TotallyDisconnectedL.C.Groups:
TheScaleandMinimizingSubgroups

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Lecture 1: The scale and minimizing subgroups for an endomorphism
   Compact open subgroups; Examples
   The scale of an endomorphism
   The scale function on $G$
   The structure of minimizing subgroups

Lecture 2: Tidy subgroups and the scale

Lecture 3: The contraction group and the nub

Lecture 4: Flat groups of automorphisms
Definition (Inductive dimension)

The empty set is defined to have inductive dimension $-1$. The topological space $(X, \mathcal{T})$ has *inductive dimension* $n$ if it does not have dimension less than $n$ and there is a base for $\mathcal{T}$ comprising sets whose boundary has inductive dimension at most $n - 1$.

The inductive dimension of $X$ is denoted $\text{ind}(X)$.

A locally compact space is totally disconnected if and only if it has inductive dimension 0, in which case the topology has a base of compact open subsets.

Totally disconnected locally compact groups are sometimes referred to as *0-dimensional groups*. 
Compact open subgroups

Theorem (van Dantzig)

Let $G$ be a totally disconnected locally compact groups and $\emptyset \ni e$ be a neighbourhood of the identity. Then there is a compact open subgroup $V \subseteq \emptyset$.

The compact open subgroup need not be normal.

Corollary

Every compact t.d. group is profinite, that is, a projective limit of finite groups. Conversely, every profinite group is compact with the profinite topology.

The set of all compact open subgroups of t.d.l.c. group $G$ will be denoted by $\mathcal{B}(G)$. 
Examples of tdlc groups

Examples

1. $F^\mathbb{Z}$, where $F$ is a finite group, with the product topology. This group is compact.

2. $(\mathbb{F}_p((t)), +)$, the additive group of the field of formal Laurent series over the field of order $p$. $t^n\mathbb{F}_p[[t]] \cong \mathbb{F}_p^\mathbb{N}$ is a compact open subgroup for each $n \in \mathbb{Z}$.

3. Aut$(T_q)$, the automorphism group of the regular tree with every vertex having valency $q$. The stabilizer of any finite subtree is compact and open.

4. SL$(n, \mathbb{Q}_p)$, the special linear group over the field of $p$-adic numbers. SL$(n, \mathbb{Z}_p)$ is a compact open subgroup.
The scale of an endomorphism

Let $G$ be a t.d.l.c. group.

- An *endomorphism* of $G$ is a continuous homomorphism $\alpha : G \to G$.

- An *automorphism* of $G$ is an endomorphism that is a bijection whose inverse is continuous.

- The automorphism $\alpha$ is *inner* if there is $g \in G$ such that $\alpha = \alpha_g$ where $\alpha_g(x) = gxg^{-1}$ for every $x \in G$.

**Definition**

Let $G$ be a t.d.l.c. group and $\alpha$ be an endomorphism of $G$. The *scale* of $\alpha$ is

$$s(\alpha) = \min \left\{ [\alpha(V) : \alpha(V) \cap V] \mid V \in \mathcal{B}(G) \right\}.$$

Any $V$ at which the minimum is attained is *minimizing for* $\alpha$. 
Properties of the scale

Theorem

Let $G$ be a t.d.l.c. group and $\alpha \in \text{End}(G)$. Then

1. $s(\alpha^n) = s(\alpha)^n$ for every $n \geq 0$.

If $\alpha$ is an automorphism, then

1. $s(\alpha) = 1 = s(\alpha^{-1})$ if and only if there is $V \in B(G)$ such that $\alpha(V) = V$; and
2. $\Delta(\alpha) = s(\alpha)/s(\alpha^{-1})$, where $\Delta(\alpha)$ is the module of $\alpha$.
3. The function $s : G \to \mathbb{Z}^+$ defined by $s(g) = s(\alpha g)$ is continuous.

These properties will be explained in the first two lectures for the case when $\alpha$ is an automorphism.
The set of topologically periodic elements is closed

An element $g \in G$ is topologically periodic if $\langle g \rangle$ is compact. Denote the set of topologically periodic elements in $G$ by $P(G)$. K. H. Hofmann asked whether $P(G)$ is closed. The answer uses only that there is a function on $G$ with the properties of $s$.

Proposition

$P(G)$ is closed.

Proof.

1. $g \in P(G)$ implies that $s(g) = 1$.
2. $s(h) = 1$ for every $h \in P(G)$.
3. If $h \in P(G)$, then $h$ normalizes some $V \in \mathcal{B}(G)$.
4. There is $g \in P(G) \cap hV$. Then $g$ normalizes $V$ and $h \in gV$. Hence $\langle h \rangle \leq \langle g \rangle V$ and has compact closure.
The scale on $G$ as a spectral radius

Proposition (R. G. Möller)
Let $\alpha \in \text{Aut}(G)$ and $V \in \mathcal{B}(G)$. Then

$$s(\alpha) = \lim_{n \to \infty} \left[ \alpha^n(V) : \alpha^n(V) \cap V \right]^{1/n}.$$

For each $V \in \mathcal{B}(G)$ define $w_V(g) = \left[ \alpha_g(V) : \alpha_g(V) \cap V \right]$.

Proposition

- $w_V$ is submultiplicative, that is, $w_V(gh) \leq w_V(g)w_V(h)$.
- All weights $w_V$ are equivalent, that is, given $U, V \in \mathcal{B}(G)$ there is $K > 0$ such that
  $$K^{-1}w_V(g) \leq w_U(g) \leq Kw_V(g) \text{ for all } g \in G.$$
The scale on $G$ as a spectral radius 2

Equivalence of the weights implies that the weighted $L^1$-space

$$L^1(G, w_V) = \left\{ \varphi \in L^1(G) \mid \int_G |\varphi| w_V < \infty \right\}$$

is independent of the weight $w_V$.

Submultiplicativity of the weights implies that $L^1(G, w_V)$ is a Banach algebra under convolution.

For $g \in G$, let $L_g$ denote the operator on $L^1(G, w_V)$ of left translation by $g$. $L_g : \varphi \mapsto \varphi_g$, where $\varphi_g(x) = \varphi(gx)$. Then $L_g$ is a bounded operator and

$$s(g) = r(L_g) \quad (= \text{the spectral radius of } L_g).$$
Subgroups tidy for an endomorphism

Let \( \alpha \in \text{End}(G) \) and \( V \in \mathcal{B}(G) \). Define

\[
V_+ = \{ v \in V \mid \exists \{ v_n \}_{n \geq 0} \subset V \text{ with } v_0 = v \text{ and } \alpha(v_{n+1}) = v_n \}
\]

and

\[
V_- = \{ v \in V \mid \alpha^n(v) \in V \quad \forall n \geq 0 \}.
\]

**Theorem**

*The subgroup \( V \in \mathcal{B}(G) \) is minimizing for \( \alpha \in \text{End}(G) \) iff*

\[
\text{TA}(\alpha) \quad V = V_+ V_-;
\]

\[
\text{TB}_1(\alpha) \quad V_{++} := \bigcup_{n \geq 0} \alpha^n(V_+) \text{ is closed}; \text{ and}
\]

\[
\text{TB}_2(\alpha) \quad \{ [\alpha^{n+1}(V_+) : \alpha^n(V_+)] \}_{n \geq 0} \text{ is constant}.
\]

*In this case, \( s(\alpha) = [\alpha(V_+) : V_+] \).*

\( V \) is *tidy above* for \( \alpha \) if it satisfies \( \text{TA}(\alpha) \) and *tidy below* if it satisfies \( \text{TB}_1(\alpha) \) and \( \text{TB}_2(\alpha) \).
OUTLINE OF PROOF

1. Given $V \in \mathcal{B}(G)$, reduce to a subgroup $U$ that satisfies $TA(\alpha)$.
   \[w_U(\alpha) \leq w_V(\alpha), \text{ with equality iff } V \text{ satisfies } TA(\alpha).\]

2. Given $V \in \mathcal{B}(G)$ satisfying $TA(\alpha)$, augment $V$ to obtain a subgroup $U$ satisfying $TB(\alpha)$ as well.
   \[w_U(\alpha) \leq w_V(\alpha), \text{ with equality iff } V \text{ satisfies } TB(\alpha).\]

3. Show that, if $U$ and $V$ are both tidy for $\alpha$, then $w_U(\alpha) = w_V(\alpha)$.
Achieving tidiness above

From now on, $\alpha$ will be an automorphism.

In this case, $V_+ = \bigcap_{n \geq 0} \alpha^n(V)$ and $V_- = \bigcap_{n \geq 0} \alpha^{-n}(V)$.

Lemma

Let $V \in \mathcal{B}(G)$. There is $N \geq 0$ such that the subgroup $U := \bigcap_{k=0}^{N} \alpha^k(V)$ satisfies

$$\alpha(U) \subseteq \alpha(U_+)U.$$  (1)
Lemma
Suppose that $\alpha(V) \subseteq \alpha(V_+) V$. Then $V = V_+ V_-$. Conversely, if $V = V_+ V_-$ then $\alpha(V) \subseteq \alpha(V_+) V$.

Lemma
Let $V$ be any compact open subgroup of $G$. Then

$$w_V(\alpha) = [\alpha(V) : \alpha(V) \cap V] \geq [\alpha(V_+) : V_+],$$

with equality if and only if $\alpha(V) \leq \alpha(V_+) V$.

Summary: Given $V \in \mathcal{B}(G)$, there is $N \geq 0$ such that $U := \bigcap_{k=0}^{N} \alpha^k(V)$ is tidy above for $\alpha$. The subgroup $U$ satisfies $w_U(\alpha) \leq w_V(\alpha)$ with equality if and only if $V$ is already tidy above.
Examples

1. Let $G = \mathbb{F}_p^\mathbb{Z}$, let $\alpha$ be the shift automorphism and $V = \{g \in \mathbb{F}_p^\mathbb{Z} | g(-1) = e = g(1) = g(2)\}$. Then $N = 1$ and $U = \{g \in \mathbb{F}_p^\mathbb{Z} | g(n) = e \text{ if } |n| < 3\}$.

2. Let $G = (\mathbb{F}_p((t)), +)$, let $\alpha$ be multiplication by $t^{-1}$ and $V = \text{span}\{t^{-4}, t^{-3}, t^{-2}\} + \mathbb{F}_p[[t]]$. Then $N = 3$ and $U = \mathbb{F}_p[[t]]$. 
Examples

3. Let $G = \text{Aut}(T_q)$, let $\alpha$ be the inner automorphism $\alpha_g$ where $g$ is a translation with axis $\ell$, and $V = \text{Fix}(a)$, where $a$ is a vertex distance 4 from $\ell$. Then $N = 1$ and $U = \text{Fix}([a, g.a])$, where $[a, g.a]$ is the path of length 9 from $a$ to $g.a$ (which intersects $\ell$ in an edge).

4. Let $G = \text{SL}(n, \mathbb{Q}_p)$, let $\alpha$ conjugation by $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, and $V = \text{SL}(n, \mathbb{Z}_p)$. Then $N = 1$ and $U = \left\{ \begin{pmatrix} a_{11} & p a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid a_{ij} \in \mathbb{Z}_p \right\}$. 
References


