Towards Generalized Hydrodynamic Integrability via the Characteristic Variety

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The traditional representation theory of Lie groups begins with the question “how hard is it to integrate the infinitesimal structure of the Lie algebra into the local structure of the Lie group?”

Therefore, examination of regularity and micro-local analysis (“how difficult is integration?”) of involutive PDE/EDS should help build our knowledge Lie pseudogroups.

Where to find invariant notions of “difficult to integrate?”
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Short talk – Get to the point!

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Where to find invariant notions of “difficult to integrate?”
Dans certains cas, le nombre de ces familles de caractéristiques peut être réduit, certaines familles devenant double, triples, etc. Ces cas de réduction sont analogues à ceux qui se présentent dans la réduction d’une substitution linéaire à sa forme normale et la recherche des caractéristiques dépend d’ailleurs d’une telle réduction. —Cartan, 102 years ago.
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Moreover, those multiplicities in the characteristic variety can be accessed via the incidence correspondence given by the **rank-one variety** of the tableau.

For involutive systems with higher Cartan int, submanifolds secant to the rank-one cone give hydrodynamic reductions, and the secant system indicates **hydrodynamic integrability** in local coordinates.
Two interwoven stories: rank-one variety & hydrodynamic integrability.

Tableau and Symbol:

\[
M^{(2)}, \mathcal{I}^{(2)} \quad \xrightarrow{T_E M^{(2)}} \quad Z^{(1)} \\
M^{(1)}, \mathcal{I}^{(1)} \quad \xrightarrow{T_e M^{(1)}} \quad Z \\
M, \mathcal{I} \quad \xrightarrow{T_p M} \quad W \\
V
\]
Two interwoven stories: rank-one variety & hydrodynamic integrability.

Tableau and Symbol:

\[ d\theta^a \equiv \left( \tau(\eta) \right)_i^a \wedge \omega^i + \frac{1}{2} T^a_{ij} \omega^i \wedge \omega^j \mod \theta \]
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\[ 0 \rightarrow Z \xrightarrow{\tau} W \otimes V^* \xrightarrow{\sigma} U^* \rightarrow 0 \]
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Characteristic Variety and Rank-One Variety:
Two interwoven stories: rank-one variety & hydrodynamic integrability.

$M^{(2)}, I^{(2)} \xrightarrow{\kappa} Z^{(1)}$

$M^{(1)}, I^{(1)} \xrightarrow{\eta} Z$

$M, I \xrightarrow{\theta} W$

$V$

Tableau and Symbol:

$$d\theta^a \equiv \left(\tau(\eta)\right)^a_i \wedge \omega^i + \frac{1}{2} T^a_{ij} \omega^i \wedge \omega^j \mod \theta$$

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Characteristic Variety and Rank-One Variety:

$$\Xi = \{\xi \in V^* : \exists w, \sigma_\xi(w) = \sigma(w \otimes \xi) = 0\}$$

$$C = \{z \in Z : \tau(z) = w \otimes \xi, \text{ has rank } 1\}$$

(slides sloppy about $\mathbb{P}$'s)
Characteristic and Rank-One Variety

Some properties if $\mathcal{I}$ is involutive:

1. The eikonal system $E(\Xi_C)$ is involutive on any ordinary integral $N$. (Typically, difficult.)
2. $\Xi$ is essentially preserved under prolongation.
3. The last Cartan character is $\dim \hat{\Xi}_C = \deg \hat{\Xi}_C$.
4. If $\mathcal{I}$ has no Cauchy characteristics, then $\hat{\Xi}_C$ spans $\bar{e}^C$.
5. If $\Xi_C = \emptyset$, then $\mathcal{I}$ is Frobenius (totally integrable).
6. If $\hat{\Xi}_R = \emptyset$, then $\mathcal{I}$ is elliptic.
7. If $\hat{\Xi}_R$ has appropriate space-like hyperplanes, then $\mathcal{I}$ is hyperbolic.
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$$
\begin{align*}
Gr_\bullet(W) & \quad \Xi \\
\Downarrow & \\
C & \\
\Downarrow & \\
\Xi & \\
\Downarrow & \\
W_\xi \otimes \xi & \quad \xi
\end{align*}
$$

$$
W_\xi = \ker \sigma_\xi
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\begin{align*}
\text{Gr}_*(W) & \xrightarrow{\Xi} C \\
W_\xi \otimes \xi & \xrightarrow{\xi} W_\xi = \ker \sigma_\xi
\end{align*}
Some properties if $\mathcal{I}$ is involutive:

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Characteristic and Rank-One Variety

\[ C \quad \xrightarrow{\text{Gr}.(W)} \quad \Xi \quad \xrightarrow{W_\xi \otimes \xi} \quad W_\xi = \ker \sigma_\xi \]

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7. If \( \Xi_R \) has appropriate space-like hyperplanes, then \( \mathcal{I} \) is hyperbolic.
Some examples with \( \dim \mathcal{Z} = s = s_1 = \dim \mathcal{Z}^{(1)} = 4 \)

<table>
<thead>
<tr>
<th>Characteristic and Rank-One Variety</th>
</tr>
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<tbody>
<tr>
<td>involutive tableau ( \iff ) commuting symbol relations ( \iff ) compatible primary decompositions.</td>
</tr>
</tbody>
</table>

\[
\begin{pmatrix}
\eta^1 \\
\lambda_1 \eta^1 \\
\mu_1 \eta^1 \\
\eta^2 \\
\lambda_2 \eta^2 \\
\mu_2 \eta^2 \\
\eta^3 \\
\lambda_3 \eta^3 \\
\mu_3 \eta^3 \\
\eta^4 \\
\lambda_4 \eta^4 \\
\mu_4 \eta^4
\end{pmatrix}
\]

- distinct
- \[ [1 : \lambda_1 : \mu_1], [1 : \lambda_2 : \mu_2], [1 : \lambda_3 : \mu_3], [1 : \lambda_4 : \mu_4] \]
- \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
- \( \text{Sec}_3(C) = 4 \mathbb{P}^0 \)

\[
\begin{pmatrix}
\eta^1 \\
\eta^2 \\
\eta^3 \\
\eta^4
\end{pmatrix}
\]

- duplicates
- \[ [1 : \lambda_1 : \mu_1], [1 : \lambda_1 : \mu_1], [1 : \lambda_3 : \mu_3], [1 : \lambda_4 : \mu_4] \]
- \[
\begin{bmatrix}
* & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]
- \( \text{Sec}_3(C) = 1 \mathbb{P}^1 \sqcup 2 \mathbb{P}^0 \)

\[
\begin{pmatrix}
\eta^1 + \eta^2 \\
\lambda_1 \eta^1 + \eta^2 \\
\mu_1 \eta^1 + \eta^2 \\
\eta^3 \\
\lambda_3 \eta^3 \\
\mu_3 \eta^3 \\
\eta^4 \\
\lambda_4 \eta^4 \\
\mu_4 \eta^4
\end{pmatrix}
\]

- nilpotents
- \[ [1 : \lambda_1 : \mu_1], [1 : \lambda_1 : \mu_1], [1 : \lambda_3 : \mu_3], [1 : \lambda_4 : \mu_4] \]
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\begin{bmatrix}
1 & 0 & 0 & 0 \\
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These systems are easier to distinguish with \( C \) than with \( \Xi \).
The Secants of the Rank-One Cone

Fix $e \in M^{(1)}$. 

Note that $\text{Sec}(C) \subset \text{Gr}(Z) = \text{Gr}(T M^{(1)})$, defined by some ideal. 

Also, as usual, $M^{(2)} \subset \text{Gr}(T M^{(1)})$ defined by $I^{(1)}$. 

So, there is an ideal on $M^{(1)}$ whose variety is $\text{Sec}(C) \cap M^{(2)}$. 

**(how to compute it?)** Prolong once and use Terracini's Lemma: If $E = l_1 + l_2 + \cdots + l_n \in \text{Sec}(C)$, then

$$T E \text{Sec}(C) = \bigcap_i T l_i C \subset T^* E M^{(2)}$$

Call it $A(I)$. 

A.Smith (Fordham)
The Secants of the Rank-One Cone

Fix $e \in M^{(1)}$. $\Xi_e \subset e^* \cong V^*$, defined by characteristic ideal
The Secants of the Rank-One Cone

Fix $e \in M^{(1)}$. $\Xi_e \subset e^* \cong V^*$, defined by characteristic ideal $C_e \subset Z_e$, defined by $2 \times 2$ minors on $\tau(Z_e) \subset W \otimes V^*$. 

Note that $\text{Sec}_n(C_e) \subset \text{Gr}_n(Z_e) = \text{Gr}_n(T_e M^{(1)})$, defined by some ideal.

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A. Smith (Fordham)
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Fix $e \in M^{(1)}$. $\Xi_e \subset e^* \cong V^*$, defined by characteristic ideal $C_e \subset \mathcal{Z}_e$, defined by $2 \times 2$ minors on $\tau(\mathcal{Z}_e) \subset W \otimes V^*$.

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Prolong once and use Terracini’s Lemma: If $E = l_1 + l_2 + \cdots + l_n \in \text{Sec}_n(C)$, then

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Call it $A(\mathcal{I})$. 
\( \mathcal{I} \rightarrow \mathcal{I}^{(1)} \rightarrow \mathcal{A}(\mathcal{I}) \). What does this give?
\( \mathcal{I} \rightarrow \mathcal{I}^{(1)} \rightarrow \mathcal{A}(\mathcal{I}) \). What does this give?

\[
\begin{align*}
\mathcal{A}^{k-1}(\mathcal{I})^{(1)} & \rightarrow \mathcal{A}^k(\mathcal{I}) \\
\mathcal{A}^2(\mathcal{I})^{(1)} & \rightarrow \mathcal{A}^3(\mathcal{I}) \\
\mathcal{A}(\mathcal{I})^{(1)} & \rightarrow \mathcal{A}(\mathcal{A}(\mathcal{I})) \\
\mathcal{I}^{(1)} & \rightarrow \mathcal{A}(\mathcal{I}) \\
\mathcal{I} & \rightarrow \mathcal{I}^{(1)} \\
\end{align*}
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(Where) Does this end?
Wanted: A *general* notion of integrability.
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1. for PDEs already seen in the literature, the notion must reproduce the observed geometry;
2. the notion must be applicable to all PDEs of all orders & dimensions (perhaps trivially so);
3. the notion must extend naturally to generic EDS or \(D\)-modules (or provide obvious obstructions to such an extension);
4. the notion must be contact invariant;
5. the notion must be preserved under prolongation; and
6. the notion should be equally applicable in the real or complex cases, with the usual algebraic caveats.

Additionally, the following properties would be convenient:

1. the notion should be testable in real-world examples;
2. the notion should provide a means of constructing actual solutions; and
3. the notion should provide a means for constructing Lax pairs, \(\tau\) functions, or loop group formulations when those theories also apply.

That is, integrable systems should be viewed as a subvariety of involutive/regular systems. What is their defining ideal?
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4. the notion must be contact invariant;
5. the notion must be preserved under prolongation; and
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1. the notion should be testable in real-world examples;
2. the notion should provide a means of constructing actual solutions; and
3. the notion should provide a means for constructing Lax pairs, $\tau$ functions, or loop group formulations when those theories also apply.
Wanted: A general notion of integrability. To me, this means:

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That is, integrable systems should be viewed as a subvariety of involutive/regular systems. What is their defining ideal?
Consider this 1st-order system of PDE on functions $(X^n, x^i) \rightarrow (Y^r, y^a)$:

$$\frac{\partial y^a}{\partial x^i} = F^a_i(y) \frac{\partial y^a}{\partial x^1} \quad \text{(no sum!)}$$
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This system is called a semi-Hamiltonian or rich system of conservation laws (Tsarëv and D.Serre). They:

- are uninteresting in \(r \leq 2\).
- describe systems of commuting wavefronts
- admit \(C^\infty\) solutions using the generalized hodograph method
- are characterized as orthogonal coordinate webs (Darboux, Tsarëv) (more on this later)
- appear in the linearizations of many “integrable” PDEs (more on this later)
Let $h^a = \frac{\partial y^a}{\partial x^1} \neq 0$, with $(h^a)$ valued in some space $H$. Consider the EDS on $M = X \times (Y \times H)$ generated by

$$\{\theta^a\} = \left\{ dy^a - h^a F_i^a(y) \, dx^i \right\}$$

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\vdots & \vdots & \vdots & \ddots & \vdots \\
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3. Best of all possible $s_1$ involutive systems. Every localization of $\mathcal{A}(\mathcal{I})$ is Frobenius, maximal.
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\mu & \quad M \subset J^p(\mathbb{R}^n, \mathbb{R}^q) \\
Y = \mathbb{R}^r & \quad \pi \\
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\mu_* \left( \frac{\partial}{\partial y^a} \right) = \left( \frac{\partial x^i}{\partial y^a}, \frac{\partial u}{\partial y^a}, \frac{\partial p_1}{\partial y^a} F_i ^a, \frac{\partial p_{11}}{\partial y^a} F_i ^a F_j ^a, \frac{\partial p_{111}}{\partial y^a} F_i ^a F_j ^a F_k ^a, \ldots \right)
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But can we characterize as EDS with no other restrictions?
Proposition

Suppose that $\mathcal{I}$ is a PDE-type involutive EDS with no Cauchy characteristics or unabsorbable torsion. Then

1. $\mathcal{I}$ is Frobenius (over $\mathbb{C}$) if and only if $A(\mathcal{I})$ is always empty. [trivial to prove.]

2. $\mathcal{I}$ is semi-Hamiltonian if and only if $A(\mathcal{I})$ is Frobenius [see Cartan’s abstract.]

3. $\mathcal{I}$ is hydro int if and only if $A(\mathcal{I})$ is semi-Hamiltonian [** in known subcases].

Therefore, the condition "$\mathcal{I}$ is Frobenius for some $k$" appears to be a generalization of hydrodynamic integrability that is manifestly invariant.

Dear experts: Has this condition been used or named before?

Reminder of the motivation from Lie algebras:

- trivial
- abelian
- solvable
- semi-simple

$D(\mathfrak{g}) = 0$

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