Arnold diffusion for convex nearly integrable systems

V. Kaloshin

November 24, 2014
Plan of the talk

- Motivation: Ergodic and quasiergodic hypothesis.
  - Nearly integrable systems and the problem of Arnold diffusion
  - Results in 3, 4, and more degrees of freedom
  - Indication of Arnold diffusion in the Solar system
  - Stochastic aspects of Arnold diffusion
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Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function, $(q, p) \in \mathbb{R}^n \times \mathbb{R}^n$. Let $X_H$ be the Hamiltonian flow associated to $H$.

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\begin{align*}
\dot{q} &= \partial_p H \\
\dot{p} &= -\partial_q H
\end{align*}
\]

Let $S_E = \{(q, p) \in T^* M : H(q, p) = E\}$ be an energy surface.

Ergodic Hypothesis (Boltzmann, Maxwell) Is a generic Hamiltonian flow $X_H$ on a generic energy surface $S_E$ ergodic?

Numerical doubts (Fermi-Pasta-Ulam) Chains of nonlinear springs
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the $\alpha$-term — nonlinearity. Most “small” solutions are almost periodic!
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**KAM theory**  Each *nearly integrable* systems has collections of invariant tori of positive measure $\Rightarrow$ no ergodicity!

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**Quasiergodic Hypothesis** (Birkhoff, Ehrenfest) Does a generic Hamiltonian flow on a generic energy surface $S_E$ have a dense orbit?
Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian, $\varphi \in \mathbb{T}^n$ be angle, $I \in \mathbb{R}^n$ be action.

A Hamiltonian system is **Arnold-Liouville integrable** if for an open set $U \subset \mathbb{R}^n$ there exists a symplectic map $\Phi : \mathbb{T}^n \times U \to \mathbb{R}^{2n}$ s. t. $H \circ \Phi(\varphi, I)$ depends only on $I$ and

$$\begin{cases} \dot{\varphi} = \partial_I(H \circ \Phi)(I) = \omega(I), \\ I = 0. \end{cases}$$

$(\varphi, I)$–action-angle coordinates

In particular, $\Phi(\mathbb{T}^n \times U)$ is foliated by invariant $n$-dim’l tori & on each torus $\mathbb{T}^n$ the flow is linear.
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Integrable systems & action-angles coordinates

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Integrable systems

- Newtonian two body problem.
- Pendulum \( H = \frac{I^2}{2} - \cos 2\pi \varphi, \ (\varphi, I) \in T^*T = T \times \mathbb{R} \).
- Harmonic oscillator \( \ddot{q} = -kq \) or \( H = \frac{p^2}{2} + \frac{kq^2}{2} \).
- Motion in a central force field \( \ddot{q} = F(||q||)q \).
- Newtonian two center problem.
- Lagrange’s top, Kovaleskaya’s top, Euler top.
- Toda lattice: chain \( \cdots < x_0 < x_1 < \cdots \) with the neighbor interaction \( \sum_i \exp(x_i - x_{i+1}) \).
- Calogero-Moser system: chain of harmonic oscillators with a neighbor repulsive interaction.
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Arnold diffusion

Arnold, 63: Let \((\varphi, I) \in T^*T^n = T^n \times \mathbb{R}^n, \ t \in T.\)

(weak form) Does there exist a real instability in many-dimensional problems of perturbation theory when the invariant tori do not divide the phase space? More precisely, for a generic perturbation \(\varepsilon H_1(\varphi, I, t)\) the Hamiltonian

\[ H_\varepsilon(\varphi, I, t) = H_0(I) + \varepsilon H_1(\varphi, I, t) \]

has an orbit whose action component “travels” in action space, in particular, \(\max_t \| I(t) - I(0) \| = O(1).\)

(strong form) For any two open sets \(U, U' \subset B^n\) the Hamiltonian \(H_\varepsilon(\varphi, I, t)\) has an orbit whose action component “travels” from \(U\) to \(U'\), i.e. \(I(0) \in U\) and \(I(T) \in U'\) for some \(T > 0.\)
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V. Kaloshin (University of Maryland)
Let $H_0(l) = \frac{l^2}{2}$. Time one map $(\varphi, l) \rightarrow (\varphi + l, l) \pmod{1}$.

Let $H_\varepsilon(\varphi, l, t) = H_0(l) + \varepsilon H_1(\varphi, l, t)$. The model time one map

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KAM Theorem, obstacles to instability

Let $H_0(l)$ have non-degenerate Hessian, e.g. $H_0(l) = \sum l_j^2 / 2$.

KAM Theorem

Let $H_\varepsilon(\varphi, l, t) = H_0(l) + \varepsilon H_1(\varphi, l, t)$ be a smooth perturbation. Then with probability $1 - O(\sqrt{\varepsilon})$ has an initial condition in $\mathbb{T}^n \times B^n \times \mathbb{T}$ having a quasiperiodic orbit. Moreover, $\mathbb{T}^n \times B^n \times \mathbb{T}$ with certain neighborhood of rational lines deleted is laminated by invariant $(n + 1)$-dimensional tori, one for each diophantine $\omega$. 
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Diagram showing a torus lattice with width $\sim \sqrt{\epsilon}$. 

V. Kaloshin (University of Maryland) 
Arnold diffusion 
November 24, 2014
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For $n = 1$ they confine orbits!
For $n > 1$ they do not!
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In $(2n + 1)$-dimensional space there are $(n + 1)$-dimensional tori. For $n = 1$ they confine orbits! For $n > 1$ they do not!
Let \( H_0(l) \) be smooth and strictly convex, \( l \in B^n \).

**The First Main Result** For any \( \gamma > 0 \) & a generic smooth perturbation \( \varepsilon H_1(\phi, l, t) \) the Hamiltonian

\[
H_\varepsilon(\phi, l, t) = H_0(l) + \varepsilon H_1(\phi, l, t)
\]

has an orbit \( (\phi_\varepsilon, l_\varepsilon, t)(t) \) which is \( \gamma \)-dense in \( \mathbb{T}^n \times B^n \times \mathbb{T} \). Namely, \( \gamma \)-neighbourhood of \( \cup_t (\phi_\varepsilon, l_\varepsilon, t)(t) \) contains \( \mathbb{T}^n \times B^n \times \mathbb{T} \).

[K-Zhang, 12] \( n=2 \) (arxiv)

In 2002 a version of this result was announced by Mather.

There is an announcement of Cheng.

[K-Zhang, 14] \( n=3 \) (my webpage)

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A weak form of quasiergodic hypothesis

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Unstable orbits exist in the 3 : 1 Kirkwood gap.

V. Kaloshin (University of Maryland) Arnold diffusion November 24, 2014 18 / 22
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Diffusion conjecture Let

\[ H_\varepsilon = \frac{p^2}{2} + \left( \frac{p^2}{2} + \cos q \right) + \varepsilon H_1(\varphi, l, q, p, t), \quad \varphi, q, t \in \mathbb{T}, \quad l, p \in \mathbb{R}, \]

where \( H_1 \) is a generic perturbation. Let \( \text{Leb}_\varepsilon \) be the norm Lebesgue measure on the \( \sqrt{\varepsilon} \)-ball around 0. Then \( I(\frac{-t \cdot \ln \varepsilon}{\varepsilon^2}) \) converges to a diffusion process wrt \( \text{Leb}_{\sqrt{\varepsilon}} \).
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Chirikov, ... , Guzzo
Model Problem Let

\[ f_0 : (\varphi, I) \rightarrow (\varphi + I + \varepsilon \cos \varphi, I + \varepsilon \cos \varphi), \]
\[ f_1 : (\varphi, I) \rightarrow (\varphi + I + \varepsilon \sin \varphi, I + \varepsilon \sin \varphi), \]

be a pair of standard maps.

Consider random composition of these maps

\[ f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1} (\varphi_0, I_0) = (\varphi_n, I_n). \]

**Theorem** (joint work with O. Castejon) For \( n \sim \varepsilon^{-2} \) such compositions satisfy the Central Limit Theorem, i.e.

\[ I_n - I_0 \rightarrow \mathcal{N}(0, \sigma), \]

where \( \mathcal{N}(0, \sigma) \) is a normal random variable with some variance \( \sigma > 0 \).
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Find invariant sets inside Normally Hyperbolic Invariant Cylinders w. transverse invariant manifolds
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Preprints contributing to the talk

- V. Kaloshin, K. Zhang, Arnold diffusion for three and a half degrees of freedom, April 2014, 25pp.;
- V. Kaloshin, K. Zhang, Dynamics of the dominant Hamiltonian, with applications to Arnold diffusion, October 2014, 75pp.;