $\aleph_2$-Souslin tree.
A concept of “being productive” for Souslin trees

Definition
A $\kappa$-Souslin tree $T$ is free, if for every nonzero $n < \omega$ and any sequence of distinct nodes $\langle t_i \mid i < n \rangle$ from a fixed level $\delta < \kappa$, the product tree of the upper cones $\bigotimes_{i<n} t_i^{\uparrow}$ is again $\kappa$-Souslin.

Observation
$\text{CH} \vdash \diamondsuit(E_{\aleph_1}^{\aleph_2})$ entails a free $\aleph_2$-Souslin tree.

Construct a $\aleph_1$-complete tree by recursion, where the nodes of level $\alpha$ of uncountable cofinality are obtained by forcing with countable conditions over some $\aleph_1$-sized elementary submodel that knows about anything relevant.
A concept of “being productive” for Souslin trees

Definition
A $\kappa$-Souslin tree $T$ is free, if for every nonzero $n < \omega$ and any sequence of distinct nodes $\langle t_i \mid i < n \rangle$ from a fixed level $\delta < \kappa$, the product tree of the upper cones $\bigotimes_{i<n} t_i \uparrow$ is again $\kappa$-Souslin.

Observation
$\text{CH} + \Diamond(E^{\aleph_2}_{\aleph_1})$ entails a free $\aleph_2$-Souslin tree.

Construct a $\aleph_1$-complete tree by recursion, where the nodes of level $\alpha$ of uncountable cofinality are obtained by forcing with countable conditions over some $\aleph_1$-sized elementary submodel that knows about anything relevant. The model is of size $\aleph_1$ to accompany all relevant dense sets.
A concept of “being productive” for Souslin trees

Definition
A $\kappa$-Souslin tree $T$ is free, if for every nonzero $n < \omega$ and any sequence of distinct nodes $\langle t_i \mid i < n \rangle$ from a fixed level $\delta < \kappa$, the product tree of the upper cones $\bigotimes_{i<n} t_i^{\uparrow}$ is again $\kappa$-Souslin.

Observation
$\text{CH} + \Diamond(E_{\aleph_1}^{\aleph_2})$ entails a free $\aleph_2$-Souslin tree.

Construct a $\aleph_1$-complete tree by recursion, where the nodes of level $\alpha$ of uncountable cofinality are obtained by forcing with countable conditions over some $\aleph_1$-sized elementary submodel that knows about anything relevant.
The model is of size $\aleph_1$ to accompany all relevant dense sets. The $\aleph_1$-completeness of the tree and the countable conditions are necessary for the existence of a generic over the $\aleph_1$-sized model.
A concept of “being productive” for Souslin trees

Question
How about free $\lambda^+$-Souslin tree for $\lambda$ singular?

Observation
$\text{CH} + \diamondsuit(\mathcal{E}_{\aleph_1}^{\aleph_2})$ entails a free $\aleph_2$-Souslin tree.

Construct a $\aleph_1$-complete tree by recursion, where the nodes of level $\alpha$ of uncountable cofinality are obtained by forcing with countable conditions over some $\aleph_1$-sized elementary submodel that knows about anything relevant. The model is of size $\aleph_1$ to accompany all relevant dense sets. The $\aleph_1$-completeness of the tree and the countable conditions are necessary for the existence of a generic over the $\aleph_1$-sized model.
A concept of “being productive” for Souslin trees

Question

How about free $\lambda^+$-Souslin tree for $\lambda$ singular? Freeness requires that the generic meet $\lambda$ many dense sets, but the tree cannot be $\lambda$-complete, and there cannot be a generic for the relevant poset over a model of size $\lambda$. 
A concept of “being productive” for Souslin trees

Question

How about free $\lambda^+$-Souslin tree for $\lambda$ singular? Freeness requires that the generic meet $\lambda$ many dense sets, but the tree cannot be $\lambda$-complete, and there cannot be a generic for the relevant poset over a model of size $\lambda$. But, there is another way:

Theorem

$P(\kappa, \mu, \subseteq, \kappa)$ entails a $\mu$-slim, free $\kappa$-Souslin tree.
A concept of “being productive” for Souslin trees

Theorem

\[ P(\kappa, \mu, \subseteq, \kappa) \text{ entails a } \mu\text{-slim, free } \kappa\text{-Souslin tree.} \]

Corollary

If \( \Box \lambda + \text{CH}_\lambda \) holds for \( \lambda \) singular, then there exists a free \( \lambda^+\text{-Souslin tree.} \)
A concept of “being productive” for Souslin trees

Corollary

If $\square^+ \lambda + \text{CH}_{\lambda}$ holds for $\lambda$ singular, then there exists a free $\lambda^+$-Souslin tree.

Corollary

If $V = L$, then any regular uncountable $\kappa$ is not weakly compact iff there exists a free $\kappa$-Souslin tree.
A concept of “being productive” for Souslin trees

Definition
A $\kappa$-Souslin tree $T$ is $\chi$-free, if for every nonzero $\nu < \chi$ and any sequence of distinct nodes $\langle t_i \mid i < \nu \rangle$ from a fixed level $\delta < \kappa$, the product tree of the upper cones $\bigotimes_{i < \nu} t_i^{\uparrow}$ is again $\kappa$-Souslin.
A concept of “being productive” for Souslin trees

Definition
A $\kappa$-Souslin tree $T$ is $\chi$-free, if for every nonzero $\nu < \chi$ and any sequence of distinct nodes $\langle t_i \mid i < \nu \rangle$ from a fixed level $\delta < \kappa$, the product tree of the upper cones $\bigotimes_{i < \nu} t_i \uparrow$ is again $\kappa$-Souslin.

From GCH-type assumption, we can also construct $\chi$-free trees for uncountable $\chi$. For instance:

Corollary
If $\square_\lambda + CH_\lambda$ holds for $\lambda$ singular, then there exists a $\log_\lambda(\lambda^+)$-free $\lambda^+$-Souslin tree.
Calibrating

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$P(\kappa, 2, \sqsubseteq, \kappa)$ entails a coherent $\kappa$-Souslin tree.
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Theorem

$P(\kappa, \kappa, \sqsubseteq^*, 1)$ entails a $\kappa$-Souslin tree.
Specializable Souslin trees
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Recall (implicit in David, 1990)

If $V = L$, then for every regular $\lambda$, there exists a $\lambda^+$-Souslin tree which is specializable.
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Proposition
If $\lambda^{<\lambda} = \lambda$, then any coherent $\lambda$-splitting $\lambda^+$-Souslin tree is specializable.
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If $\lambda^{< \lambda} = \lambda$, then any coherent $\lambda$-splitting $\lambda^+$-Souslin tree is specializable.

Corollary
If $\lambda^{< \lambda} = \lambda$, $P(\lambda^+, 2, \subseteq, \lambda^+)$ entails a specializable $\lambda^+$-Souslin tree.
Specializable Souslin trees

Theorem
If $\lambda^{<\lambda} = \lambda$, $P(\lambda^+, \lambda^+, \lambda \sqsubseteq^*, 1, \{ E^\lambda_\lambda \}, \lambda^+, 1, 1)$ entails a $\lambda$-complete, specializable $\lambda^+$-Souslin tree.

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If $2^\lambda = \lambda^+$ and there exists a nonreflecting stationary subset of $E^\lambda_\lambda$, then $P(\lambda^+, 2, \lambda \subseteq^*, \{E^\lambda_\lambda\})$ holds.
Theorem
If $\lambda < \lambda = \lambda$, $P(\lambda^+, \lambda^+, \lambda \subseteq^*, 1, \{ E_{\lambda}^{\lambda^+} \}, \lambda^+, 1, 1)$ entails a $\lambda$-complete, specializable $\lambda^+$-Souslin tree.

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If $\lambda < \lambda = \lambda$, $2^\lambda = \lambda^+$ and there exists a nonreflecting stationary subset of $E_{<\lambda}^{\lambda^+}$, then there exists a $\lambda$-complete, specializable $\lambda^+$-Souslin tree.
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non-Specializable Souslin trees

Let \( \chi < \lambda \) denote infinite cardinals.

**Theorem**

\( P(\lambda^+, 2, \sqsubseteq_{\chi}, 1, \{\lambda^+\}, 2, \omega) \) entails a *non-Specializable \( \lambda^+ \)-Souslin tree.*
Let $\chi \prec \lambda$ denote infinite cardinals.

**Theorem**

$P(\lambda^+, 2, \sqsubseteq_\chi, 1, \{E^{\lambda^+}_{\geq \kappa}\}, 2, \omega)$ entails a non-Specializable $\lambda^+$-Souslin tree, which is $\kappa$-complete, provided that $\lambda^\kappa = \lambda$. 

non-Specializable Souslin trees
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This covers the Baumgartner and Cummings constructions from $GCH + \Box_\lambda$ and $\lozenge_\lambda$. 
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This covers the Baumgartner and Cummings constructions from $\text{GCH} + \Box_\lambda$ and $\lozenge_\lambda$. In addition, the relation $\sqsubseteq_\chi$ is weak enough to make $P(\lambda^+, 2, \sqsubseteq_\chi, 1, \{\lambda^+\}, 2, \omega)$ consistent with large cardinals that refute $\Box^*_\lambda$. 
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This covers the Baumgartner and Cummings constructions from GCH + $\square_\lambda$ and $\diamond_\lambda$. In addition, the relation $\subseteq \chi$ is weak enough to make $P(\lambda^+, 2, \subseteq \chi, 1, \{\lambda^+\}, 2, \omega)$ consistent with large cardinals that refute $\square^*_\lambda$. Thereby, covering a seemingly unrelated scenario of Shelah and Ben-David.

**A model of “all Aronszajn trees are nonspecial”**

It is consistent that $\kappa$ is supercompact, $\lambda = \kappa^+\omega$, and there exists a non-Specializable $\lambda^+$-Souslin tree.
non-Specializable Souslin trees

Let $\chi < \lambda$ denote infinite cardinals.

**Theorem**

$P(\lambda^+, 2, \sqsubseteq_\chi, 1, \{E^\lambda_{\geq \kappa}\}, 2, \omega)$ entails a *non-Specializable* $\lambda^+$-Souslin tree, which is $\kappa$-complete, provided that $\lambda^{<\kappa} = \lambda$.

**Theorem**

$P(\lambda^+, 2, \sqsubseteq_\chi, \lambda^+, \{\lambda^+\}, 2, \omega)$ entails a *free, non-Specializable* $\lambda^+$-Souslin tree.
Let $\chi < \lambda$ denote infinite cardinals.

**Theorem**

$P(\lambda^+, 2, \succeq_\chi, 1, \{E^{\lambda^+}_{\geq \kappa}\}, 2, \omega)$ entails a **non-Specializable** $\lambda^+$-Souslin tree, which is $\kappa$-complete, provided that $\lambda^{<\kappa} = \lambda$.

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$P(\lambda^+, 2, \preceq_\chi, \lambda^+, \{ E^{\lambda^+}_{\geq \kappa} \}, 2, \omega)$ entails a free, non-Specializable $\lambda^+$-Souslin tree, which is $\kappa$-complete, provided that $\lambda^{<\kappa} = \lambda$.

A model of “all Aronszajn trees are nonspecial”
It is consistent that $\kappa$ is supercompact, $\lambda = \kappa^{+\omega}$, and there exists a free, non-Specializable $\lambda^+$-Souslin tree.
Some more
Recall (Gregory, 1976)

If $\lambda^{<\lambda} = \lambda$, $\text{CH}_\lambda$ and there exists a nonreflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then there exists a $\lambda^+$-Souslin tree.

**Theorem**

If $2^{<\lambda} = \lambda$, $\text{CH}_\lambda + \square^*_\lambda$ and exists a nonreflecting stationary subset of $E^{\lambda^+}_{\neq \text{cf}(\lambda)}$, then $P(\lambda^+, \lambda^+, \sqsubseteq)$ holds.
Generalizing Gregory’s theorem to singular cardinals

Recall (Gregory, 1976)

If $\lambda^{<\lambda} = \lambda$, CH$\lambda$ and there exists a nonreflecting stationary subset of $E_{<\lambda}^{\lambda^+}$, then there exists a $\lambda^+$-Souslin tree.

Theorem

If $2^{<\lambda} = \lambda$, CH$\lambda + \Box^*$ and exists a nonreflecting stationary subset of $E_{<\lambda}^{\lambda^+}$, then $P(\lambda^+, \lambda^+, \subseteq)$ holds.

Theorem (Cummings-Foreman-Magidor, 2001)

After Prikry forcing over a supercompact cardinal $\lambda$, $\Box^*_\lambda$ holds, yet, any stationary subset of $E_{<\lambda}^{\lambda^+}$ reflects.
Generalizing Gregory’s theorem to singular cardinals

Recall (Gregory, 1976)
If $\lambda^{<\lambda} = \lambda$, $CH_{\lambda}$ and there exists a nonreflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then there exists a $\lambda^+$-Souslin tree.

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If $2^{<\lambda} = \lambda$, $CH_{\lambda} + \Box^*_{\lambda}$ and exists a nonreflecting stationary subset of $E^{\lambda^+}_{\neq cf(\lambda)}$, then $P(\lambda^+, \lambda^+, \subseteq)$ holds.

Theorem (Cummings-Foreman-Magidor, 2001)
After Prikry forcing over a supercompact cardinal $\lambda$, $\Box^*_{\lambda}$ holds, yet, any stationary subset of $E^{\lambda^+}_{\neq cf(\lambda)}$ reflects.

Theorem
After Prikry forcing over a measurable cardinal $\lambda$ satisfying $CH_{\lambda}$, $P(\lambda^+, \lambda^+, \subseteq, \lambda^+)$ holds.
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If $2^{<\lambda} = \lambda$, $\text{CH}_{\lambda} + \square^*_\lambda$ and exists a nonreflecting stationary subset of $E^{\lambda^+}_{\neq \text{cf}(\lambda)}$, then $P(\lambda^+, \lambda^+, \sqsubseteq)$ holds.

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After Prikry forcing over a measurable cardinal $\lambda$ satisfying $\text{CH}_\lambda$, $P(\lambda^+, \lambda^+, \sqsubseteq, \lambda^+)$ holds.
Generalizing Gregory’s theorem to singular cardinals

The derived trees

- $P(\lambda^+, \lambda^+, \sqsubseteq)$ entails a rigid $\lambda^+$-Souslin tree;
- $P(\lambda^+, \lambda^+, \sqsubseteq, \lambda^+)$ entails a free $\lambda^+$-Souslin tree;
- $P(\lambda^+, \lambda^+, \sqsubseteq, \lambda^+)$ entails an homogeneous $\lambda^+$-Souslin tree.

Theorem

If $2^{<\lambda} = \lambda$, $\text{CH}_\lambda + \Box^*_\lambda$ and exists a nonreflecting stationary subset of $E_{\lambda^+, \neq}^{\lambda^+}$, then $P(\lambda^+, \lambda^+, \sqsubseteq)$ holds.

Theorem

After Prikry forcing over a measurable cardinal $\lambda$ satisfying $\text{CH}_\lambda$, $P(\lambda^+, \lambda^+, \sqsubseteq, \lambda^+)$ holds.
More results

Let $\lambda^{<\lambda} = \lambda$ denote a regular uncountable cardinal.

- If $\text{CH}_\lambda$, then adding a single $\lambda$-Cohen set entails $P(\lambda^+, \lambda^+, \subseteq, \lambda^+, \{E^\lambda_{\lambda^+}\})$, and hence free/homogeneous/specializable trees.
More results

Let $\lambda^{<\lambda} = \lambda$ denote a regular uncountable cardinal.

- If $\text{CH}_\lambda$, then adding a single $\lambda$-Cohen set entails $P(\lambda^+, \lambda^+, \sqsubseteq, \lambda^+, \{E_\lambda^{\lambda^+}\})$, and hence free/homogeneous/specializable trees.

- If $\Box\lambda + \text{CH}_\lambda$, then a single $\lambda$-Cohen set entails $P(\lambda^+, 2, \sqsubseteq, \lambda^+, \{E_\lambda^{\lambda^+}\}, 2, \omega)$, and hence free/coherent/specializable/non-specializable trees.
Let $\lambda^{<\lambda} = \lambda$ denote a regular uncountable cardinal.

- If $\text{CH}_\lambda$, then adding a single $\lambda$-Cohen set entails $P(\lambda^+, \lambda^+, \subseteq, \lambda^+, \{E^\lambda_\lambda\})$, and hence free/homogeneous/specializable trees.

- If $\square_\lambda + \text{CH}_\lambda$, then a single $\lambda$-Cohen set entails $P(\lambda^+, 2, \subseteq, \lambda^+, \{E^\lambda_\lambda\}, 2, \omega)$, and hence free/coherent/specializable/non-specializable trees.

- If $\square_\lambda + \diamondsuit^*(\lambda^+)$, then there exists a (free) $\lambda^+$-Souslin tree $T$, whose $\omega$-reduced power tree $\omega T/\mathcal{U}$ is $\lambda^+$-Kurepa for any nonprincipal ultrafilter $\mathcal{U}$ over $\omega$. 
The microscopic approach
Diamond for $H_\kappa$

Recall that $P(\kappa, \cdots)$ asserts that $\diamondsuit(\kappa) + P^-(\kappa, \cdots)$ holds.
Diamond for $H_\kappa$

Recall that $P(\kappa, \cdots)$ asserts that $\Diamond(\kappa) + P^-(\kappa, \cdots)$ holds.

**Proposition**

For $\kappa$ regular uncountable, $\Diamond(\kappa)$ iff $\Diamond(H_\kappa)$.

**Definition**

$\Diamond(H_\kappa)$ asserts the existence of $\varphi_0 : \kappa \to H_\kappa$ and $\varphi_1 : \kappa \to H_\kappa$ as follows. For every $a \in H_\kappa$, $A \subseteq H_\kappa$, and $p \in H_{\kappa^{++}}$, there exists an elementary submodel $M \prec H_{\kappa^{++}}$ such that:

- $p \in M$;
- $M \cap \kappa \in \kappa$;
- $\varphi_0(M \cap \kappa) = a$;
- $\varphi_1(M \cap \kappa) = M \cap A$. 
Diamond for $H_\kappa$
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A construction à la microscopic approach

#include <NormalTree.h>
#include <SealAntichain.h>
#include <SealAutomorphism.h>
#include <SealProductTree.h>