Adaptable colouring and colour critical graphs

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Adapted $k$-colouring of graphs

**Definitions.** A graph $G$ is **adaptably $k$-colourable** if for every $k$-edge colouring $c'$, there is a $k$-vertex colouring $c$ such that for every edge $xy$ in $G$, not all of $c(x)$, $c(y)$, and $c'(xy)$ are the same.

The edge $xy$ is **monochromatic** if $c(x)=c(y)=c'(xy)$.

The **adaptable chromatic number** of $G$, $\chi_a(G)$, is the least $k$ such that $G$ is adaptably $k$-colourable.
Adapted $k$-colouring as a game

- There are two players $E$ and $V$.
- Player $E$ colours the edges of a graph $G$ first using colours in $\{1,2,\ldots,k\}$.
- Player $V$ then colours vertices of $G$ using the same set of colours.
- Player $V$ wins if he can colour the vertices without creating any monochromatic edges.
- Otherwise $E$ wins.
Adapted $k$-colouring as a game

- The least number of colours that player V always has a winning strategy is the adaptable chromatic number of $G$, $\chi_a(G)$. 
Example. $K_4$

- Consider the graph $K_4$:
A 2-edge colouring of $K_4$.

- E colours the edges in two colours:
An adapted 2-colouring

- $V$ colours the vertices in two colours:

There is no monochromatic edge.
A winning strategy for E with 2 colours

- E has a winning strategy with two colours:

Therefore $\chi_a(K4) > 2$. 
A winning strategy of $V$ with 3 colours

\[ \chi_a(K_4) = 3. \]
Colour critical graphs

• A graph $G$ is $k$-critical if $\chi(G) = k$ and $\chi(G \square e) = k - 1$ for every edge $e$ in $G$.

• A $k$-critical graph can be coloured with $k - 1$ colours such that there is only one edge joining two vertices of the same colour.

Fact. If $G$ is $k$-critical then $\chi_a(G) \leq k - 1$.

Problem. (Molloy and Thron 2012) Are there any critical graphs $G$ with $\chi_a(G) = \chi(G) - 1$?
Construction 1

The Hajós’ construction.

\[ G_1 = \begin{array}{cc}
  x_1 & \\ \\
y_1 & \\ \\
\end{array} \quad G_2 = \begin{array}{cc}
  x_2 & \\ \\
y_2 & \\ \\
\end{array} \]

The result is
Construction 1

Let $G$ be the graph obtained by applying the Hajós’ construction to two graphs $G_1$ and $G_2$.

**Fact.** If both $G_1$ and $G_2$ are $k$-critical, then $G$ is also $k$-critical.

**Fact.** (Huizenga 2008) If $\chi_a(G_1) \geq k$ and $\chi_a(G_2) \geq k$, then $\chi_a(G) \geq k$.

**Implication.** If there is a $k$-critical graph $G$ with $\chi_a(G) = k + 1$ then there are infinitely many such graphs.
Construction 2

$G_1 \vee G_2$, the join of $G_1$ and $G_2$

$G_1 = \circ$

$G_2 =$

$G_1 \vee G_2 = \equiv W_5$
Construction 2

- If $G_1$ is a $k_1$-critical graph and $G_2$ is a $k_2$-critical graph, then $G_1 \lor G_2$ is a $(k_1 + k_2)$-critical graph.
- However, it can happen that

$$
\chi_a(G) < \chi_a(G_1) + \chi_a(G_2).
$$
The graph $W_5$

$W_5$ is 4-critical.

$\chi_a(W_5) \geq 3$.

Therefore, $\chi_a(W_5) = 3$. 
An important property of $W_5$

$W_5$ has a proper subgraph $H_4$ such that

$$\chi_a(H_4) = 3.$$
The construction for \( k = 5 \). (1)

We apply Hajós’ construction to two copies of \( W_5 \).
The construction for \( k = 5 \). (2)

We apply Hajós’ construction one more time.
The construction for $k = 5$. (3)

We continue applying Hajós’ construction to get this graph $F_4$.

$F_4$ is 4-critical.
The construction for $k = 5$. (4)

$F_4$ contains three disjoint copies of $H_4$.

$G_5 = K_1 \lor F_4$. 
The construction for $k = 5$. (5)

$G_5$ is 5-critical. Therefore $\chi_a(G_5) \leq 4$.
Claim. $\chi_a(G_5) \geq 4$.

We show that Player E has a winning strategy with 3 colours on $G_5$. 
General case

**Theorem.** For every integer $k$ such that $k \geq 4$, there is a $k$-critical graph $G_k$ that contains a proper subgraph $H_k$ such that

$$\chi a(H_k) \geq k - 1.$$
\( K4 \) again.

\[
\begin{align*}
\cdot \chi(K4) &= 4 \text{ and } \chi(K4 \square e) = 3 \text{ for every edge } e \text{ in } K4. \\
\cdot \chi_a(K4) &= 3. \\
\cdot \chi_a (K4 \square e) &= 2 \text{ for every edge } e \text{ in } K4.
\end{align*}
\]

**Question.** Are there any other such “double critical” graphs \( G \) with \( \chi_a = \chi(G) \square 1 \)?
The Grötzsch graph

Let $G$ be the Grötzsch graph.

**Fact.** $G$ is 4-critical.

**Fact.** $G$ is triangle-free.
The Grötzsch graph

**Fact.** $\chi_a(G) = 3$.
Player E has a winning strategy if there are two colours.

**Fact.** There are triangle-free 4-critical graphs with adaptable chromatic number 3.
More questions

- Question 1: Are there triangle-free $k$-critical graphs with adaptable chromatic number $k-1$ for every $k \geq 5$?

- Question 2: Are there $k$-critical graphs with adaptable chromatic number $k-1$ and girth $g$ for every $k \geq 4$ and $g \geq 4$?
Lower bound

\[
\chi_a(G) \geq \frac{\chi(G)}{\sqrt{n \log(\chi(G))}}
\]

where \( n \) is the number of vertices in \( G \).

**Conjecture.** (Greene) There is a function \( f \) such that \( \chi_a(G) \geq f(\chi(G)) \) and \( \lim_{k \to \infty} f(k) = \infty \).
Lower bound (2)

- **Theorem.** (Huizanga, 2008) There is an unbounded function $f$ such that $\chi_a(G) \geq f(\chi(G))$ for almost every graph $G$.
- **Theorem.** (Molloy and Thron, 2011) There is a function $h$ tending to infinity such that $ch_a(G) \geq h(ch(G))$.
- **Theorem.** (BZ 2013)
  \[ \chi_a(G) \geq K \log \log \chi(G) \]
  where $K$ is a positive integer.
Still more questions

• **Problem.** (Molloy, Thron) Are there any graph \( G \) such that \( \chi_a(G) \) is less than the order of \( \sqrt{\chi(G)} \)?

• **Problem.** Can the lower bound \( \log \log \chi(G) \) be improved?