Quotients of strongly proper posets, and related topics

Sean Cox

Virginia Commonwealth University
scox9@vcu.edu

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Joint work with John Krueger.
A conjecture of Viale-Weiss

The principle ISP(\(\omega_2\)):

- introduced by Weiss
- follows from PFA (Viale-Weiss), and many consequences of PFA factor through ISP(\(\omega_2\)).

Conjecture (Viale-Weiss): ISP(\(\omega_2\)) is consistent with large continuum (i.e. \(> \omega_2\)).
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- introduced by Weiss
- follows from PFA (Viale-Weiss), and many consequences of PFA factor through ISP($\omega_2$).
- **Conjecture (Viale-Weiss):** ISP($\omega_2$) is consistent with large continuum (i.e. $> \omega_2$).

**Theorem (C.-Krueger 2014)**

*Proved the conjecture of Viale-Weiss. Developed general theory of quotients of strongly proper forcings.*
Outline

1 Approximation property and guessing models

2 Strongly proper forcings and their quotients

3 an application: the Viale-Weiss conjecture

4 Specialized guessing models, and a question
Definition (Hamkins)
Let \((W, W')\) be transitive models of set theory such that:

- \(W \subseteq W'\)
- \(\mu\) is regular in \(W\)

We say \((W, W')\) has the \(\mu\)-approximation property iff whenever:

1. \(X \in W'\);
2. \(X\) is a bounded subset of \(W\);
3. \(\forall z \in W \mid z^W < \mu \implies z \cap X \in W\)

then \(X \in W\).
Approximation property

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then \(X \in W\).

We will focus on the case \(\mu = \omega_1\) throughout this talk.
The class $G_{\omega_1}$

**Definition (Viale-Weiss)**

$M$ is $\omega_1$-guessing, denoted $M \in G_{\omega_1}$, iff $|M| = \omega_1 \subset M$ and $(H_M, V)$ has the $\omega_1$-approximation property (where $H_M$ is transitive collapse of $M$).

**Definition (Viale-Weiss)**

$\text{ISP}(\omega_2)$ is the statement: for all regular $\theta \geq \omega_2$:

$$G_{\omega_1} \cap P_{\omega_2}(H_\theta) \text{ is stationary}$$
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The Proper Forcing Axiom (PFA) implies $\text{ISP}(\omega_2)$. 
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**Theorem (Viale-Weiss)**

The Proper Forcing Axiom (PFA) implies $ISP(\omega_2)$.

**Generalization of theorems of Baumgartner, Krueger**
Consequences of PFA that factor through ISP

- $TP(\omega_2)$
- Every tree of height and size $\omega_1$ has at most $\omega_1$ many cofinal branches (in particular no Kurepa trees)
  - together with $2^{\omega_1} = \omega_2$ this yields $\diamondsuit^+(S^2_1)$ (Foreman-Magidor)
- Failure of $\square(\theta)$ for all $\theta \geq \omega_2$ (Weiss; actually failure of weaker forms of square)
- SCH (Viale)
- IA$^{\omega_1} \neq * \text{ Unif}_{\omega_1}$ and stronger separations (Krueger)
- Laver Diamond at $\omega_2$ (Viale from PFA, Cox from ISP plus $2^\omega = \omega_2$)
Consequences of PFA that factor through ISP

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- Every tree of height and size $\omega_1$ has at most $\omega_1$ many cofinal branches (in particular no Kurepa trees)
  - together with $2^{\omega_1} = \omega_2$ this yields $\lozenge^+(S^2_1)$ (Foreman-Magidor)
- Failure of $\Box(\theta)$ for all $\theta \geq \omega_2$ (Weiss; actually failure of weaker forms of square)
- SCH (Viale)
- $\text{IA}_{\omega_1} \not\equiv \text{Unif}_{\omega_1}$ and stronger separations (Krueger)
- Laver Diamond at $\omega_2$ (Viale from PFA, Cox from ISP plus $2^\omega = \omega_2$)

Even more consequences of PFA factor through “specialized” ISP; more on that later.
Let $T$ be a tree of height $\omega_2$ and width $< \omega_2$. By stationarity of $G_{\omega_1}$ there is an $M \in G_{\omega_1}$ such that $M \prec (H_{\omega_3}, \in, T)$. Let $\sigma : H_M \rightarrow M \prec H_{\omega_3}$ be inverse of collapsing map of $M$; let

$$\alpha := M \cap \omega_2 = \text{crit}(\sigma)$$

and $T_M := \sigma^{-1}(T)$

Our goal is to prove that $H_M \models "T_M \text{ has a cofinal branch}"$. 

Example: $ISP(\omega_2)$ implies $TP(\omega_2)$

Let $T$ be a tree of height $\omega_2$ and width $< \omega_2$. By stationarity of $G_{\omega_1}$ there is an $M \in G_{\omega_1}$ such that $M \prec (H_{\omega_3}, \in, T)$. Let $\sigma : H_M \to M \prec H_{\omega_3}$ be inverse of collapsing map of $M$; let

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Our goal is to prove that $H_M \models \text{“} T_M \text{ has a cofinal branch”}$.

Since $(H_M, V)$ has the $\omega_1$-approximation property, it suffices to find (in $V$) a cofinal $b$ through $T_M$ such that every proper initial segment of $b$ is an element of $H_M$. But since $T$ is thin, then $T_M = T|\alpha$. Pick any $t$ on the $\alpha$-th level of $T$; then $t \downarrow$ is a cofinal branch through $T_M = T|\alpha$ and every proper initial segment is of course in $H_M$. 
1. Approximation property and guessing models

2. Strongly proper forcings and their quotients

3. an application: the Viale-Weiss conjecture

4. Specialized guessing models, and a question
A suborder $\mathbb{P}$ of $\mathbb{Q}$ is regular iff maximal antichains in $\mathbb{P}$ remain maximal antichains in $\mathbb{Q}$. 
A suborder \( P \) of \( Q \) is regular iff maximal antichains in \( P \) remain maximal antichains in \( Q \).

**Definition**

Suppose \( P \) is a regular suborder of \( Q \) and \( G_P \) is \( P \)-generic. In \( V[G_P] \) the (possibly nonseparable) quotient \( Q/G_P \) is the set of \( q \in Q \) which are compatible with every member of \( G_P \). Order is inherited from \( Q \).

\[
Q \sim P \ast \dot{Q}/\dot{G}_P
\]
A suborder $\mathbb{P}$ of $\mathbb{Q}$ is \textit{regular} iff maximal antichains in $\mathbb{P}$ remain maximal antichains in $\mathbb{Q}$.

**Definition**

Suppose $\mathbb{P}$ is a regular suborder of $\mathbb{Q}$ and $G_\mathbb{P}$ is $\mathbb{P}$-generic. In $V[G_\mathbb{P}]$ the (possibly nonseparative) quotient $\mathbb{Q}/G_\mathbb{P}$ is the set of $q \in \mathbb{Q}$ which are compatible with every member of $G_\mathbb{P}$. Order is inherited from $\mathbb{Q}$.

$$\mathbb{Q} \sim \mathbb{P} \ast \check{\mathbb{Q}}/\check{G}_\mathbb{P}$$

**Important variation:** “$\mathbb{P}$ is regular in $\mathbb{Q}$ below $q$”
The following notion is due to Mitchell.

**Definition**

Given a poset $\mathbb{P}$ and a model $M$, a condition $p \in \mathbb{P}$ is an $(M, \mathbb{P})$ strong master condition iff “$M \cap \mathbb{P}$ is a regular suborder of $\mathbb{P}$ below $p$”.

*(we focus only on countable $M$)*
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*(we focus only on countable $M$)*

“$\mathbb{P}$ is strongly proper”: defined similarly to properness, using strong master condition instead of master condition.
Examples and properties of strongly proper forcings

Examples:
- Todorcevic’s finite $\in$-collapse
- Baumgartner’s adding a club with finite conditions
- adding any number of Cohen reals
- Various (pure) side condition posets of Mitchell, Friedman, Neeman, Krueger, and others.

Key properties (Mitchell):
$\text{Add}(\omega, V, V_P)$ has the $\omega_1$-approximation property

Remark: To get $\omega_1$ approx, suffices to be strongly proper wrt stationarily many countable models.
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Remark: To get $\omega_1$ approx, suffices to be strongly proper wrt stationarily many countable models.
Suppose $1_{\mathbb{P}}$ forces that $\dot{b}$ is a **new** subset of $\theta$ and that $z \cap \dot{b} \in V$ for every $V$-countable set $z$. Let $M \prec (H_{\theta^+}, \in, \dot{b}, \ldots)$ be countable and let $p$ be a strong master condition for $M$. Since $M$ is countable then by assumption $\check{M} \cap \dot{b}$ is forced to be in the ground model. Let $p' \leq p$ decide this value.
Suppose $1_p$ forces that $\dot{b}$ is a **new** subset of $\theta$ and that $z \cap \dot{b} \in V$ for every $V$-countable set $z$. Let $M \prec (H_{\theta^+}, \in, \dot{b}, \ldots)$ be countable and let $p$ be a strong master condition for $M$. Since $M$ is countable then by assumption $\check{M} \cap \dot{b}$ is forced to be in the ground model. Let $p' \leq p$ decide this value.

Let $p'|M$ be a **reduct** of $p'$ into $M \cap P$. Since $\dot{b}$ is forced to be new and $\dot{b}, p'|M \in M$, then there are $r, s \in M$ below $p'|M$ which disagree about some member of $M$ being an element of $\dot{b}$. Then clearly they cannot both be compatible with a condition which decides $\check{M} \cap \dot{b}$. In particular they cannot both be compatible with $p'$. Contradiction.
Quotients of strongly proper forcings

Question

Suppose $Q$ is strongly proper and $P$ is a regular suborder. When does the quotient $Q / \dot{G}_P$ have the following properties?

- strongly proper “wrt $V$ models”?
- $\omega_1$-approximation property?
Quotients of strongly proper forcings

**Question**

*Suppose $\mathbb{Q}$ is strongly proper and $\mathbb{P}$ is a regular suborder. When does the quotient $\mathbb{Q}/\dot{G}_\mathbb{P}$ have the following properties?*

- *strongly proper “wrt V models”?*
- *$\omega_1$-approximation property?*

**Remark:** There are well-known examples of quotients of proper forcings that aren’t proper.
From now on we only deal with “well-met” posets: if $p \parallel q$ then they have a GLB.
The star condition

From now on we only deal with “well-met” posets: if \( p \parallel q \) then they have a GLB

**Definition (Krueger)**

Assume \( P \) is a suborder of \( Q \).

\( \star(P, Q) \) denotes the statement: whenever \( p \in P \) and \( q_1, q_2 \in Q \) and \( p, q_1, q_2 \) are pairwise compatible, then there is a lower bound for all three.

\( \star(Q) \) is the stronger statement that \( \star(Q, Q) \) holds.

Examples where \( \star(Q) \) holds:
- \( \text{Col}(\mu, \theta) \)
- Todorcevic’s \( \in \)-collapse
- Krueger’s adequate set forcing
Lemma

Assume \( \star(\mathbb{P}, \mathbb{Q}) \) and let \( G_{\mathbb{P}} \) be generic for \( \mathbb{P} \). Then in \( V[G_{\mathbb{P}}] \):

\[
\left( \forall q_1, q_2 \in \mathbb{Q}/G_{\mathbb{P}} \right) \left( q_1 \parallel_{\mathbb{Q}} q_2 \implies q_1 \parallel_{\mathbb{Q}/G_{\mathbb{P}}} q_2 \right)
\]
Lemma

Assume $\star(\mathbb{P}, \mathbb{Q})$ and let $G_\mathbb{P}$ be generic for $\mathbb{P}$. Then in $V[G_\mathbb{P}]$:

$$(\forall q_1, q_2 \in \mathbb{Q}/G_\mathbb{P}) \ (q_1 \parallel q_2 \implies q_1 \parallel q_2')$$

Proof: let $q_1, q_2 \in \mathbb{Q}/G_\mathbb{P}$ and suppose $q_1 \land q_2 \neq 0$ in $\mathbb{Q}$; we will prove that $q_1 \land q_2 \in \mathbb{Q}/G_\mathbb{P}$, i.e. that $q_1 \land q_2$ is compatible with every member of $G_\mathbb{P}$. Let $p \in G_\mathbb{P}$. Then $q_1 \land p \neq 0 \neq q_2 \land p$. By $\star(\mathbb{P}, \mathbb{Q})$ we have $q_1 \land q_2 \land p \neq 0$. 

\[(\mathbb{P}, \mathbb{Q}) \text{ implies strong master conditions survive in the quotient}\]

**Lemma**

Suppose \(\star(\mathbb{P}, \mathbb{Q})\) holds and \(q\) is \((M, \mathbb{Q})\) strong master condition. Then

\[
\Vdash_{\mathbb{P}} \check{q} \in \mathbb{Q}/\dot{G}_{\mathbb{P}} \implies \check{q} \text{ is } (M[\dot{G}_{\mathbb{P}}], \mathbb{Q}/\dot{G}_{\mathbb{P}}) \text{ s.m.c.}
\]
\((P, Q)\) implies strong master conditions survive in the quotient

**Lemma**

Suppose \(\star(P, Q)\) holds and \(q\) is \((M, Q)\) strong master condition. Then

\[ \vdash_P \exists q \in Q / \dot{G}_P \implies \exists q \text{ is } (M[\dot{G}_P], Q / \dot{G}_P) \text{ s.m.c.} \]

Proof sketch: Suppose \(p \in P\) forces that \(\exists q \in Q / \dot{G}_P\) (i.e. \(\exists q \parallel \dot{G}_P\)). Then \(p\) must force that \(M[\dot{G}_P] \cap V = M\); otherwise there is some \(p' \leq p\) forcing \(M \subset M[\dot{G}_P] \cap V\), but \(p'\) still forces \(\exists q \in Q / \dot{G}_P\). So let \(G_P \ast H\) be generic (in the 2-step iteration) with \((p', q) \in G_P \ast H\). But \(q\) is in particular an \((M, Q)\) master condition, so \(M = M[G_P \ast H] \cap V \supset M[G_P] \cap V\). Contradiction.
Recall $q$ is $(M, \mathcal{Q})$ strong master condition, and we showed that if $q \in \mathcal{Q}/G_P$ then in particular $\mathcal{Q} \cap M = \mathcal{Q} \cap M[G_P] =: \mathcal{Q}_M$. Now $\mathcal{Q}_M$ is regular in $\mathcal{Q}$ below $q$ (this is $\Sigma_0$ statement).

Suppose $q' \leq q$, where $q' \in \mathcal{Q}/G_P$. Let $q'|M$ be a reduct of $q'$ into $\mathcal{Q}_M$. We need to see that:

- $q'|M \parallel G_P$; this is straightforward, especially if $q'|M \geq q'$ as is usually the case; and

- any extension of $q'|M$ in $\mathcal{Q}_M/G_P$ is compatible with $q'$ in $\mathcal{Q}/G_P$. Suppose $q''$ is such a condition; so $q'' \parallel G_P$ and is $\mathcal{Q}$-compatible with $q'$. By the previous lemma (using the $\star(\mathcal{P}, \mathcal{Q})$ assumption), $q'$ and $q''$ are compatible in $\mathcal{Q}/G_P$. 
A sufficient condition

Theorem (C.-Krueger)

Suppose:

- $Q$ is well-met;
- There is a stationary set $S$ of countable models $M$ for which $Q$ has universal strong master conditions;
- $P$ is a regular suborder of $Q$ (possibly “below a condition”);
- $\star(\mathcal{P}, Q)$ holds

Then $P$ forces that $Q/\dot{G}_P$ is strongly proper for the stationary set of models of the form $M[\dot{G}_P]$ where $M \in S$. In particular, the quotient has the $\omega_1$ approximation property.

REMARK: universality isn’t needed if you only want $\omega_1$-approx property.
A sufficient condition

**Theorem (C.-Krueger)**

Suppose:
- \( Q \) is well-met;
- There is a stationary set \( S \) of countable models \( M \) for which \( Q \) has *universal* strong master conditions;
- \( P \) is a regular suborder of \( Q \) (possibly “below a condition”)
- \( *\!( P, Q) \) holds

Then \( P \) forces that \( Q/G_P \) is strongly proper for the stationary set of models of the form \( M[G_P] \) where \( M \in S \). In particular, the quotient has the \( \omega_1 \) approximation property.

**REMARK:** universality isn’t needed if you only want \( \omega_1 \)-approx property.
A counterexample

Quotients of strongly proper posets may fail to have the $\omega_1$-approximation property:

**Theorem (Krueger)**

Assume $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$. Let $Q$ be the forcing with coherent adequate sets of countable submodels of $H_{\omega_3}$. Then $Q$ has the following properties:

- $Q$ is strongly proper and $\omega_2$-cc;
- $Q$ forces CH
- $Q$ adds a Kurepa tree on $\omega_1$ with $\omega_3$ many cofinal branches
- There is a regular suborder $P$ of size $\omega_2$ such that

$$\models_P Q/\dot{G}_P \text{ fails to have the } \omega_1 \text{ approximation property}$$
Outline

1. Approximation property and guessing models
2. Strongly proper forcings and their quotients
3. an application: the Viale-Weiss conjecture
4. Specialized guessing models, and a question
Recall Viale-Weiss:

- proved PFA implies ISP(ω₂);
- conjectured that ISP(ω₂) is consistent with large continuum.
Recall Viale-Weiss:
- proved PFA implies ISP(\(\omega_2\));
- conjectured that ISP(\(\omega_2\)) is consistent with large continuum.

**Theorem (C.-Krueger)**

Assume \(\kappa\) is a supercompact cardinal and \(\theta \geq \kappa\) arbitrary. Let:
- \(\mathbb{P}\) be “adequate set forcing” to turn \(\kappa\) into \(\kappa_2\); (or Neeman’s side condition forcing; or Friedman’s; ...)
- \(\mathbb{Q} = \text{Add}(\omega, \theta)\)

Then \(V^{\mathbb{P} \times \mathbb{Q}} \models \text{ISP}(\omega_2)\) and \(2^\omega = \theta\).
Proof outline

Let $G \times H$ be generic for $\mathbb{P} \times \mathbb{Q}$. Let $\theta \geq \omega_2 = \kappa$ be regular and $\mathcal{A} = (H_\theta[G \times H], \in, \ldots)$ be an algebra.
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$\mathcal{A} = (H_\theta[G \times H], \in, \ldots)$ be an algebra.

Back in $V$ let $j : V \rightarrow N$ be sufficiently supercompact with 
crit$(j) = \kappa$ so that $j[H_\theta] \in N$. $\mathbb{P} \times \mathbb{Q}$ is $\kappa$-cc and crit$(j) = \kappa$, so 
j : $\mathbb{P} \times \mathbb{Q} \rightarrow j(\mathbb{P} \times \mathbb{Q})$ is a regular embedding; so we can force with the quotient 

$$j(\mathbb{P} \times \mathbb{Q})/j[G \times H]$$

and lift $j$ to 

$$j : V[G \times H] \rightarrow N[G' \times H']$$
Proof outline

Let $G \times H$ be generic for $\mathbb{P} \times \mathbb{Q}$. Let $\theta \geq \omega_2 = \kappa$ be regular and $\mathcal{A} = (H_\theta[G \times H], \in, \ldots)$ be an algebra.

Back in $V$ let $j : V \to N$ be sufficiently supercompact with $\text{crit}(j) = \kappa$ so that $j[H_\theta] \in N$. $\mathbb{P} \times \mathbb{Q}$ is $\kappa$-cc and $\text{crit}(j) = \kappa$, so $j : \mathbb{P} \times \mathbb{Q} \to j(\mathbb{P} \times \mathbb{Q})$ is a regular embedding; so we can force with the quotient

$$j(\mathbb{P} \times \mathbb{Q})/j[G \times H]$$

and lift $j$ to

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$N$ believes that $j(\mathbb{P} \times \mathbb{Q})$ is strongly proper and the pair

$$j[\mathbb{P} \times \mathbb{Q}], j(\mathbb{P} \times \mathbb{Q})$$

satisfies the star property. So $N[j[G \times H]]$ believes that the quotient in (1) has the $\omega_1$-approximation property; so $(H^V_\theta[G \times H], N[G' \times H'])$ has $\omega_1$-a.p., and also

$$j[H^V_\theta[G \times H]] \prec j(\mathcal{A})$$. Then use elementarity of $j$. 

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What Viale-Weiss really proved

Definition
Let’s call $M$ a specialized $\omega_1$ guessing model, and write $M \in sG_{\omega_1}$, iff a certain tree related to $M$ is specialized; in particular $M \in G_{\omega_1}$ and remains so in any outer model with the same $\omega_1$.

They proved that under PFA, $sG_{\omega_1} \cap P_{\omega_2}(H_\theta) (\cap IC_{\omega_1})$ is stationary for all $\theta \geq \omega_2$. 
Consequences of PFA which factor through specialized guessing models

- If $T$ is a tree of height and size $\omega_1$ then forcing with $T$ collapses $\omega_1$ (Baumgartner)
- (together with assumption $2^\omega = \omega_2$) Every forcing which adds a new subset of $\omega_1$ either adds a real or collapses $\omega_2$ (Todorcevic)
In $V$ consider the stationary set $S := sG_{\omega_1} \cap P_{\omega_2}(H_{\omega_2})$. Using stationarity of $S$ and the assumption that $2^\omega = \omega_2$, fix a $\subset$-increasing (non-continuous) chain $\langle M_\alpha \mid \alpha < \omega_2 \rangle$ of elements of $S$ whose union contains $H_{\omega_1}$. 

Suppose $W$ is an outer model of $V$ which adds a new subset $b$ of $\omega_1$, and doesn’t add a real. Then it doesn’t add new subsets of countable ordinals either, so for all $\xi < \omega_1$ we have $b \cap \xi \in H_{\omega_1} \subset \bigcup_{\alpha < \omega_2} M_\alpha$. In $W$ define a function $f: \omega_1 \to \omega_{\omega_2}$ by sending $\xi$ to the least $\alpha$ such that $b \cap \xi \in M_\alpha$. This is a cofinal map from $\omega_1 \to \omega_{\omega_2}$ since for any $\alpha < \omega_2$, since $b \not\in M_\alpha$ and $M_\alpha$ is $G_{\omega_1}$-closed, there is some $\xi < \omega_1$ such that $b \cap \xi \not\in M_\alpha$. 

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Sketch of proof

In \( V \) consider the stationary set \( S := sG_{\omega_1} \cap P_{\omega_2}(H_{\omega_2}) \). Using stationarity of \( S \) and the assumption that \( 2^\omega = \omega_2 \), fix a \( \subset \)-increasing (non-continuous) chain \( \langle M_\alpha \mid \alpha < \omega_2 \rangle \) of elements of \( S \) whose union contains \( H_{\omega_1} \).

Suppose \( W \) is an outer model of \( V \) which adds a new subset \( b \) of \( \omega_1 \), and doesn’t add a real. Then it doesn’t add new subsets of countable ordinals either, so for all \( \xi < \omega_1 \) we have

\[
b \cap \xi \in H_{\omega_1}^V \subset \bigcup_{\alpha < \omega_2} M_\alpha
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In $V$ consider the stationary set $S := sG_{\omega_1} \cap P_{\omega_2}(H_{\omega_2})$. Using stationarity of $S$ and the assumption that $2^\omega = \omega_2$, fix a $\subseteq$-increasing (non-continuous) chain $\langle M_\alpha \mid \alpha < \omega_2 \rangle$ of elements of $S$ whose union contains $H_{\omega_1}$.

Suppose $W$ is an outer model of $V$ which adds a new subset $b$ of $\omega_1$, and doesn’t add a real. Then it doesn’t add new subsets of countable ordinals either, so for all $\xi < \omega_1$ we have

$$b \cap \xi \in H_{\omega_1}^V \subset \bigcup_{\alpha < \omega_2} M_\alpha$$

In $W$ define a function $f : \omega_1 \to \omega_2^V$ by sending $\xi$ to the least $\alpha$ such that $b \cap \xi \in M_\alpha$. This is a cofinal map from $\omega_1 \to \omega_2^V$ since for any $\alpha < \omega_2$, since $b \notin M_\alpha$ and $M_\alpha$ is $G_{\omega_1}^W$ then there is some $\xi < \omega_1$ such that $b \cap \xi \notin M_\alpha$. 
A new question

Our model of ISP(\(\omega_2\)) plus large continuum is NOT a model of the “specialized” version (because it has a tree of height and size \(\omega_1\) whose forcing doesn’t collapse \(\omega_1\)).
This suggests a natural modification of the Viale-Weiss question:

**Question**

Assume “specialized” ISP(\(\omega_2\)); i.e. suppose \(sG_{\omega_1}\) is stationary for all \(P_{\omega_2}(H_\theta)\). Does this imply \(2^\omega = \omega_2\)?