Cohomology of Banach Algebras
Fields Mini-Course

Michael C. White
Newcastle University
15 - 16 May, 2014
1.1 Summary

List of Topics

1. Basic Definitions and Notation
2. Amenability and Weak Amenability
3. Derivations
4. Injective Modules
5. Submodules and Quotient Modules
6. Higher Cohomology
7. Resolutions
8. Bifunctorial cohomology: Ext
9. Cyclic cohomology
1.1 Summary

List of Topics

1. Basic Definitions and Notation
1.1 Summary

List of Topics

1. Basic Definitions and Notation
2. Amenability and Weak Amenability
1.1 Summary

List of Topics

1. Basic Definitions and Notation
2. Amenability and Weak Amenability
3. Derivations
1.1 Summary

### List of Topics

1. Basic Definitions and Notation
2. Amenability and Weak Amenability
3. Derivations
4. Injective Modules
1.1 Summary

List of Topics

1. Basic Definitions and Notation
2. Amenability and Weak Amenability
3. Derivations
4. Injective Modules
5. Submodules and Quotient Modules
1.1 Summary

List of Topics

1. Basic Definitions and Notation
2. Amenability and Weak Amenability
3. Derivations
4. Injective Modules
5. Submodules and Quotient Modules
6. Higher Cohomology
1.1 Summary

List of Topics

1. Basic Definitions and Notation
2. Amenability and Weak Amenability
3. Derivations
4.Injective Modules
5. Submodules and Quotient Modules
6. Higher Cohomology
7. Resolutions
### 1.1 Summary

#### List of Topics

1. Basic Definitions and Notation
2. Amenability and Weak Amenability
3. Derivations
4. Injective Modules
5. Submodules and Quotient Modules
6. Higher Cohomology
7. Resolutions
8. Bifunctorial cohomology: $\text{Ext}$
1.1 Summary

List of Topics

1. Basic Definitions and Notation
2. Amenability and Weak Amenability
3. Derivations
4. Injective Modules
5. Submodules and Quotient Modules
6. Higher Cohomology
7. Resolutions
8. Bifunctorial cohomology: Ext
9. Cyclic cohomology
2.1 Standing assumptions

Cavets aka Excuses

All Banach algebras will have a unit, or bounded approximate identity
All modules will be Banach modules
In general a unit should be added before using the definitions
All maps will be continuous and linear
Algebras will act unitally on modules, or the b.a.i. will act as such
Sub- and quotient modules usually need to be (weakly) complemented as Banach spaces

For more details see:
- Helemskii for detailed treatment topological algebras
- Weibel for the algebraic background
- Loday for detailed treatment of cyclic cohomology
2.1 Standing assumptions

Cavets aka Excuses

- All Banach algebras will have a unit, or bounded approximate identity.
2.1 Standing assumptions

Cavets aka Excuses

- All Banach algebras will have a unit, or bounded approximate identity
- All modules will be Banach modules

For more details see:
- Helemskii for detailed treatment topological algebras
- Weibel for the algebraic background
- Loday for detailed treatment of cyclic cohomology
2.1 Standing assumptions

Cavets aka Excuses

- All Banach algebras will have a unit, or bounded approximate identity
- All modules will be Banach modules
- In general a unit should be added before using the definitions
2.1 Standing assumptions

Cavets aka Excuses

- All Banach algebras will have a unit, or bounded approximate identity
- All modules will be Banach modules
- In general a unit should be added before using the definitions
- All maps will be continuous and linear

For more details see:
- Helemskii for detailed treatment topological algebras
- Weibel for the algebraic background
- Loday for detailed treatment of cyclic cohomology
## Cavets aka Excuses

- All Banach algebras will have a unit, or bounded approximate identity
- All modules will be Banach modules
- In general a unit should be added before using the definitions
- All maps will be continuous and linear
- Algebras will act unitally on modules, or the b.a.i. will act as such

Sub- and quotient modules usually need to be (weakly) complemented as Banach spaces

For more details see:
- Helemskii for detailed treatment topological algebras
- Weibel for the algebraic background
- Loday for detailed treatment of cyclic cohomology
2.1 Standing assumptions

Cavets aka Excuses

- All Banach algebras will have a unit, or bounded approximate identity
- All modules will be Banach modules
- In general a unit should be added before using the definitions
- All maps will be continuous and linear
- Algebras will act unitally on modules, or the b.a.i. will act as such
- Sub- and quotient modules usually need to be (weakly) complemented as Banach spaces

For more details see:
- Helemskii for detailed treatment topological algebras
- Weibel for the algebraic background
- Loday for detailed treatment of cyclic cohomology
2.1 Standing assumptions

**Cavets aka Excuses**

- All Banach algebras will have a unit, or bounded approximate identity
- All modules will be Banach modules
- In general a unit should be added before using the definitions
- All maps will be continuous and linear
- Algebras will act unitally on modules, or the b.a.i. will act as such
- Sub- and quotient modules usually need to be (weakly) complemented as Banach spaces
- For more details see:

  - Helemskii for detailed treatment topological algebras
  - Weibel for the algebraic background
  - Loday for detailed treatment of cyclic cohomology
2.1 Standing assumptions

Cavets aka Excuses

- All Banach algebras will have a unit, or bounded approximate identity
- All modules will be Banach modules
- In general a unit should be added before using the definitions
- All maps will be continuous and linear
- Algebras will act unitally on modules, or the b.a.i. will act as such
- Sub- and quotient modules usually need to be (weakly) complemented as Banach spaces
- For more details see:
  - Helemskii for detailed treatment topological algebras
2.1 Standing assumptions

Cavets aka Excuses

- All Banach algebras will have a unit, or bounded approximate identity
- All modules will be Banach modules
- In general a unit should be added before using the definitions
- All maps will be continuous and linear
- Algebras will act unitally on modules, or the b.a.i. will act as such
- Sub- and quotient modules usually need to be (weakly) complemented as Banach spaces
- For more details see:
  - Helemskii for detailed treatment topological algebras
  - Weibel for the algebraic background
2.1 Standing assumptions

Cavets aka Excuses

- All Banach algebras will have a unit, or bounded approximate identity
- All modules will be Banach modules
- In general a unit should be added before using the definitions
- All maps will be continuous and linear
- Algebras will act unitally on modules, or the b.a.i. will act as such
- Sub- and quotient modules usually need to be (weakly) complemented as Banach spaces
- For more details see:
  - Helemskii for detailed treatment topological algebras
  - Weibel for the algebraic background
  - Loday for detailed treatment of cyclic cohomology
2.1 Standing assumptions

Cavets aka Excuses

- All Banach algebras will have a unit, or bounded approximate identity
- All modules will be Banach modules
- In general a unit should be added before using the definitions
- All maps will be continuous and linear
- Algebras will act unitally on modules, or the b.a.i. will act as such
- Sub- and quotient modules usually need to be (weakly) complemented as Banach spaces
- For more details see:
  - Helemskii for detailed treatment topological algebras
  - Weibel for the algebraic background
  - Loday for detailed treatment of cyclic cohomology
2.1 Standing assumptions

Cavets aka Excuses

- All Banach algebras will have a unit, or bounded approximate identity
- All modules will be Banach modules
- In general a unit should be added before using the definitions
- All maps will be continuous and linear
- Algebras will act unitally on modules, or the b.a.i. will act as such
- Sub- and quotient modules usually need to be (weakly) complemented as Banach spaces

For more details see:

- Helemskii for detailed treatment topological algebras
- Weibel for the algebraic background
- Loday for detailed treatment of cyclic cohomology
2.2 Definitions and Notation

**Derivations**

\[ Z_1(A, Y) = \{ D: A \to Y, s, t, D(ab) = aD(b) + D(a)b \} \]

**Inner derivations**

\[ B_1(A, Y) = \{ \delta_y(a) = ay - ya \} \]

Commutative bimodules, where \( ay = ya \) satisfy

\[ D(a^n) = a^n - 1D(a) \]

\[ H_1(A, Y) := Z_1(A, Y) / B_1(A, Y) \]

Note \( H_1(A, Y) \) a Banach space iff \( B_1(A, Y) \) closed. \[Ex\], \( Z_1(A, Y) \) is always closed.

For semigroup algebras this is related to reversal depth.
2.2 Definitions and Notation

Definitions and Notation

- Derivations $\mathcal{Z}^1(A, Y) = \{D : A \rightarrow Y, \text{s.t. } D(ab) = aD(b) + D(a)b\}$
### 2.2 Definitions and Notation

**Definitions and Notation**

- **Derivations** \( \mathcal{Z}^1(A, Y) = \{ D : A \to Y, \text{s.t. } D(ab) = aD(b) + D(a)b \} \)
- **Inner derivations** \( \mathcal{B}^1(A, Y) = \{ \delta_Y(a) = ay - ya \} \)
2.2 Definitions and Notation

Definitions and Notation

- Derivations $\mathcal{Z}^1(A, Y) = \{ D : A \rightarrow Y, \text{s.t. } D(ab) = aD(b) + D(a)b \}$
- Inner derivations $\mathcal{B}^1(A, Y) = \{ \delta_y(a) = ay - ya \}$
- Commutative bimodules, where $ay = ya$ satisfy $D(a^n) = a^{n-1}D(a)$
2.2 Definitions and Notation

**Definitions and Notation**

- **Derivations** \( \mathcal{Z}_1(A, Y) = \{ D : A \to Y, \text{s.t. } D(ab) = aD(b) + D(a)b \} \)
- **Inner derivations** \( \mathcal{B}_1(A, Y) = \{ \delta_y(a) = ay - ya \} \)
- **Commutative bimodules**, where \( ay = ya \) satisfy \( D(a^n) = a^{n-1}D(a) \)
- \( \mathcal{H}_1(A, Y) := \mathcal{Z}_1(A, Y)/\mathcal{B}_1(A, Y) \)
2.2 Definitions and Notation

<table>
<thead>
<tr>
<th>Definitions and Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Derivations $\mathcal{Z}^1(A, Y) = {D : A \to Y, \text{s.t. } D(ab) = aD(b) + D(a)b}$</td>
</tr>
<tr>
<td>Inner derivations $\mathcal{B}^1(A, Y) = {\delta_y(a) = ay - ya}$</td>
</tr>
<tr>
<td>Commutative bimodules, where $ay = ya$ satisfy $D(a^n) = a^{n-1}D(a)$</td>
</tr>
<tr>
<td>$\mathcal{H}^1(A, Y) := \mathcal{Z}^1(A, Y)/\mathcal{B}^1(A, Y)$</td>
</tr>
<tr>
<td>Note $\mathcal{H}^1(A, Y)$ a Banach space iff $\mathcal{B}^1(A, Y)$ closed [Ex],</td>
</tr>
</tbody>
</table>

Note $\mathcal{H}^1(A, Y)$ a Banach space iff $\mathcal{B}^1(A, Y)$ closed [Ex],
2.2 Definitions and Notation

Definitions and Notation

- Derivations $\mathcal{Z}^1(A, Y) = \{D : A \to Y, \text{s.t. } D(ab) = aD(b) + D(a)b\}$
- Inner derivations $\mathcal{B}^1(A, Y) = \{\delta_y(a) = ay - ya\}$
- Commutative bimodules, where $ay = ya$ satisfy $D(a^n) = a^{n-1}D(a)$
- $\mathcal{H}^1(A, Y) := \mathcal{Z}^1(A, Y)/\mathcal{B}^1(A, Y)$
- Note $\mathcal{H}^1(A, Y)$ a Banach space iff $\mathcal{B}^1(A, Y)$ closed [Ex].
### 2.2 Definitions and Notation

- **Definitions and Notation**
  - Derivations $\mathcal{Z}^1(A, Y) = \{D : A \to Y, \text{s.t. } D(ab) = aD(b) + D(a)b\}$
  - Inner derivations $\mathcal{B}^1(A, Y) = \{\delta_y(a) = ay - ya\}$
  - Commutative bimodules, where $ay = ya$ satisfy $D(a^n) = a^{n-1}D(a)$
  - $\mathcal{H}^1(A, Y) := \mathcal{Z}^1(A, Y)/\mathcal{B}^1(A, Y)$
  - Note $\mathcal{H}^1(A, Y)$ a Banach space iff $\mathcal{B}^1(A, Y)$ closed [Ex], $\mathcal{Z}^1(A, Y)$ is always closed
# 2.2 Definitions and Notation

<table>
<thead>
<tr>
<th>Definitions and Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Derivations $\mathcal{Z}^1(A, Y) = { D : A \to Y, \text{s.t. } D(ab) = aD(b) + D(a)b }$</td>
</tr>
<tr>
<td>- Inner derivations $\mathcal{B}^1(A, Y) = { \delta_y(a) = ay - ya }$</td>
</tr>
<tr>
<td>- Commutative bimodules, where $ay = ya$ satisfy $D(a^n) = a^{n-1}D(a)$</td>
</tr>
<tr>
<td>- $\mathcal{H}^1(A, Y) := \mathcal{Z}^1(A, Y)/\mathcal{B}^1(A, Y)$</td>
</tr>
<tr>
<td>- Note $\mathcal{H}^1(A, Y)$ a Banach space iff $\mathcal{B}^1(A, Y)$ closed [Ex], $\mathcal{Z}^1(A, Y)$ is always closed</td>
</tr>
<tr>
<td>- For semigroup algebras this is related to <em>reversal depth</em></td>
</tr>
</tbody>
</table>
2.2 Definitions and Notation

Definitions and Notation

- Derivations $\mathcal{Z}^1(A, Y) = \{ D : A \to Y, \text{s.t. } D(ab) = aD(b) + D(a)b \}$
- Inner derivations $\mathcal{B}^1(A, Y) = \{ \delta_y(a) = ay - ya \}$
- Commutative bimodules, where $ay = ya$ satisfy $D(a^n) = a^{n-1}D(a)$
- $\mathcal{H}^1(A, Y) := \mathcal{Z}^1(A, Y)/\mathcal{B}^1(A, Y)$
- Note $\mathcal{H}^1(A, Y)$ is a Banach space iff $\mathcal{B}^1(A, Y)$ closed [Ex], $\mathcal{Z}^1(A, Y)$ is always closed
- For semigroup algebras this is related to *reversal depth*
### 2.2 Definitions and Notation

#### Definitions and Notation

- **Derivations** $\mathcal{Z}^1(A, Y) = \{D : A \to Y, \text{s.t. } D(ab) = aD(b) + D(a)b\}$
- **Inner derivations** $\mathcal{B}^1(A, Y) = \{\delta_Y(a) = ay - ya\}$
- **Commutative bimodules**, where $ay = ya$ satisfy $D(a^n) = a^{n-1}D(a)$
- $\mathcal{H}^1(A, Y) := \mathcal{Z}^1(A, Y)/\mathcal{B}^1(A, Y)$
- Note $\mathcal{H}^1(A, Y)$ a Banach space iff $\mathcal{B}^1(A, Y)$ closed [Ex], $\mathcal{Z}^1(A, Y)$ is always closed
- For semigroup algebras this is related to *reversal depth*
2.3 Connections with Amenability

Amenability and Cohomology

A is amenable if \( H_1(A, X) = 0 \), for example \( L_1(G) \), for amenable \( G \).

A is weakly amenable if \( H_1(A, A') = 0 \), for example all \( L_1(G) \).

Why do we consider derivations into the dual module? The Singer-Wermer Theorem says that semi-simple algebras have no non-zero derivations \( D: A \to A \).

It is Functorial, \( \theta: A \to B \), gives \( \theta^*: \text{HH}^1(B) \to \text{HH}^1(A) \) [Ex].

Simplicially trivial algebras have \( \text{HH}_n(A) = H_n(A, A') = 0 \), e.g. semilattice algebras.

Cyclically amenable algebras have \( \text{HC}_1(A) = 0 \), e.g. \( \ell_1(Z) \) [Ex].

Cyclic derivations also satisfy \( D(a)(b) = -D(b)(a) \).
Amenability and Cohomology

- $A$ is *amenable* iff $\mathcal{H}^1(A, X') = 0$, for example $L^1(G)$, for amenable $G$
Amenability and Cohomology

- $A$ is \textit{amenable} iff $\mathcal{H}^1(A, X') = 0$, for example $L^1(G)$, for amenable $G$
- $A$ is \textit{weakly amenable} if $\mathcal{H}^1(A, A') = 0$, for example all $L^1(G)$
2.3 Connections with Amenability

Amenability and Cohomology

- $A$ is *amenable* iff $H^1(A, X') = 0$, for example $L^1(G)$, for amenable $G$
- $A$ is *weakly amenable* if $H^1(A, A') = 0$, for example all $L^1(G)$
- Why do we consider derivations into the dual module?
2.3 Connections with Amenability

Amenability and Cohomology

- $A$ is *amenable* iff $\mathcal{H}^1(A, X') = 0$, for example $L^1(G)$, for amenable $G$
- $A$ is *weakly amenable* if $\mathcal{H}^1(A, A') = 0$, for example all $L^1(G)$
- Why do we consider derivations into the dual module?
- The Singer-Wermer Theorem says that semi-simple algebras have no non-zero derivations $D : A \rightarrow A$, 

\[ \text{Simplicially trivial algebras have } \mathcal{H}^n(A) = H^n(A, A') = 0, \text{ e.g. semilattice algebras} \]

\[ \text{Cyclically amenable algebras have } \mathcal{H}C^1(A) = 0, \text{ e.g. } \ell^1(\mathbb{Z} +) \]
2.3 Connections with Amenability

Amenability and Cohomology

- \( A \) is *amenable* iff \( \mathcal{H}^1(A, X') = 0 \), for example \( L^1(G) \), for amenable \( G \).
- \( A \) is *weakly amenable* if \( \mathcal{H}^1(A, A') = 0 \), for example all \( L^1(G) \).
- Why do we consider derivations into the dual module?
- The Singer-Wermer Theorem says that semi-simple algebras have no non-zero derivations \( D : A \to A \),
- It is *Functoral*, \( \theta : A \to B \), gives \( \theta^* : \mathcal{H}\mathcal{H}^1(B) \to \mathcal{H}\mathcal{H}^1(A) \) [Ex]
2.3 Connections with Amenability

### Amenability and Cohomology

- A is *amenable* iff \( \mathcal{H}^1(A, X') = 0 \), for example \( L^1(G) \), for amenable \( G \)
- A is *weakly amenable* if \( \mathcal{H}^1(A, A') = 0 \), for example all \( L^1(G) \)
- Why do we consider derivations into the dual module?
- The Singer-Wermer Theorem says that semi-simple algebras have no non-zero derivations \( D : A \to A \),
- It is *Functoral*, \( \theta : A \to B \), gives \( \theta^* : \mathcal{H}\mathcal{H}^1(B) \to \mathcal{H}\mathcal{H}^1(A) \) [Ex]
- *Simplicially trivial* algebras have \( \mathcal{H}\mathcal{H}^n(A) = \mathcal{H}^n(A, A') = 0 \), e.g. semilattice algebras
2.3 Connections with Amenability

Amenability and Cohomology

- A is *amenable* iff $H^1(A, X') = 0$, for example $L^1(G)$, for amenable $G$
- A is *weakly amenable* if $H^1(A, A') = 0$, for example all $L^1(G)$
- Why do we consider derivations into the dual module?
- The Singer-Wermer Theorem says that semi-simple algebras have no non-zero derivations $D : A \to A$,
- It is *Functoral*, $\theta : A \to B$, gives $\theta^* : HH^1(B) \to HH^1(A)$ [Ex]
- *Simplicially trivial* algebras have $HH^n(A) = H^n(A, A') = 0$, e.g. semilattice algebras
- *Cyclically amenable* algebras have $HC^1(A) = 0$, e.g. $\ell^1(\mathbb{Z}_+)$ [Ex]
### Amenability and Cohomology

- A is *amenable* iff $\mathcal{H}^1(A, X') = 0$, for example $L^1(G)$, for amenable $G$
- A is *weakly amenable* if $\mathcal{H}^1(A, A') = 0$, for example all $L^1(G)$
- Why do we consider derivations into the dual module?
- The Singer-Wermer Theorem says that semi-simple algebras have no non-zero derivations $D : A \to A$
- It is *Functoral*, $\theta : A \to B$, gives $\theta^* : \mathcal{HH}^1(B) \to \mathcal{HH}^1(A)$ [Ex]
- *Simplicially trivial* algebras have $\mathcal{HH}^n(A) = \mathcal{H}^n(A, A') = 0$, e.g. semilattice algebras
- *Cyclically amenable* algebras have $\mathcal{HC}^1(A) = 0$, e.g. $\ell^1(\mathbb{Z}_+)$ [Ex]
- *Cyclic derivations* also satisfy $D(a)(b) = -D(b)(a)$,
2.3 Connections with Amenability

Amenability and Cohomology

- A is amenable iff $\mathcal{H}^1(A, X') = 0$, for example $L^1(G)$, for amenable $G$
- A is weakly amenable if $\mathcal{H}^1(A, A') = 0$, for example all $L^1(G)$
- Why do we consider derivations into the dual module?
- The Singer-Wermer Theorem says that semi-simple algebras have no non-zero derivations $D : A \to A$,
- It is Functoral, $\theta : A \to B$, gives $\theta^* : \mathcal{H}H^1(B) \to \mathcal{H}H^1(A)$ [Ex]
- Simplicially trivial algebras have $\mathcal{H}H^n(A) = \mathcal{H}^n(A, A') = 0$, e.g. semilattice algebras
- Cyclically amenable algebras have $\mathcal{H}C^1(A) = 0$, e.g. $\ell^1(\mathbb{Z}^+) \ [Ex]$
- Cyclic derivations also satisfy $D(a)(b) = -D(b)(a)$,
Amenability and Cohomology

- **A is amenable** iff $\mathcal{H}^1(A, X') = 0$, for example $L^1(G)$, for amenable $G$
- **A is weakly amenable** if $\mathcal{H}^1(A, A') = 0$, for example all $L^1(G)$
- Why do we consider derivations into the dual module?
- The Singer-Wermer Theorem says that semi-simple algebras have no non-zero derivations $D : A \to A$,
- It is *Functoral*, $\theta : A \to B$, gives $\theta^* : \mathcal{HH}^1(B) \to \mathcal{HH}^1(A)$ [Ex]
- *Simplicially trivial* algebras have $\mathcal{HH}^n(A) = \mathcal{H}^n(A, A') = 0$, e.g. semilattice algebras
- **Cyclically amenable** algebras have $\mathcal{HC}^1(A) = 0$, e.g. $\ell^1(\mathbb{Z}_+)$ [Ex]
- **Cyclic derivations** also satisfy $D(a)(b) = -D(b)(a)$,
2.4 Core Facts and Conventions

Multi-linear Miscellany

A will be a Banach algebra

Y will be a Banach space which is a left, right or bi-module over A

All linear maps will be bounded

X′ is the dual module of X in the usual way

\( \hat{E} \otimes \hat{F} \) is the projective tensor product of Banach spaces,

\((\hat{E} \otimes \hat{F})′ \sim = L(E, F; C) \sim = L(F, E′) \sim = L(E, F)\)

Also \( L(\hat{E} \otimes \hat{F}; G) \sim = BL(E, F; G) \sim = L(E, L(F, G))\)

Where \( L \) and \( BL \) denote spaces of (bounded) linear and bilinear maps

\( h(A(X, Y)) \) is the space of left A-module maps, i.e. \( T(ax) = aT(x) \)

\( h(A(E, F)) \) and \( A h(A(M, N)) \) denote the right and bimodule morphisms
2.4 Core Facts and Conventions

Multi-linear Miscellany

- \( A \) will be a Banach algebra
2.4 Core Facts and Conventions

Multi-linear Miscellany

- $A$ will be a Banach algebra
- $Y$ will be a Banach space which is a left, right or bi-module over $A$
Multi-linear Miscellany

- $A$ will be a Banach algebra
- $Y$ will be a Banach space which is a left, right or bi-module over $A$
- All linear maps will be bounded
Multi-linear Miscellany

- $A$ will be a Banach algebra
- $Y$ will be a Banach space which is a left, right or bi-module over $A$
- All linear maps will be bounded
- $X'$ is the dual module of $X$ in the usual way
2.4 Core Facts and Conventions

Multi-linear Miscellany

- $A$ will be a Banach algebra
- $Y$ will be a Banach space which is a left, right or bi-module over $A$
- All linear maps will be bounded
- $X'$ is the dual module of $X$ in the usual way
- $E \hat{\otimes} F$ is the projective tensor product of Banach spaces,
2.4 Core Facts and Conventions

Multi-linear Miscellany

- $A$ will be a Banach algebra
- $Y$ will be a Banach space which is a left, right or bi-module over $A$
- All linear maps will be bounded
- $X'$ is the dual module of $X$ in the usual way
- $E \hat{\otimes} F$ is the \textit{projective tensor product} of Banach spaces,
- $(E \hat{\otimes} F)' \cong BL(E, F; C) \cong L(E \hat{\otimes} F; C) \cong L(E, F') \cong L(F, E')$
2.4 Core Facts and Conventions

### Multi-linear Miscellany

- $A$ will be a Banach algebra
- $Y$ will be a Banach space which is a left, right or bi-module over $A$
- All linear maps will be bounded
- $X'$ is the dual module of $X$ in the usual way
- $E \hat{\otimes} F$ is the *projective tensor product* of Banach spaces,
- $(E \hat{\otimes} F)' \cong BL(E, F; C) \cong L(E \hat{\otimes} F; C) \cong L(E, F') \cong L(F, E')$
- $E \hat{\otimes} F$, and the above, sometimes inherit a bimodule structure
Multi-linear Miscellany

- \( A \) will be a Banach algebra
- \( Y \) will be a Banach space which is a left, right or bi-module over \( A \)
- All linear maps will be bounded
- \( X' \) is the dual module of \( X \) in the usual way
- \( E \hat{\otimes} F \) is the projective tensor product of Banach spaces,
- \( (E \hat{\otimes} F)' \cong BL(E, F; C) \cong L(E \hat{\otimes} F; C) \cong L(E, F') \cong L(F, E') \)
- \( E \hat{\otimes} F \), and the above, sometimes inherit a bimodule structure
- e.g., \( A \) is a left and also a right module, and \( A \hat{\otimes} A \) is a bimodule
Multi-linear Miscellany

- $A$ will be a Banach algebra
- $Y$ will be a Banach space which is a left, right or bi-module over $A$
- All linear maps will be bounded
- $X'$ is the dual module of $X$ in the usual way
- $E \hat{\otimes} F$ is the projective tensor product of Banach spaces,
  \[(E \hat{\otimes} F)' \cong BL(E, F; C) \cong L(E \hat{\otimes} F; C) \cong L(E, F') \cong L(F, E')\]
- $E \hat{\otimes} F$, and the above, sometimes inherit a bimodule structure
- e.g., $A$ is a left and also a right module, and $A \hat{\otimes} A$ is a bimodule
- Also $L(E \hat{\otimes} F; G) \cong BL(E, F; G) \cong L(E, L(F, G)),$
2.4 Core Facts and Conventions

Multi-linear Miscellany

- \( A \) will be a Banach algebra
- \( Y \) will be a Banach space which is a left, right or bi-module over \( A \)
- All linear maps will be bounded
- \( X' \) is the dual module of \( X \) in the usual way
- \( E \hat{\otimes} F \) is the projective tensor product of Banach spaces,
- \((E \hat{\otimes} F)' \cong BL(E, F; C) \cong L(E \hat{\otimes} F; C) \cong L(E, F') \cong L(F, E')\)
- \( E \hat{\otimes} F \), and the above, sometimes inherit a bimodule structure
- e.g., \( A \) is a left and also a right module, and \( A \hat{\otimes} A \) is a bimodule
- Also \( L(E \hat{\otimes} F; G) \cong BL(E, F; G) \cong L(E, L(F, G)) \),
- Where \( L \) and \( BL \) denote spaces of (bounded) linear and bilinear maps
2.4 Core Facts and Conventions

Multi-linear Miscellany

- $A$ will be a Banach algebra
- $Y$ will be a Banach space which is a left, right or bi-module over $A$
- All linear maps will be bounded
- $X'$ is the dual module of $X$ in the usual way
- $E \hat{\otimes} F$ is the \textit{projective tensor product} of Banach spaces,
- $(E \hat{\otimes} F)' \cong BL(E, F; C) \cong L(E \hat{\otimes} F; C) \cong L(E, F') \cong L(F, E')$
- $E \hat{\otimes} F$, and the above, sometimes inherit a bimodule structure
  - e.g., $A$ is a left and also a right module, and $A \hat{\otimes} A$ is a bimodule
  - Also $L(E \hat{\otimes} F; G) \cong BL(E, F; G) \cong L(E, L(F, G))$,
  - Where $L$ and $BL$ denote spaces of (bounded) linear and bilinear maps
  - $A h(X, Y)$ is the space of left $A$-module maps, i.e. $T(ax) = aT(x)$
2.4 Core Facts and Conventions

**Multi-linear Miscellany**

- $A$ will be a Banach algebra
- $Y$ will be a Banach space which is a left, right or bi-module over $A$
- All linear maps will be bounded
- $X'$ is the dual module of $X$ in the usual way
- $E \hat{\otimes} F$ is the *projective tensor product* of Banach spaces,
  
  $$(E \hat{\otimes} F)' \cong BL(E, F; C) \cong L(E \hat{\otimes} F; C) \cong L(E, F') \cong L(F, E')$$

- $E \hat{\otimes} F$, and the above, sometimes inherit a bimodule structure
- e.g., $A$ is a left and also a right module, and $A \hat{\otimes} A$ is a bimodule
- Also $L(E \hat{\otimes} F; G) \cong BL(E, F; G) \cong L(E, L(F, G))$,
- Where $L$ and $BL$ denote spaces of (bounded) linear and bilinear maps
- $Ah(X, Y)$ is the space of left $A$-module maps, i.e. $T(ax) = aT(x)$
- $h_A(E, F)$ and $Ah_A(M, N)$ denote the right and bimodule morphisms
2.4 Core Facts and Conventions

Multi-linear Miscellany

- $A$ will be a Banach algebra
- $Y$ will be a Banach space which is a left, right or bi-module over $A$
- All linear maps will be bounded
- $X'$ is the dual module of $X$ in the usual way
- $E \hat{\otimes} F$ is the projective tensor product of Banach spaces,
  
  $(E \hat{\otimes} F)' \cong BL(E, F; C) \cong L(E \hat{\otimes} F; C) \cong L(E, F') \cong L(F, E')$

- $E \hat{\otimes} F$, and the above, sometimes inherit a bimodule structure
- e.g., $A$ is a left and also a right module, and $A \hat{\otimes} A$ is a bimodule
- Also $L(E \hat{\otimes} F; G) \cong BL(E, F; G) \cong L(E, L(F, G))$,
- Where $L$ and $BL$ denote spaces of (bounded) linear and bilinear maps
- $A h(X, Y)$ is the space of left $A$-module maps, i.e. $T(ax) = aT(x)$
- $h_A(E, F)$ and $A h_A(M, N)$ denote the right and bimodule morphisms
2.4 Core Facts and Conventions

Multi-linear Miscellany

- $A$ will be a Banach algebra
- $Y$ will be a Banach space which is a left, right or bi-module over $A$
- All linear maps will be bounded
- $X'$ is the dual module of $X$ in the usual way
- $E \hat{\otimes} F$ is the \textit{projective tensor product} of Banach spaces,
  \[(E \hat{\otimes} F)' \cong BL(E, F; C) \cong L(E \hat{\otimes} F; C) \cong L(E, F') \cong L(F, E')\]
- $E \hat{\otimes} F$, and the above, sometimes inherit a bimodule structure
- e.g., $A$ is a left and also a right module, and $A \hat{\otimes} A$ is a bimodule
- Also $L(E \hat{\otimes} F; G) \cong BL(E, F; G) \cong L(E, L(F, G))$
- Where $L$ and $BL$ denote spaces of (bounded) linear and bilinear maps
- $A h(X, Y)$ is the space of left $A$-module maps, i.e. $T(ax) = aT(x)$
- $h_A(E, F)$ and $A h_A(M, N)$ denote the right and bimodule morphisms
2.5 Cohomology 101: Derivations

- $A$ a Banach algebra

A derivation $D : A \to \mathcal{Y}$ satisfies the 1-cocycle equation:

$$\delta D(a, b) = a \cdot D(b) - D(ab) + D(a) \cdot b.$$
2.5 Cohomology 101: Derivations

- $A$ a Banach algebra
- $Y$ a Banach $A$-bimodule
2.5 Cohomology 101: Derivations

- $A$ a Banach algebra
- $Y$ a Banach $A$-bimodule
- $D$ is a Derivation if $D : A \to Y$ and it satisfies the 1-cocycle equation 

$$(\delta D)(a, b) := +a \cdot D(b) - D(ab) + D(a) \cdot b.$$
2.5 Cohomology 101: Derivations

- A a Banach algebra
- Y a Banach A-bimodule
- $D$ is a Derivation if $D : A \rightarrow Y$ and it satisfies the 1-cocycle equation $(\delta D) = 0$, where

$$(\delta D)(a, b) := +a \cdot D(b) - D(ab) + D(a) \cdot b.$$
2.5 Cohomology 101: Derivations

- $A$ a Banach algebra
- $Y$ a Banach $A$-bimodule
- $D$ is a Derivation if $D : A \rightarrow Y$ and it satisfies the 1-cocycle equation $(\delta D) = 0$, where

$$(\delta D)(a, b) := +a \cdot D(b) − D(ab) + D(a) \cdot b.$$ 

We write $D \in Z^1(A; Y)$. 
A a Banach algebra

Y a Banach $A$-bimodule

$D$ is a *Derivation* if $D : A \to Y$ and it satisfies the 1-*cocycle equation* $(\delta D) = 0$, where

$$(\delta D)(a, b) := +a \cdot D(b) - D(ab) + D(a) \cdot b.$$ 

We write $D \in \mathbb{Z}^1(A; Y)$.

An *inner derivation* is a map given by a 1-*coboundary* of an element of $Y$

$$\delta y := (a \mapsto a \cdot y - y \cdot a)$$ 

these are always derivations.
2.5 Cohomology 101: Derivations

- A a Banach algebra
- Y a Banach A-bimodule
- D is a Derivation if \( D : A \to Y \) and it satisfies the 1-cocycle equation \((\delta D) = 0\), where

\[
(\delta D)(a, b) := +a \cdot D(b) - D(ab) + D(a) \cdot b.
\]

We write \( D \in Z^1(A; Y) \).

- An inner derivation is map given by a 1-coboundary of an element of Y

\[
\delta y := (a \mapsto a \cdot y - y \cdot a)
\]

these are always derivations.
A a Banach algebra

Y a Banach A-bimodule

$D$ is a Derivation if $D : A \rightarrow Y$ and it satisfies the 1-cocycle equation $(\delta D) = 0$, where

$$(\delta D)(a, b) := + a \cdot D(b) - D(ab) + D(a) \cdot b.$$ 

We write $D \in Z^1(A; Y)$.

An inner derivation is map given by a 1-coboundary of an element of $Y$

$$\delta y := (a \mapsto a \cdot y - y \cdot a)$$

these are always derivations. We write $\delta y \in B^1(A; Y)$. 

2.5 Cohomology 101: Derivations

- \( A \) a Banach algebra
- \( Y \) a Banach \( A \)-bimodule
- \( D \) is a Derivation if \( D : A \to Y \) and it satisfies the 1-cocycle equation
  \[(\delta D)(a, b) := +a \cdot D(b) - D(ab) + D(a) \cdot b.\]

We write \( D \in \mathcal{Z}^1(A; Y) \).

- An inner derivation is map given by a 1-coboundary of an element of \( Y \)
  \[\delta y := (a \mapsto a \cdot y - y \cdot a)\]
  these are always derivations. We write \( \delta y \in \mathcal{B}^1(A; Y) \).

- We measure how far from being all derivations are the inner derivations by
  \[\mathcal{H}^1(A; Y) = \frac{\mathcal{Z}^1(A; Y)}{\mathcal{B}^1(A; Y)}.\]
2.6 Examples of Derivations

- e.g. 1: $A = \ell^1(\mathbb{Z}_+)$, $Y = \mathbb{C}_0$, given by the character at 0.

\[ Df = f'(0) \]

is a derivation,
2.6 Examples of Derivations

- e.g. 1: $A = \ell^1(Z_+)$, $Y = C_0$, given by the character at 0.
  \[ Df = f'(0) \]
  is a derivation,
e.g. 1: $A = \ell^1(Z_+)$, $Y = \mathbf{C}_0$, given by the character at 0.

$$Df = f'(0)$$

is a derivation,

which cannot be inner as the module is commutative ($ay = ya$).
2.6 Examples of Derivations

- e.g. 1: \( A = \ell^1(Z_+) \), \( Y = C_0 \), given by the character at 0.
  \[
  Df = f'(0)
  \]
  is a derivation,
  which cannot be inner as the module is commutative (\( ay = ya \)).

- e.g. 2: \( A = \ell^1(Z_+) \), \( Y = A \hat{\otimes} A \), where \( D \) is given by
  \[
  Da = a \otimes 1 - 1 \otimes a.
  \]
  This derivation is always inner,
2.6 Examples of Derivations

- e.g. 1: $A = \ell^1(Z_+)$, $Y = \mathbb{C}_0$, given by the character at 0.
  
  $$Df = f'(0)$$

  is a derivation, which cannot be inner as the module is commutative ($ay = ya$).

- e.g. 2: $A = \ell^1(Z_+)$, $Y = A \hat{\otimes} A$, where $D$ is given by
  
  $$Da = a \otimes 1 - 1 \otimes a.$$ 

  This derivation is always inner,
2.6 Examples of Derivations

- e.g. 1: \( A = \ell^1(Z_+) \), \( Y = C_0 \), given by the character at 0.
  \[
  Df = f'(0)
  \]
  is a derivation, which cannot be inner as the module is commutative (\( ay = ya \)).

- e.g. 2: \( A = \ell^1(Z_+) \), \( Y = A \hat{\otimes} A \), where \( D \) is given by
  \[
  Da = a \otimes 1 - 1 \otimes a.
  \]
  This derivation is always inner, as
  \[
  D(a) = a \cdot (1 \otimes 1) - (1 \otimes 1) \cdot a.
  \]
2.6 Examples of Derivations

- e.g. 1: \( A = \ell^1(Z_+) \), \( Y = \mathbb{C}_0 \), given by the character at 0.
  \[
  Df = f'(0)
  \]
  is a derivation,
  which cannot be inner as the module is commutative (\( ay = ya \)).

- e.g. 2: \( A = \ell^1(Z_+) \), \( Y = A\hat{\otimes}A \), where \( D \) is given by
  \[
  Da = a \otimes 1 - 1 \otimes a.
  \]
  This derivation is always inner, as
  \[
  D(a) = a \cdot (1 \otimes 1) - (1 \otimes 1) \cdot a.
  \]

- e.g. 3: \( A = \ell^1(Z_+) \), \( Y = \text{Ker } \pi \subset A\hat{\otimes}A \), where \( D \) is given by
  \[
  Da = a \otimes 1 - 1 \otimes a.
  \]
  This derivation is \textit{not} inner
3.1 Injective Motivation

If a derivation $D : A \rightarrow Y$ is not inner, then

*Who is to blame: the derivation or the module?*
3.1 Injective Motivation

If a derivation $D : A \to Y$ is not inner, then

*Who is to blame: the derivation or the module?*

The module is the blame
3.1 Injective Motivation

If a derivation $D : A \rightarrow Y$ is not inner, then

*Who is to blame: the derivation or the module?*

---

**The module is the blame**

- Every bimodule $Y$ is canonically embedded in $L(A \hat{\otimes} A; Y)$,
3.1 Injective Motivation

If a derivation $D : A \rightarrow Y$ is not inner, then

Who is to blame: the derivation or the module?

The module is the blame

- Every bimodule $Y$ is canonically embedded in $L(A\hat{\otimes}A; Y)$,
- ... by $y \mapsto ((a \otimes b) \mapsto b.y.a)$ [Ex]

Theorem: Every derivation $D : A \rightarrow Y$ induces a derivation $\tilde{D} : A \rightarrow L(A\hat{\otimes}A; Y)$, which is inner

Proof: [Ex] or by later theory "$L(A\hat{\otimes}A; Y)$ is cofree, so biinjective"

Fact: all dual modules for amenable algebras are biinjective
3.1 Injective Motivation

If a derivation $D : A \rightarrow Y$ is not inner, then

*Who is to blame: the derivation or the module?*

---

**The module is the blame**

- Every bimodule $Y$ is canonically embedded in $L(A\hat{\otimes}A; Y)$,
- ... by $y \mapsto ((a \otimes b) \mapsto b.y.a)$ [Ex]
- **Theorem:** Every derivation $D : A \rightarrow Y$ induces a derivation $	ilde{D} : A \rightarrow L(A\hat{\otimes}A; Y)$, which is inner
3.1 Injective Motivation

If a derivation $D : A \to Y$ is not inner, then

*Who is to blame: the derivation or the module?*

---

**The module is the blame**

- Every bimodule $Y$ is canonically embedded in $L(A\hat{\otimes}A; Y)$,
- ... by $y \mapsto ((a \otimes b) \mapsto b.y.a)$ [Ex]
- **Theorem**: Every derivation $D : A \to Y$ induces a derivation $	ilde{D} : A \to L(A\hat{\otimes}A; Y)$, which is inner
- **Proof**: [Ex] or by later theory “$L(A\hat{\otimes}A; Y)$ is cofree, so bijective”
3.1 Injective Motivation

If a derivation $D : A \rightarrow Y$ is not inner, then

*Who is to blame: the derivation or the module?*

---

The module is the blame

- Every bimodule $Y$ is canonically embedded in $L(A \hat{\otimes} A; Y)$,
- ... by $y \mapsto ((a \otimes b) \mapsto b.y.a)$ [Ex]

**Theorem:** Every derivation $D : A \rightarrow Y$ induces a derivation $	ilde{D} : A \rightarrow L(A \hat{\otimes} A; Y)$, which is inner

**Proof:** [Ex] or by later theory “$L(A \hat{\otimes} A; Y)$ is cofree, so biinjective”

**Fact:** all dual modules for amenable algebras are biinjective
3.2 Injective modules - Definition

Defining properties
3.2 Injective modules - Definition

Defining properties

- The Hahn-Banach Theorem for $E \subseteq F$ says that every linear map $L(E, \mathbb{C})$ extends, or lifts to a map from $F$:

$$L(E, \mathbb{C}) \text{ extends, or lifts to a map from } F.$$
3.2 Injective modules - Definition

Defining properties

- The Hahn-Banach Theorem for $E \subseteq F$ says that every linear map $L(E, C)$ extends, or \textit{lifts} to a map from $F$:
3.2 Injective modules - Definition

Defining properties

- The Hahn-Banach Theorem for $E \subseteq F$ says that every linear map $L(E, \mathbb{C})$ extends, or *lifts* to a map from $F$: $L(E, \mathbb{C}) \uparrow \downarrow \hookrightarrow F \quad \rightarrow \quad \hookrightarrow \quad E \rightarrow \mathbb{C}$

We say the module $I$ is injective, if for every admissible $Z \hookrightarrow Y$, $\theta \in h_A(Z, I)$ lifts to $\tilde{\theta} \in h_A(Y, I)$: $Y \uparrow \downarrow Z \rightarrow I$

We say $F$ is a flat module iff $F'$ is injective. Projective implies flat.
### 3.2 Injective modules - Definition

**Defining properties**

- The Hahn-Banach Theorem for $E \subseteq F$ says that every linear map $L(E, \mathbb{C})$ extends, or *lifts* to a map from $F$:
  
  $\uparrow \quad \downarrow$
  
  $E \quad \rightarrow \quad \mathbb{C}$

- We say the module $I$ is *injective*, if for every admissible $Z \hookrightarrow Y$,

  $\theta \in h_{\mathcal{A}}(Z, I)$ lifts to $\tilde{\theta} \in h_{\mathcal{A}}(Y, I)$:
3.2 Injective modules - Definition

Defining properties

- The Hahn-Banach Theorem for $E \subseteq F$ says that every linear map $L(E, \mathbb{C})$ extends, or lifts to a map from $F$:

  $L(E, \mathbb{C}) \to \mathbb{C}
  \begin{array}{c}
  \uparrow & \Rightarrow \\
  E & \to
  \end{array}
  \begin{array}{c}
  \uparrow & \Rightarrow \\
  F & \to
  \end{array}$

- We say the module $I$ is injective, if for every admissible $Z \hookrightarrow Y$,

  $\theta \in h_A(Z, I)$ lifts to $\tilde{\theta} \in h_A(Y, I)$:

  $\begin{array}{c}
  \theta \in h_A(Z, I) \to \tilde{\theta} \in h_A(Y, I)
  \end{array}$
The Hahn-Banach Theorem for $E \subseteq F$ says that every linear map $L(E, \mathbb{C})$ extends, or lifts to a map from $F$: $E \rightarrow \mathbb{C}$.

We say the module $I$ is injective, if for every admissible $Z \hookrightarrow Y$, $\theta \in h_A(Z, I)$ lifts to $\tilde{\theta} \in h_A(Y, I)$: $Z \rightarrow I$.
3.2 Injective modules - Definition

Defining properties

- The Hahn-Banach Theorem for $E \subseteq F$ says that every linear map $L(E, C)$ extends, or lifts to a map from $F$: $L(E, C) \uparrow \downarrow E \rightarrow C$

- We say the module $I$ is injective, if for every admissible $Z \rightarrow Y$, $\theta \in h_A(Z, I)$ lifts to $\tilde{\theta} \in h_A(Y, I)$: $\uparrow \downarrow Y \rightarrow I$

- A module is called projective if we can lift maps, if for every admissible $Y \rightarrow X$: $\uparrow \downarrow Y \rightarrow X$
3.2 Injective modules - Definition

**Defining properties**

- The Hahn-Banach Theorem for $E \subseteq F$ says that every linear map $L(E, \mathbb{C})$ extends, or *lifts* to a map from $F$: 

  \[
  \begin{array}{c}
  L(E, \mathbb{C}) \quad \uparrow \quad \downarrow \\
  E \quad \rightarrow \quad \mathbb{C}
  \end{array}
  \]

- We say the module $I$ is *injective*, if for every admissible $Z \hookrightarrow Y$, $\theta \in h_A(Z, I)$ lifts to $\tilde{\theta} \in h_A(Y, I)$:

  \[
  \begin{array}{c}
  \theta \quad \uparrow \quad \downarrow \\
  Z \quad \rightarrow \quad I
  \end{array}
  \]

- A module is called *projective* if we can lift maps, if for every admissible $Y \rightarrow X$:

  \[
  \begin{array}{c}
  \theta \quad \uparrow \quad \downarrow \\
  P \quad \rightarrow \quad X
  \end{array}
  \]
3.2 Injective modules - Definition

**Defining properties**

- The Hahn-Banach Theorem for $E \subseteq F$ says that every linear map $L(E, \mathbb{C})$ extends, or *lifts* to a map from $F$:

  $E \rightarrow \mathbb{C}$

  $F$

  $L(E, \mathbb{C})$ extends, or *lifts* to a map from $F$.

- We say the module $I$ is *injective*, if for every admissible $Z \hookrightarrow Y$,

  $Y$

  $\theta \in h_A(Z, I)$ lifts to $\tilde{\theta} \in h_A(Y, I)$:

  $Z \rightarrow I$

- A module is called *projective* is we can lift maps,

  $Y$

  if for every admissible $Y \twoheadrightarrow X$:

  $P \rightarrow X$

- We say $F$ is a *flat module* iff $F'$ is injective.
3.2 Injective modules - Definition

Defining properties

- The Hahn-Banach Theorem for $E \subseteq F$ says that every linear map $F \rightarrow \mathbb{C}$
  $L(E, \mathbb{C})$ extends, or *lifts* to a map from $F$: $E \rightarrow \mathbb{C}$

- We say the module $I$ is *injective*, if for every admissible $Z \hookrightarrow Y$,
  $Y \rightarrow I$
  $\theta \in h_A(Z, I)$ lifts to $\tilde{\theta} \in h_A(Y, I)$:

- A module is called *projective* is we can lift maps,
  $Y \rightarrow X$
  if for every admissible $Y \twoheadrightarrow X$:
  $P \twoheadrightarrow X$

- We say $F$ is a *flat module* iff $F'$ is injective.
3.2 Injective modules - Definition

Defining properties

- The Hahn-Banach Theorem for $E \subseteq F$ says that every linear map $L(E, C)$ extends, or lifts to a map from $F$:

$$
\begin{array}{ccc}
F & \uparrow & \downarrow \\
E & \rightarrow & C
\end{array}
$$

- We say the module $I$ is **injective**, if for every admissible $Z \hookrightarrow Y$,

$$
\begin{array}{ccc}
Y & \uparrow & \downarrow \\
Z & \rightarrow & I
\end{array}
$$

- A module is called **projective** is we can lift maps,

$$
\begin{array}{ccc}
Y & \uparrow & \downarrow \\
P & \rightarrow & X
\end{array}
$$

- We say $F$ is a **flat module** iff $F'$ is injective. Projective implies flat. [Ex]
3.2 Injective modules - Definition

Defining properties

- The Hahn-Banach Theorem for $E \subseteq F$ says that every linear map $L(E, \mathbb{C})$ extends, or lifts to a map from $F$: $E \rightarrow F$

- We say the module $I$ is injective, if for every admissible $Z \hookrightarrow Y$,$Y \uparrow \downarrow Z \rightarrow I$

- $\theta \in h_A(Z, I)$ lifts to $\tilde{\theta} \in h_A(Y, I)$:$Y \uparrow \downarrow Z \rightarrow I$

- A module is called projective is we can lift maps, $Y \rightarrow X$ if for every admissible $Y \rightarrow X$: $P \rightarrow X$

- We say $F$ is a flat module iff $F'$ is injective. Projective implies flat [Ex]
3.3 Injective modules - Properties

Basic Properties of Injective modules
3.3 Injective modules - Properties

Basic Properties of Injective modules

- **Theorem:** The right module \( I \cong L(A, E) \) is injective
3.3 Injective modules - Properties

Basic Properties of Injective modules

- **Theorem:** The right module $I \cong L(A, E)$ is injective
- **Proof ...** [Ex]

Aside: the projection takes the place of averaging arguments in applications
### Basic Properties of Injective modules

- **Theorem:** The right module $I \cong L(A, E)$ is injective
  
  **Proof** . . . [Ex]

- **Theorem:** The right module $I$ is injective iff it is a module summand in $I \oplus I_2 \cong L(A, I)$

---

---
3.3 Injective modules - Properties

Basic Properties of Injective modules

- **Theorem:** The right module $l \cong L(A, E)$ is injective
- **Proof** ... [Ex]

- **Theorem:** The right module $l$ is injective iff it is a module summand in $l \oplus l_2 \cong L(A, l)$
- **Proof** ... [Ex]
### Basic Properties of Injective modules

**Theorem:** The right module $I \cong L(A, E)$ is injective

**Proof ...** [Ex]

**Theorem:** The right module $I$ is injective iff it is a module summand in $I \oplus I_2 \cong L(A, I)$

**Proof ...** [Ex]

**Aside:** the projection takes the place of *averaging* arguments in applications
3.3 Injective modules - Properties

Basic Properties of Injective modules

- **Theorem:** The right module $I \cong L(A, E)$ is injective
- PROOF ...[Ex]

- **Theorem:** The right module $I$ is injective iff it is a module summand in $I \oplus I_2 \cong L(A, I)$
- PROOF ...[Ex]

- Aside: the projection takes the place of *averaging* arguments in applications
3.3 Injective modules - Properties

Basic Properties of Injective modules

- **Theorem**: The right module $I \cong L(A, E)$ is injective
  - PROOF . . . [Ex]
- **Theorem**: The right module $I$ is injective iff it is a module summand in $I \oplus I_2 \cong L(A, I)$
  - PROOF . . . [Ex]
- Aside: the projection takes the place of *averaging* arguments in applications

Basic Properties of Projective modules
3.3 Injective modules - Properties

Basic Properties of Injective modules

- **Theorem:** The right module $I \cong L(A, E)$ is injective
  - PROOF ... [Ex]

- **Theorem:** The right module $I$ is injective iff it is a module summand in $I \oplus I_2 \cong L(A, I)$
  - PROOF ... [Ex]

- Aside: the projection takes the place of *averaging* arguments in applications

Basic Properties of Projective modules

- **Theorem:** The left module $P \cong A \hat{\otimes} E$ is projective
3.3 Injective modules - Properties

Basic Properties of Injective modules

- **Theorem:** The right module $I \cong L(A, E)$ is injective
- PROOF ... [Ex]

- **Theorem:** The right module $I$ is injective iff it is a module summand in $I \oplus I_2 \cong L(A, I)$
- PROOF ... [Ex]

- Aside: the projection takes the place of averaging arguments in applications

Basic Properties of Projective modules

- **Theorem:** The left module $P \cong A\hat{\otimes}E$ is projective
- PROOF ... [Ex]
3.3 Injective modules - Properties

**Basic Properties of Injective modules**

- **Theorem:** The right module $I \cong L(A, E)$ is injective
- PROOF ... [Ex]

- **Theorem:** The right module $I$ is injective iff it is a module summand in $I \oplus I_2 \cong L(A, I)$
- PROOF ... [Ex]

- Aside: the projection takes the place of averaging arguments in applications

**Basic Properties of Projective modules**

- **Theorem:** The left module $P \cong A \hat{\otimes} E$ is projective
- PROOF ... [Ex]

- **Theorem:** The left module $P$ is projective iff it is a module summand in $P \oplus P_2 \cong A \hat{\otimes} P$ [Ex]
3.3 Injective modules - Properties

Basic Properties of Injective modules

- **Theorem:** The right module $I \cong L(A, E)$ is injective
- PROOF . . . [Ex]
- **Theorem:** The right module $I$ is injective iff it is a module summand in $I \oplus I_2 \cong L(A, I)$
- PROOF . . . [Ex]
- Aside: the projection takes the place of *averaging* arguments in applications

Basic Properties of Projective modules

- **Theorem:** The left module $P \cong A \hat{\otimes} E$ is projective
- PROOF . . . [Ex]
- **Theorem:** The left module $P$ is projective iff it is a module summand in $P \oplus P_2 \cong A \hat{\otimes} P$ [Ex]
3.3 Injective modules - Properties

Basic Properties of Injective modules

- **Theorem:** The right module $I \cong L(A, E)$ is injective
  - PROOF . . .[Ex]
- **Theorem:** The right module $I$ is injective iff it is a module summand in $I \oplus I_2 \cong L(A, I)$
  - PROOF . . .[Ex]
- Aside: the projection takes the place of *averaging* arguments in applications

Basic Properties of Projective modules

- **Theorem:** The left module $P \cong A \hat{\otimes} E$ is projective
  - PROOF . . .[Ex]
- **Theorem:** The left module $P$ is projective iff it is a module summand in $P \oplus P_2 \cong A \hat{\otimes} P$ [Ex]
3.4 Bimodules

Biinjective definitions come for free
Biinjective definitions come for free

- Recall that $A \hat{\otimes} A = A^{ev}$ can be made into an algebra by using the product:

\[(a_1 \otimes b_1) \times (a_2 \otimes b_2) = (a_1 a_2 \otimes b_2 b_1)\]
3.4 Bimodules

Bijective definitions come for free

- Recall that $A \hat{\otimes} A = A^{ev}$ can be made into an algebra by using the product:

$$(a_1 \otimes b_1) \times (a_2 \otimes b_2) = (a_1 a_2 \otimes b_2 b_1)$$
Biinjective definitions come for free

- Recall that \( A \hat{\otimes} A = A^{ev} \) can be made into an algebra by using the product: 
  \[
  (a_1 \otimes b_1) \times (a_2 \otimes b_2) = (a_1 a_2 \otimes b_2 b_1)
  \]
Biinjective definitions come for free

- Recall that $A \hat{\otimes} A = A^{ev}$ can be made into an algebra by using the product: 
  \[(a_1 \otimes b_1) \times (a_2 \otimes b_2) = (a_1 a_2 \otimes b_2 b_1) \]
Biinjective definitions come for free

- Recall that $A \hat{\otimes} A = A^{ev}$ can be made into an algebra by using the product: $(a_1 \otimes b_1) \times (a_2 \otimes b_2) = (a_1 a_2 \otimes b_2 b_1)$ [Ex]
- Any $A$ bimodule $Y$ is now a right $A^{ev}$ module by:

\[ Y(a \otimes b) = ba \]
3.4 Bimodules

Biinjective definitions come for free

- Recall that $A \hat{\otimes} A = A^{ev}$ can be made into an algebra by using the product: $(a_1 \otimes b_1) \times (a_2 \otimes b_2) = (a_1 a_2 \otimes b_2 b_1)$ [Ex]
- Any $A$ bimodule $Y$ if now a right $A^{ev}$ module by:

\[ y(a \otimes b) = bya \]
3.4 Bimodules

Biinjective definitions come for free

- Recall that $A \hat{\otimes} A = A^{ev}$ can be made into an algebra by using the product:
  \[(a_1 \otimes b_1) \times (a_2 \otimes b_2) = (a_1 a_2 \otimes b_2 b_1)\] [Ex]

- Any $A$ bimodule $Y$ if now a right $A^{ev}$ module by:
  $y(a \otimes b) = bya$
3.4 Bimodules

Biinjective definitions come for free

- Recall that $A \hat{\otimes} A = A^{ev}$ can be made into an algebra by using the product: $(a_1 \otimes b_1) \times (a_2 \otimes b_2) = (a_1 a_2 \otimes b_2 b_1)$ [Ex]
- Any $A$ bimodule $Y$ is now a right $A^{ev}$ module by: $y(a \otimes b) = bya$
- We can now define a module to be biinjective if it is right injective as an $A^{ev}$ module
Biinjective definitions come for free

- Recall that $A \hat{\otimes} A = A^{ev}$ can be made into an algebra by using the product: $(a_1 \otimes b_1) \times (a_2 \otimes b_2) = (a_1a_2 \otimes b_2b_1)$ [Ex]
- Any $A$ bimodule $Y$ if now a right $A^{ev}$ module by: $y(a \otimes b) = bya$
- We can now define a module to be biinjective if it is right injective as an $A^{ev}$ module
- Similarly, we can consider left module to be right modules over $A^{op}$, and vice versa
Biinjective definitions come for free

- Recall that $A \hat{\otimes} A = A^{ev}$ can be made into an algebra by using the product: $(a_1 \otimes b_1) \times (a_2 \otimes b_2) = (a_1 a_2 \otimes b_2 b_1)$ [Ex]
- Any $A$ bimodule $Y$ if now a right $A^{ev}$ module by: $y(a \otimes b) = bya$
- We can now define a module to be biinjective if it is right injective as an $A^{ev}$ module
- Similarly, we can consider left module to be right modules over $A^{op}$, and vice versa
- It is natural to ask:
  
  *if $I$ is left injective and right injective, then is it biinjective?*
3.4 Bimodules

Biinjective definitions come for free

- Recall that $A \hat{\otimes} A = A^{ev}$ can be made into an algebra by using the product: $(a_1 \otimes b_1) \times (a_2 \otimes b_2) = (a_1 a_2 \otimes b_2 b_1)$ [Ex]
- Any $A$ bimodule $Y$ if now a right $A^{ev}$ module by: $y(a \otimes b) = bya$
- We can now define a module to be biinjective if it is right injective as an $A^{ev}$ module
- Similarly, we can consider left module to be right modules over $A^{op}$, and vice versa
- It is natural to ask:
  
  if $I$ is left injective and right injective, then is it biinjective?
- $L(A \hat{\otimes} A, I) \cong L(A, L(A, I)) \cong L(A, I \oplus I_1) \cong L(A, I) \oplus L(A, I_1) \cong I \oplus I_2 \oplus L(A, I_1)$ [Ex]
3.4 Bimodules

Biinjective definitions come for free

- Recall that $A \hat{\otimes} A = A^{ev}$ can be made into an algebra by using the product: $(a_1 \otimes b_1) \times (a_2 \otimes b_2) = (a_1 a_2 \otimes b_2 b_1)$ [Ex]
- Any $A$ bimodule $Y$ if now a right $A^{ev}$ module by: $y(a \otimes b) = bya$
- We can now define a module to be biinjective if it is right injective as an $A^{ev}$ module
- *Similarly*, we can consider left module to be right modules over $A^{op}$, and vice versa
- It is natural to ask:
  - *if $I$ is left injective and right injective, then is it biinjective?*
- $L(A \hat{\otimes} A, I) \cong L(A, L(A, I)) \cong L(A, I \oplus I_1) \cong L(A, I) \oplus L(A, I_1) \cong I \oplus I_2 \oplus L(A, I_1)$ [Ex]
3.4 Bimodules

Biinjective definitions come for free

- Recall that $A \hat{\otimes} A = A^{ev}$ can be made into an algebra by using the product: $(a_1 \otimes b_1) \times (a_2 \otimes b_2) = (a_1 a_2 \otimes b_2 b_1)$ [Ex]
- Any $A$ bimodule $Y$ if now a right $A^{ev}$ module by: $y(a \otimes b) = bya$
- We can now define a module to be *biinjective* if it is right injective as an $A^{ev}$ module
- *Similarly*, we can consider left module to be right modules over $A^{op}$, and vice versa
- It is natural to ask:
  *if $I$ is left injective and right injective, then is it biinjective?*
- $L(A \hat{\otimes} A, I) \cong L(A, L(A, I)) \cong L(A, I \oplus I_1) \cong L(A, I) \oplus L(A, I_1) \cong I \oplus I_2 \oplus L(A, I_1)$ [Ex]
3.5 Averaging with Biinjectives

Derivations into biinjective modules are inner

Proof:
\[ h(y(b, c)) = cyb \]
plays the role of the inner derivation

Recall that \( \bar{h}y = y \) for the bimodule map \( \bar{\cdot} : L(A^{\hat{\otimes}} \otimes A; Y) \to Y \)

\[ g(b, c) = cD(b) \]
plays the role of the linear map to be averaged

\[(a \cdot g)(b, c) = g(b, ca) = caD(b) \]

\[(g \cdot a)(b, c) = g(ab, c) = cD(ab) = caD(b) + cD(a)b \]

\[ a \cdot \bar{g} - \bar{g} \cdot a = D(a) \]

Michael C. White (Newcastle University)
3.5 Averaging with Biinjectives

Derivations into biinjective modules are inner

Proof:

$$h_y(b, c) := cyb$$ plays the role of the inner derivation

Recall that $$\bar{h}_y = y$$ for the bimodule map $$\bar{\cdot}$$:

$$L(A \hat{\otimes} A; Y) \to Y$$

$$g(b, c) := c \cdot D(b)$$ plays the role of the linear map to be averaged

$$(a \cdot g)(b, c) = g(b, ca) = ca \cdot D(b)$$

$$(g \cdot a)(b, c) = g(ab, c) = c \cdot D(ab) = ca \cdot D(b) + c \cdot D(a)$$

$$[(a \cdot g) - (g \cdot a)](b, c) = h_D(a)(b, c)$$

Michael C. White (Newcastle University) Cohomology of Banach Algebras 15 - 16 May, 2014
Derivations into biinjective modules are inner

- **Proof:**
- \( h_y(b, c) := cyb \) plays the role of the inner derivation
3.5 Averaging with Biinjectives

Derivations into biinjective modules are inner

- **Proof:**
  - \( h_y(b, c) := cyb \) plays the role of the inner derivation
  - Recall that \( \bar{h}_y = y \) for the bimodule map \( \bar{\cdot} : L(A \hat{\otimes} A; Y) \to Y \)
3.5 Averaging with Biinjectives

Derivations into biinjective modules are inner

- **Proof:**
- $h_y(b, c) := cyb$ plays the role of the inner derivation
- Recall that $\bar{h}_y = y$ for the bimodule map $\bar{\cdot}: L(A\hat{\otimes}A; Y) \to Y$
- $g(b, c) := c.D(b)$ plays the role of the linear map to be averaged
3.5 Averaging with Biinjectives

Derivations into biinjective modules are inner

**Proof:**

1. $h_y(b, c) := cyb$ plays the role of the inner derivation
2. Recall that $\tilde{h}_y = y$ for the bimodule map $\tilde{\cdot} : L(A\hat{\otimes} A; Y) \to Y$
3. $g(b, c) := c.D(b)$ plays the role of the linear map to be averaged
4. $(a.g)(b, c) = g(b, ca) = ca.D(b)$
3.5 Averaging with Biinjectives

Derivations into biinjective modules are inner

**Proof:**

- $h_y(b, c) := cyb$ plays the role of the inner derivation
- Recall that $\bar{h}_y = y$ for the bimodule map $\bar{\cdot} : L(A\hat{\otimes}A; Y) \to Y$
- $g(b, c) := c.D(b)$ plays the role of the linear map to be averaged
- $(a.g)(b, c) = g(b, ca) = ca.D(b)$
- $(g.a)(b, c) = g(ab, c) = c.D(ab) = ca.D(b) + c.D(a).b$
3.5 Averaging with Biinjectives

Derivations into bijective modules are inner

**Proof:**
- $h_y(b, c) := cyb$ plays the role of the inner derivation
- Recall that $\tilde{h}_y = y$ for the bimodule map $\tilde{\cdot}: L(A \hat{\otimes} A; Y) \to Y$
- $g(b, c) := c.D(b)$ plays the role of the linear map to be averaged
- $(a.g)(b, c) = g(b, ca) = ca.D(b)$
- $(g.a)(b, c) = g(ab, c) = c.D(ab) = ca.D(b) + c.D(a).b$
- $[(a.g) - (g.a)](b, c) = h_{D(a)}(b, c)$
3.5 Averaging with Biinjectives

Derivations into bijective modules are inner

- **Proof:**
  - $h_y(b, c) := cyb$ plays the role of the inner derivation
  - Recall that $\bar{h}y = y$ for the bimodule map $\bar{\cdot}: L(A\hat{\otimes}A; Y) \to Y$
  - $g(b, c) := c.D(b)$ plays the role of the linear map to be averaged
  - $(a.g)(b, c) = g(b, ca) = ca.D(b)$
  - $(g.a)(b, c) = g(ab, c) = c.D(ab) = ca.D(b) + c.D(a).b$
  - $[(a.g) - (g.a)](b, c) = h_{D(a)}(b, c)$
  - $a.\bar{g} - \bar{g}.a = D(a)$
3.5 Averaging with Biinjectives

Derivations into bijective modules are inner

- **Proof:**
- \( h_y(b, c) := cyb \) plays the role of the inner derivation
- Recall that \( \bar{h}_y = y \) for the bimodule map \( \bar{\cdot} : L(A \hat{\otimes} A; Y) \to Y \)
- \( g(b, c) := c.D(b) \) plays the role of the linear map to be averaged
- \( (a.g)(b, c) = g(b, ca) = ca.D(b) \)
- \( (g.a)(b, c) = g(ab, c) = c.D(ab) = ca.D(b) + c.D(a).b \)
- \( [(a.g) - (g.a)](b, c) = h_{D(a)}(b, c) \)
- \( a.\bar{g} - \bar{g}.a = D(a) \)
3.5 Averaging with Biinjectives

Derivations into biinjective modules are inner

- **Proof:**
- \( h_y(b, c) := cyb \) plays the role of the inner derivation
- Recall that \( \bar{h}_y = y \) for the bimodule map \( \bar{\cdot} : L(A \hat{\otimes} A; Y) \to Y \)
- \( g(b, c) := c.D(b) \) plays the role of the linear map to be averaged
- \( (a.g)(b, c) = g(b, ca) = ca.D(b) \)
- \( (g.a)(b, c) = g(ab, c) = c.D(ab) = ca.D(b) + c.D(a).b \)
- \( [(a.g) - (g.a)](b, c) = h_{D(a)}(b, c) \)
- \( a.\bar{g} - \bar{g}.a = D(a) \)
Dual modules over amenable algebras are injective
Dual modules over amenable algebras are injective

- Recall a Banach algebra $A$ is *amenable* if it has an approximate diagonal: a bounded net $m_\lambda \in A\hat{\otimes}A$, so that

\[
\text{Proof: Denote the approximate diagonal by } m_\lambda = \sum_{i=1}^{\infty} a_{\lambda i} \otimes b_{\lambda i} \text{ and assume the net to be an ultrafilter (or take subnets later)}.
\]

Given a left module $X$ with dual right module $X'$, we define a module projection $\overline{\cdot} : L(A, X') \to X'$ by

\[
\overline{T}(x) = \lim_{\lambda} \sum_{i=1}^{\infty} T(a_{\lambda i})(b_{\lambda i}x).
\]

$\overline{\cdot}$ is a module map:

\[
(\overline{T}.c)(x) = \lim_{\lambda} \sum_{i=1}^{\infty} (T.c)(a_{\lambda i})(b_{\lambda i}x) = \lim_{\lambda} \sum_{i=1}^{\infty} T(ca_{\lambda i})(b_{\lambda i}x) = \overline{T}(c.x).
\]

$\overline{\cdot}$ is a projection: recall $h_y(a) = y a(h_y)(x) = \lim_{\lambda} \sum_{i=1}^{\infty} (h_y)(a_{\lambda i})(b_{\lambda i}x) = \lim_{\lambda} \sum_{i=1}^{\infty} (y.a_{\lambda i})(b_{\lambda i}x) = \lim_{\lambda} y(\pi(m_\lambda)x) = y(x) = h_y(x)$.
3.6 Amenable gives Biinjectives

Dual modules over amenable algebras are injective

- Recall a Banach algebra $A$ is *amenable* if it has an approximate diagonal: a bounded net $m_\lambda \in \hat{A} \hat{\otimes} A$, so that
  
  $a \cdot m_\lambda - m_\lambda \cdot a \to 0$ and $\pi(m_\lambda) \cdot a \to a$
Dual modules over amenable algebras are injective

- Recall a Banach algebra $A$ is *amenable* if it has an approximate diagonal: a bounded net $m_\lambda \in A^\hat{\otimes} A$, so that

$$a \cdot m_\lambda - m_\lambda \cdot a \to 0 \text{ and } \pi(m_\lambda) \cdot a \to a$$

- **Proof:**
3.6 Amenable gives Biinjectives

Dual modules over amenable algebras are injective

- Recall a Banach algebra $A$ is amenable if it has an approximate diagonal: a bounded net $m_\lambda \in A \hat{\otimes} A$, so that

\[ a \cdot m_\lambda - m_\lambda \cdot a \to 0 \] \[ \pi(m_\lambda) \cdot a \to a \]

- **Proof:**
Dual modules over amenable algebras are injective

- Recall a Banach algebra $A$ is *amenable* if it has an approximate diagonal: a bounded net $m_\lambda \in \hat{A} \otimes A$, so that $a \cdot m_\lambda = m_\lambda \cdot a \to 0$ and $\pi(m_\lambda) \cdot a \to a$

- **Proof:** Denote the approximate diagonal by $m_\lambda = \sum_{i=1}^\infty a_i^\lambda \otimes b_i^\lambda$ and assume the net to be an ultrafilter (or take subnets later)
Dual modules over amenable algebras are injective

- Recall a Banach algebra $A$ is *amenable* if it has an approximate diagonal: a bounded net $m_{\lambda} \in A\hat{\otimes}A$, so that
  
  $a \cdot m_{\lambda} - m_{\lambda} \cdot a \to 0$ and $\pi(m_{\lambda}) \cdot a \to a$

- **Proof:** Denote the approximate diagonal by $m_{\lambda} = \sum_{i=1}^{\infty} a_{i}^{\lambda} \otimes b_{i}^{\lambda}$ and assume the net to be an ultrafilter (or take subnets later)

- Given a left module $X$ with dual right module $X'$
Amenable gives Biinjectives

Dual modules over amenable algebras are injective

- Recall a Banach algebra $A$ is amenable if it has an approximate diagonal: a bounded net $m_{\lambda} \in A^\wedge \otimes A$, so that
  
  \[ a \cdot m_{\lambda} - m_{\lambda} \cdot a \to 0 \text{ and } \pi(m_{\lambda}) \cdot a \to a \]

- **Proof:** Denote the approximate diagonal by $m_{\lambda} = \sum_{i=1}^{\infty} a_i^\lambda \otimes b_i^\lambda$
  and assume the net to be an ultrafilter (or take subnets later)

- Given a left module $X$ with dual right module $X'$

- We define a module projection $\tilde{\cdot} : L(A, X') \to X'$ by
3.6 Amenable gives Biinjectives

Dual modules over amenable algebras are injective

- Recall a Banach algebra $A$ is *amenable* if it has an approximate diagonal: a bounded net $m_\lambda \in A^\hat{} \otimes A$, so that
  $$a \cdot m_\lambda - m_\lambda \cdot a \to 0 \quad \text{and} \quad \pi(m_\lambda) \cdot a \to a$$

**Proof:** Denote the approximate diagonal by $m_\lambda = \sum_{i=1}^\infty a_i^\lambda \otimes b_i^\lambda$
and assume the net to be an ultrafilter (or take subnets later)
- Given a left module $X$ with dual right module $X'$
- We define a module projection $\tilde{T} : L(A, X') \to X'$ by
  $$\tilde{T}(x) = \lim_\lambda \sum_{i=1}^\infty T(a_i^\lambda)(b_i^\lambda x)$$
Dual modules over amenable algebras are injective

- **Recall** a Banach algebra $A$ is *amenable* if it has an approximate diagonal: a bounded net $m_\lambda \in A\hat{\otimes}A$, so that $a.m_\lambda - m_\lambda.a \to 0$ and $\pi(m_\lambda).a \to a$

- **Proof**: Denote the approximate diagonal by $m_\lambda = \sum_{i=1}^{\infty} a_i^\lambda \otimes b_i^\lambda$ and assume the net to be an ultrafilter (or take subnets later)

- Given a left module $X$ with dual right module $X'$
- We define a module projection $\bar{\cdot} : L(A, X') \to X'$ by $\bar{T}(x) = \lim_\lambda \sum_{i=1}^{\infty} T(a_i^\lambda)(b_i^\lambda x)$
- $\bar{\cdot}$ is a module map: $(\bar{T}.c)(x) = \lim_\lambda \sum_{i=1}^{\infty} (T.c)(a_i^\lambda)(b_i^\lambda x) = \lim_\lambda \sum_{i=1}^{\infty} T(ca_i^\lambda)(b_i^\lambda x) = \lim_\lambda \sum_{i=1}^{\infty} T(a_i^\lambda)(b_i^\lambda cx) = \bar{T}(c.x)$
Amenable gives Biinjectives

Dual modules over amenable algebras are injective

- Recall a Banach algebra $A$ is amenable if it has an approximate diagonal: a bounded net $m_\lambda \in A\hat\otimes A$, so that

\[ a.m_\lambda - m_\lambda.a \to 0 \text{ and } \pi(m_\lambda).a \to a \]

- **Proof:** Denote the approximate diagonal by $m_\lambda = \sum_{i=1}^\infty a_i^\lambda \otimes b_i^\lambda$ and assume the net to be an ultrafilter (or take subnets later).

- Given a left module $X$ with dual right module $X'$

- We define a module projection $\bar \cdot : L(A, X') \to X'$ by

\[ \bar T(x) = \lim_\lambda \sum_{i=1}^\infty T(a_i^\lambda)(b_i^\lambda x) \]

- $\bar \cdot$ is a module map: $(\bar T.c)(x) = \lim_\lambda \sum_{i=1}^\infty (T.c)(a_i^\lambda)(b_i^\lambda x) = \lim_\lambda \sum_{i=1}^\infty T(ca_i^\lambda)(b_i^\lambda x) = \lim_\lambda \sum_{i=1}^\infty T(a_i^\lambda)(b_i^\lambda cx) = \bar T(c.x)$

- $\bar \cdot$ is a projection: recall $h_y(a) = ya$

\[ (\bar h_y)(x) = \lim_\lambda \sum_{i=1}^\infty (h_y)(a_i^\lambda)(b_i^\lambda x) = \lim_\lambda \sum_{i=1}^\infty (y.a_i^\lambda)(b_i^\lambda x) = \lim_\lambda \sum_{i=1}^\infty y(a_i^\lambda.b_i^\lambda x) = \lim_\lambda y(\pi(m_\lambda)x) = y(x) = h_y(x) \]
3.6 Amenable gives Biinjectives

Dual modules over amenable algebras are injective

- Recall a Banach algebra $A$ is \textit{amenable} if it has an approximate diagonal: a bounded net $m_\lambda \in A\hat{\otimes}A$, so that
  \[ a \cdot m_\lambda - m_\lambda \cdot a \to 0 \]  
  and $\pi(m_\lambda) \cdot a \to a$

- \textbf{Proof:} Denote the approximate diagonal by $m_\lambda = \sum_{i=1}^{\infty} a_i^\lambda \otimes b_i^\lambda$ and assume the net to be an ultrafilter (or take subnets later).

- Given a left module $X$ with dual right module $X'$
- We define a module projection $\bar{\cdot} : L(A, X') \to X'$ by
  \[ \bar{T}(x) = \lim_\lambda \sum_{i=1}^{\infty} T(a_i^\lambda)(b_i^\lambda x) \]

- $\bar{\cdot}$ is a module map: \( (\bar{T} \cdot c)(x) = \lim_\lambda \sum_{i=1}^{\infty} (T \cdot c)(a_i^\lambda)(b_i^\lambda x) = \lim_\lambda \sum_{i=1}^{\infty} T(ca_i^\lambda)(b_i^\lambda cx) = \bar{T}(c \cdot x) \)

- $\bar{\cdot}$ is a projection: recall $h_y(a) = ya$
  \[ (h_y)(x) = \lim_\lambda \sum_{i=1}^{\infty} (h_y)(a_i^\lambda)(b_i^\lambda x) = \lim_\lambda \sum_{i=1}^{\infty} (y \cdot a_i^\lambda)(b_i^\lambda x) = \lim_\lambda \sum_{i=1}^{\infty} y(a_i^\lambda \cdot b_i^\lambda x) = \lim_\lambda y(\pi(m_\lambda) x) = y(x) = h_y(x) \]
Dual modules over amenable algebras are injective

- Recall a Banach algebra $A$ is amenable if it has an approximate diagonal: a bounded net $m_\lambda \in A\hat{\otimes} A$, so that $a.m_\lambda - m_\lambda.a \to 0$ and $\pi(m_\lambda).a \to a$

**Proof:** Denote the approximate diagonal by $m_\lambda = \sum_{i=1}^{\infty} a_i^\lambda \otimes b_i^\lambda$ and assume the net to be an ultrafilter (or take subnets later).

- Given a left module $X$ with dual right module $X'$
- We define a module projection $\overline{\cdot}: L(A, X') \to X'$ by
  \[
  \overline{T}(x) = \lim_\lambda \sum_{i=1}^{\infty} T(a_i^\lambda)(b_i^\lambda x)
  \]
- $\overline{\cdot}$ is a module map: $(\overline{\cdot}.c)(x) = \lim_\lambda \sum_{i=1}^{\infty} (T.c)(a_i^\lambda)(b_i^\lambda x) = \lim_\lambda \sum_{i=1}^{\infty} T(ca_i^\lambda)(b_i^\lambda x) = \lim_\lambda \sum_{i=1}^{\infty} T(a_i^\lambda)(b_i^\lambda cx) = \overline{T}(c.x)$
- $\overline{\cdot}$ is a projection: recall $h_y(a) = ya$
  \[
  (\overline{h_y})(x) = \lim_\lambda \sum_{i=1}^{\infty} (h_y)(a_i^\lambda)(b_i^\lambda x) = \lim_\lambda \sum_{i=1}^{\infty} (y.a_i^\lambda)(b_i^\lambda x) = \lim_\lambda \sum_{i=1}^{\infty} y(a_i^\lambda.b_i^\lambda x) = \lim_\lambda y(\pi(m_\lambda)x) = y(x) = h_y(x)
  \]
3.7 Averaging without amenability

Averaging with Bounded Approximate Identities (bai)

Theorem: Let $J$ be a left ideal with a brai $\{e_\alpha\}$, then $(A/J)'$ is injective.

Proof: We can write $(A/J)'$ as a module direct summand of the injective module $A'$ using the module projection $f \mapsto (a \mapsto \lim_\alpha f(a - ae_\alpha))$ for $f \in A'$; there is a partial converse to this for (weakly) complemented ideals if $(A/J)'$ is injective then $J$ has a brai. [Ex]

If $A' = (A/J)' \oplus J'$, then $A'' = (A/J)' \oplus J''$ and $1 \in A''$ decomposes.
3.7 Averaging without amenability

Averaging with Bounded Approximate Identities (bai)

**Theorem:** Let $J$ be a left ideal with a brai $\{e_\alpha\}$, then $(A/J)'$ is injective.

Proof: We can write $(A/J)'$ as a module direct summand of the injective module $A'$ using the module projection $f \mapsto \lim_\alpha f(a - ae_\alpha)$ for $f \in A'$; there is a partial converse to this for (weakly) complemented ideals if $(A/J)'$ is injective, then $J$ has a brai. [Ex] If $A'' = (A/J)' \oplus J'$, then $A''$ decomposes interesting examples are the peak points in uniform algebras.
Averaging with Bounded Approximate Identities (bai)

**Theorem:** Let $J$ be a left ideal with a brai $\{e_\alpha\}$, then $(A/J)'$ is injective.

**Proof:** We can write $(A/J)'$ as a module direct summand of the injective module $A'$ \([Ex]\)
3.7 Averaging without amenability

Averaging with Bounded Approximate Identities (bai)

- **Theorem:** Let $J$ be a left ideal with a brai $\{e_\alpha\}$, then $(A/J)'$ is injective.

- **Proof:** We can write $(A/J)'$ as a module direct summand of the injective module $A'$ [Ex]

- using the module projection $f \mapsto (a \mapsto \lim_\alpha f(a - ae_\alpha))$ for $f \in A'$;
3.7 Averaging without amenability

Averaging with Bounded Approximate Identities (bai)

**Theorem:** Let $J$ be a left ideal with a brai $\{e_\alpha\}$, then $(A/J)'$ is injective.

**Proof:** We can write $(A/J)'$ as a module direct summand of the injective module $A'$ [Ex]

using the module projection $f \mapsto (a \mapsto \lim_\alpha f(a - ae_\alpha))$ for $f \in A'$;

There is a partial converse to this for (weakly) complemented ideals.
3.7 Averaging without amenability

Averaging with Bounded Approximate Identities (bai)

- **Theorem:** Let $J$ be a left ideal with a brai $\{e_\alpha\}$, then $(A/J)'$ is injective.

- **Proof:** We can write $(A/J)'$ as a module direct summand of the injective module $A'$ [Ex]
  
  using the module projection $f \mapsto (a \mapsto \lim_\alpha f(a - ae_\alpha))$ for $f \in A'$;

- There is a partial converse to this for (weakly) complemented ideals

- if $(A/J)'$ is injective then $J$ has a brai. [Ex]
3.7 Averaging without amenability

Averaging with Bounded Approximate Identities (bai)

**Theorem:** Let $J$ be a left ideal with a brai $\{e_\alpha\}$, then $(A/J)'$ is injective.

**Proof:** We can write $(A/J)'$ as a module direct summand of the injective module $A'$ [Ex]

using the module projection $f \mapsto (a \mapsto \lim_\alpha f(a - ae_\alpha))$ for $f \in A'$;

There is a partial converse to this for (weakly) complemented ideals

if $(A/J)'$ is injective then $J$ has a brai. [Ex]

If $A' = (A/J)' \oplus J'$, then $A'' = (A/J)'' \oplus J''$ and $1 \in A''$ decomposes
3.7 Averaging without amenability

Averaging with Bounded Approximate Identities (bai)

- **Theorem:** Let $J$ be a left ideal with a brai $\{e_\alpha\}$, then $(A/J)'$ is injective.

- **Proof:** We can write $(A/J)'$ as a module direct summand of the injective module $A'$ [Ex]
  
  using the module projection $f \mapsto (a \mapsto \lim_\alpha f(a - ae_\alpha))$ for $f \in A'$; 
  
  There is a partial converse to this for (weakly) complemented ideals 
  
  if $(A/J)'$ is injective then $J$ has a brai. [Ex]
  
  If $A' = (A/J)' \oplus J'$, then $A'' = (A/J)'' \oplus J''$ and $1 \in A''$ decomposes
  
  Interesting examples are the peak points in uniform algebras
3.7 Averaging without amenability

Averaging with Bounded Approximate Identities (bai)

**Theorem:** Let \( J \) be a left ideal with a brai \( \{ e_\alpha \} \), then \( (A/J)' \) is injective.

**Proof:** We can write \( (A/J)' \) as a module direct summand of the injective module \( A' \) \([Ex]\)

using the module projection \( f \mapsto (a \mapsto \lim_\alpha f(a - ae_\alpha)) \) for \( f \in A' \);

There is a partial converse to this for (weakly) complemented ideals

if \( (A/J)' \) is injective then \( J \) has a brai. \([Ex]\)

If \( A' = (A/J)' \oplus J' \), then \( A'' = (A/J)'' \oplus J'' \) and \( 1 \in A'' \) decomposes

Interesting examples are the peak points in uniform algebras
3.7 Averaging without amenability

Averaging with Bounded Approximate Identities (bai)

- **Theorem:** Let $J$ be a left ideal with a brai $\{e_\alpha\}$, then $(A/J)'$ is injective.

- **Proof:** We can write $(A/J)'$ as a module direct summand of the injective module $A'$ [Ex]

  using the module projection $f \mapsto (a \mapsto \lim_\alpha f(a - ae_\alpha))$ for $f \in A'$;

There is a partial converse to this for (weakly) complemented ideals

if $(A/J)'$ is injective then $J$ has a brai. [Ex]

If $A' = (A/J)' \oplus J'$, then $A'' = (A/J)'' \oplus J''$ and $1 \in A''$ decomposes

Interesting examples are the peak points in uniform algebras
4.1 Quotient and Submodules

How are derivations into modules related to derivations onto submodules and quotient modules?

Derivations into Submodules
How are derivations into modules related to derivations onto submodules and quotient modules?

**Derivations into Submodules**

- We have already seen that by enlarging the module we can make any derivation inner, $Y \rightarrow L(A \hat{\otimes} A, Y)$, but in general it will not become inner.
4.1 Quotient and Submodules

How are derivations into modules related to derivations onto submodules and quotient modules?

Derivations into Submodules

- We have already seen that by enlarging the module we can make any derivation inner, \( Y \to L(A \hat{\otimes} A, Y) \), but in general it will not become inner.
- \( 0 \to X \hookrightarrow Y \to Z \to 0 \), an admissible s.e.s.
How are derivations into modules related to derivations onto submodules and quotient modules?

**Derivations into Submodules**

- We have already seen that by enlarging the module we can make any derivation inner, $Y \to L(A \hat{\otimes} A, Y)$, but in general it will not become inner.
- $0 \to X \leftrightarrow Y \to Z \to 0$, an admissible s.e.s.
- $\mathcal{H}^1(A, X) \to \mathcal{H}^1(A, Y) \to \mathcal{H}^1(A, Z)$
4.1 Quotient and Submodules

How are derivations into modules related to derivations onto submodules and quotient modules?

**Derivations into Submodules**

- We have already seen that by enlarging the module we can make any derivation inner, $Y \to L(A \hat{\otimes} A, Y)$, but in general it will not become inner.
- $0 \to X \hookrightarrow Y \to Z \to 0$, an admissible s.e.s.
- $\mathcal{H}^1(A, X) \to \mathcal{H}^1(A, Y) \to \mathcal{H}^1(A, Z)$
- This is exact at $\mathcal{H}^1(A, Y)$
How are derivations into modules related to derivations onto submodules and quotient modules?

**Derivations into Submodules**

- We have already seen that by enlarging the module we can make any derivation inner, \( Y \rightarrow L(A \hat{\otimes} A, Y) \), but in general it will not become inner.
- \( 0 \rightarrow X \leftarrow Y \rightarrow Z \rightarrow 0 \), an admissible s.e.s.
- \( H^1(A, X) \rightarrow H^1(A, Y) \rightarrow H^1(A, Z) \)
- This is **exact** at \( H^1(A, Y) \)
- **Proof:** ... [Ex]
How are derivations into modules related to derivations onto submodules and quotient modules?

Derivations into Submodules

- We have already seen that by enlarging the module we can make any derivation inner, $Y \rightarrow L(A \hat{\otimes} A, Y)$, but in general it will not become inner
- $0 \rightarrow X \leftrightarrow Y \rightarrow Z \rightarrow 0$, an admissible s.e.s.
- $\mathcal{H}^1(A, X) \rightarrow \mathcal{H}^1(A, Y) \rightarrow \mathcal{H}^1(A, Z)$
- This is exact at $\mathcal{H}^1(A, Y)$
- **Proof:** ... [Ex]
4.1 Quotient and Submodules

How are derivations into modules related to derivations onto submodules and quotient modules?

Derivations into Submodules

- We have already seen that by enlarging the module we can make any derivation inner, \( Y \to L(A\hat{\otimes}A, Y) \), but in general it will not become inner.

- \( 0 \to X \to Y \to Z \to 0 \), an admissible s.e.s.

- \( H^1(A, X) \to H^1(A, Y) \to H^1(A, Z) \)

- This is exact at \( H^1(A, Y) \)

- Proof: \( \ldots \) [Ex]
4.2 Quotient and Submodules

Derivations to Quotients

Given $D: A \to Y/X$, can we lift it to $\tilde{D}: A \to Y$? As a linear map we can do this, but can this be a derivation?

$$\phi(a, b) = a \tilde{D}(b) - \tilde{D}(ab) + \tilde{D}(a)b \in X$$

It should be noted for later, that $\phi$ satisfies the rule

$$a\phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b)c = 0$$

If there exists $\psi: A \to X$ such that

$$\phi(a, b) = a\psi(b) - \psi(ab) + \psi(a)b$$

then $\bar{D}(a) = \tilde{D}(a) - \psi(a)$ is a derivation $\bar{D}: A \to Y$.

$H_1(A, Y) \to H_1(A, Y/X) \to H_2(A, X)$

This is exact at $H_1(A, Y/X)$.

Proof: ...
4.2 Quotient and Submodules

**Derivations to Quotients**

Given $D : A \rightarrow Y/X$, can we lift it to $\tilde{D} : A \rightarrow Y$?

As a linear map we can do this, but can this be a derivation?

$$\phi(a, b) = a \tilde{D}(b) - \tilde{D}(ab) + \tilde{D}(a)b \in X$$

It should be noted for later, that $\phi$ satisfies the rule

$$a \phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b)c = 0 \ [\text{Ex}]$$

If there exists $\psi : A \rightarrow X$ such that $\phi(a, b) = a \psi(b) - \psi(ab) + \psi(a)b$ then $\bar{D}(a) = \tilde{D}(a) - \psi(a)$ is a derivation $\bar{D} : A \rightarrow Y$ [Ex]

$H_1(A, Y) \rightarrow H_1(A, Y/X) \rightarrow H_2(A, X)$

This is exact at $H_1(A, Y/X)$ [Ex]

Proof: . . .
4.2 Quotient and Submodules

Derivations to Quotients

- Given $D : A \to Y/X$, can we lift it to $\tilde{D} : A \to Y$?

- As a linear map we can do this, but can this be a derivation?
  
  $\phi(a, b) = a\tilde{D}(b) - \tilde{D}(ab) + \tilde{D}(a)b \in X$
4.2 Quotient and Submodules

Derivations to Quotients

- Given \( D : A \rightarrow Y/X \), can we lift it to \( \tilde{D} : A \rightarrow Y \)?
- As a linear map we can do this, but can this be a derivation?
  \[
  \phi(a, b) = a\tilde{D}(b) - \tilde{D}(ab) + \tilde{D}(a)b \in X
  \]
- It should be noted for later, that \( \phi \) satisfies the rule
4.2 Quotient and Submodules

Derivations to Quotients

- Given $D : A \to Y/X$, can we lift it to $\tilde{D} : A \to Y$?
- As a linear map we can do this, but can this be a derivation?
  \[ \phi(a, b) = a\tilde{D}(b) - \tilde{D}(ab) + \tilde{D}(a)b \in X \]
- It should be noted for later, that $\phi$ satisfies the rule
- $a\phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b)c = 0$ [Ex]
### Derivations to Quotients

- Given $D : A \to Y/X$, can we lift it to $\tilde{D} : A \to Y$?

- As a linear map we can do this, but can this be a derivation?
  \[ \phi(a, b) = a\tilde{D}(b) - \tilde{D}(ab) + \tilde{D}(a)b \in X \]

- It should be noted for later, that $\phi$ satisfies the rule

- If there exists $\psi : A \to X$ such that
  \[ a\phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b)c = 0 \]
4.2 Quotient and Submodules

**Derivations to Quotients**

- Given $D : A \rightarrow Y/X$, can we lift it to $\tilde{D} : A \rightarrow Y$?
- As a linear map we can do this, but can this be a derivation?
  \[ \phi(a, b) = a\tilde{D}(b) - \tilde{D}(ab) + \tilde{D}(a)b \in X \]
- It should be noted for later, that $\phi$ satisfies the rule
  \[ a\phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b)c = 0 \] (Ex)
- If there exists $\psi : A \rightarrow X$ such that
  \[ \phi(a, b) = a\psi(b) - \psi(ab) + \psi(a)b \]
4.2 Quotient and Submodules

Derivations to Quotients

- Given $D : A \rightarrow Y/X$, can we lift it to $\tilde{D} : A \rightarrow Y$?
- As a linear map we can do this, but can this be a derivation?
  $$\phi(a, b) = a\tilde{D}(b) - \tilde{D}(ab) + \tilde{D}(a)b \in X$$
- It should be noted for later, that $\phi$ satisfies the rule
- $$a\phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b)c = 0 \ [\text{Ex}]}$$
- If there exists $\psi : A \rightarrow X$ such that
- $$\phi(a, b) = a\psi(b) - \psi(ab) + \psi(a)b$$
- then $\bar{D}(a) = \tilde{D}(a) - \psi(a)$ is a derivation $\bar{D} : A \rightarrow Y \ [\text{Ex}]$
4.2 Quotient and Submodules

### Derivations to Quotients

- Given $D : A \to Y/X$, can we lift it to $\tilde{D} : A \to Y$?
- As a linear map we can do this, but can this be a derivation?
  \[ \phi(a, b) = a\tilde{D}(b) - \tilde{D}(ab) + \tilde{D}(a)b \in X \]
- It should be noted for later, that $\phi$ satisfies the rule
  \[ a\phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b)c = 0 \] [Ex]
- If there exists $\psi : A \to X$ such that
  \[ \phi(a, b) = a\psi(b) - \psi(ab) + \psi(a)b \]
- then $\tilde{D}(a) = \tilde{D}(a) - \psi(a)$ is a derivation $\tilde{D} : A \to Y$ [Ex]
- $\mathcal{H}^1(A, Y) \to \mathcal{H}^1(A, Y/X) \to \mathcal{H}^2(A, X)$
4.2 Quotient and Submodules

Derivations to Quotients

- Given $D : A \rightarrow Y/X$, can we lift it to $\tilde{D} : A \rightarrow Y$?
- As a linear map we can do this, but can this be a derivation?
  \[
  \phi(a, b) = a\tilde{D}(b) - \tilde{D}(ab) + \tilde{D}(a)b \in X
  \]
- It should be noted for later, that $\phi$ satisfies the rule
  \[
  a\phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b)c = 0 \quad [\text{Ex}]
  \]
- If there exists $\psi : A \rightarrow X$ such that
  \[
  \phi(a, b) = a\psi(b) - \psi(ab) + \psi(a)b
  \]
  then $\tilde{D}(a) = \tilde{D}(a) - \psi(a)$ is a derivation $\tilde{D} : A \rightarrow Y$ [Ex]
- $\mathcal{H}^1(A, Y) \rightarrow \mathcal{H}^1(A, Y/X) \rightarrow \mathcal{H}^2(A, X)$
- This is exact at $\mathcal{H}^1(A, Y/X)$ [Ex]
4.2 Quotient and Submodules

Derivations to Quotients

- Given \( D : A \to Y/X \), can we lift it to \( \tilde{D} : A \to Y \)?

- As a linear map we can do this, but can this be a derivation?
  \[
  \phi(a, b) = a\tilde{D}(b) - \tilde{D}(ab) + \tilde{D}(a)b \in X
  \]

- It should be noted for later, that \( \phi \) satisfies the rule
  \[
  a\phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b)c = 0 \quad [\text{Ex}]
  \]

- If there exists \( \psi : A \to X \) such that
  \[
  \phi(a, b) = a\psi(b) - \psi(ab) + \psi(a)b
  \]

- then \( \tilde{D}(a) = \tilde{D}(a) - \psi(a) \) is a derivation \( \tilde{D} : A \to Y \) \([\text{Ex}]\)

- \( \mathcal{H}^1(A, Y) \to \mathcal{H}^1(A, Y/X) \to \mathcal{H}^2(A, X) \)

- This is exact at \( \mathcal{H}^1(A, Y/X) \) \([\text{Ex}]\)

- **Proof:** . . .
### 4.2 Quotient and Submodules

#### Derivations to Quotients

- **Given** \( D : A \rightarrow Y/X \), can we lift it to \( \tilde{D} : A \rightarrow Y \)?
- **As a linear map** we can do this, but can this be a derivation?
  \[
  \phi(a, b) = a\tilde{D}(b) - \tilde{D}(ab) + \tilde{D}(a)b \in X
  \]
- It should be noted for later, that \( \phi \) satisfies the rule
  \[
  a\phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b)c = 0 \quad [\text{Ex}]
  \]
- If there exists \( \psi : A \rightarrow X \) such that
  \[
  \phi(a, b) = a\psi(b) - \psi(ab) + \psi(a)b
  \]
- then \( \bar{D}(a) = \tilde{D}(a) - \psi(a) \) is a derivation \( \bar{D} : A \rightarrow Y \) \([\text{Ex}]\)
- \( \mathcal{H}^1(A, Y) \rightarrow \mathcal{H}^1(A, Y/X) \rightarrow \mathcal{H}^2(A, X) \)
- This is exact at \( \mathcal{H}^1(A, Y/X) \) \([\text{Ex}]\)
- **Proof:** . . .
4.2 Quotient and Submodules

Derivations to Quotients

- Given \( D : A \to Y/X \), can we lift it to \( \tilde{D} : A \to Y \)?
- As a linear map we can do this, but can this be a derivation?
  \[ \phi(a, b) = a\tilde{D}(b) - \tilde{D}(ab) + \tilde{D}(a)b \in X \]
- It should be noted for later, that \( \phi \) satisfies the rule
  \[ a\phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b)c = 0 \] [Ex]
- If there exists \( \psi : A \to X \) such that
  \[ \phi(a, b) = a\psi(b) - \psi(ab) + \psi(a)b \]
- then \( \tilde{D}(a) = \tilde{D}(a) - \psi(a) \) is a derivation \( \tilde{D} : A \to Y \) [Ex]
- \( \mathcal{H}^1(A, Y) \to \mathcal{H}^1(A, Y/X) \to \mathcal{H}^2(A, X) \)
- This is exact at \( \mathcal{H}^1(A, Y/X) \) [Ex]
- **Proof:** . . .
5.1 Second cohomology from 2-forms

We define

\[(\delta y)(f) := f \cdot y - y \cdot f\]

\[(\delta \psi)(f, g) := f \cdot \psi(g) - \psi(fg) + \psi(f) \cdot g\]

\[(\delta \phi)(f, g, h) := f \cdot \phi(g, h) - \phi(fg, h) + \phi(f, gh) - \phi(f, g) \cdot h\]
5.1 Second cohomology from 2-forms

- We define
  \[(\delta y)(f) := f \cdot y - y \cdot f\]
  \[(\delta \psi)(f, g) := f \cdot \psi(g) - \psi(fg) + \psi(f) \cdot g\]
  \[(\delta \phi)(f, g, h) := f \cdot \phi(g, h) - \phi(fg, h) + \phi(f, gh) - \phi(f, g) \cdot h\]

- We say \(\phi\) is a 2-cocycle if \(\delta \phi = 0\)
5.1 Second cohomology from 2-forms

- We define
  \[
  (\delta y)(f) := f \cdot y - y \cdot f
  \]
  \[
  (\delta \psi)(f, g) := f \cdot \psi(g) - \psi(fg) + \psi(f) \cdot g
  \]
  \[
  (\delta \phi)(f, g, h) := f \cdot \phi(g, h) - \phi(fg, h) + \phi(f, gh) - \phi(f, g) \cdot h
  \]
- We say \( \phi \) is a 2-cocycle if \( \delta \phi = 0 \)
- We say \( \phi \) is a 2-coboundary if \( \phi = \delta \psi \). (Think \( \psi = D \).)
5.1 Second cohomology from 2-forms

- We define
  \[(\delta y)(f) := f \cdot y - y \cdot f\]
  \[(\delta \psi)(f, g) := f \cdot \psi(g) - \psi(fg) + \psi(f) \cdot g\]
  \[(\delta \phi)(f, g, h) := f \cdot \phi(g, h) - \phi(fg, h) + \phi(f, gh) - \phi(f, g) \cdot h\]

- We say \(\phi\) is a 2-cocycle if \(\delta \phi = 0\)
- We say \(\phi\) is a 2-coboundary if \(\phi = \delta \psi\). (Think \(\psi = D\).)
- How can such a 2-cocycle arise?

Michael C. White (Newcastle University)  Cohomology of Banach Algebras  15 - 16 May, 2014  18 / 35
5.1 Second cohomology from 2-forms

- We define

\[(\delta_y)(f) := f \cdot y - y \cdot f\]

\[(\delta\psi)(f, g) := f \cdot \psi(g) - \psi(fg) + \psi(f) \cdot g\]

\[(\delta\phi)(f, g, h) := f \cdot \phi(g, h) - \phi(fg, h) + \phi(f, gh) - \phi(f, g) \cdot h\]

- We say \(\phi\) is a 2-cocycle if \(\delta\phi = 0\)
- We say \(\phi\) is a 2-coboundary if \(\phi = \delta\psi\). (Think \(\psi = D\).)
- How can such a 2-cocycle arise?
- e.g. \(A = \ell^1(Z_+^2), \; Y = C_0\), so \(f(z, w) \cdot \lambda = f(0, 0)\lambda\)

\[
\phi(f, g) = \left( \frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial f}{\partial w} \frac{\partial g}{\partial z} \right) \bigg|_{(0,0)}. 
\]
5.1 Second cohomology from 2-forms

We define

\[(\delta y)(f) := f \cdot y - y \cdot f\]

\[(\delta \psi)(f, g) := f \cdot \psi(g) - \psi(fg) + \psi(f) \cdot g\]

\[(\delta \phi)(f, g, h) := f \cdot \phi(g, h) - \phi(fg, h) + \phi(f, gh) - \phi(f, g) \cdot h\]

We say \(\phi\) is a 2-cocycle if \(\delta \phi = 0\)

We say \(\phi\) is a 2-coboundary if \(\phi = \delta \psi\). (Think \(\psi = D\).)

How can such a 2-cocycle arise?

e.g. \(A = \ell^1(\mathbb{Z}^2_+), \ Y = \mathcal{C}_0\), so \(f(z, w) \cdot \lambda = f(0, 0) \lambda\)

\[
\phi(f, g) = \left( \frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial f}{\partial w} \frac{\partial g}{\partial z} \right) \bigg|_{(0,0)}.
\]

Calculation shows this to be a 2-cocycle, which is not zero, as \(\phi(z, w) = +1\), and \(\phi(w, z) = -1\),
5.1 Second cohomology from 2-forms

- We define
  
  \[(\delta y)(f) := f \cdot y - y \cdot f\]
  
  \[(\delta \psi)(f, g) := f \cdot \psi(g) - \psi(fg) + \psi(f) \cdot g\]
  
  \[(\delta \phi)(f, g, h) := f \cdot \phi(g, h) - \phi(fg, h) + \phi(f, gh) - \phi(f, g) \cdot h\]

- We say \(\phi\) is a 2-cocycle if \(\delta \phi = 0\)
- We say \(\phi\) is a 2-coboundary if \(\phi = \delta \psi\). (Think \(\psi = D\).)
- How can such a 2-cocycle arise?
- e.g. \(A = \ell^1(Z_+^2), Y = C_0\), so \(f(z, w) \cdot \lambda = f(0, 0)\lambda\)

\[
\phi(f, g) = \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial f}{\partial w} \frac{\partial g}{\partial z}\right) |_{(0,0)}.
\]

- Calculation shows this to be a 2-cocycle, which is not zero, as \(\phi(z, w) = +1\), and \(\phi(w, z) = -1\),
- Moreover, it cannot cobound, as if \(\phi = \delta \psi\), then

\[
\phi(f, g) = f \cdot \psi(g) - \psi(fg) + \psi(f)g = \phi(g, f).
\]
We define higher cohomology, $\mathcal{H}^n(A, Y)$, by generalising the 2-cocycle formula to the complex

\[ \cdots \to L^{n-1}(A, Y) \to L^n(A, Y) \to L^{n+1}(A, Y) \to \cdots \]

It is easy to check $\delta^2 = 0$, which defines a chain complex $Z^n(A) = \text{Ker}\delta$, are the $n$-cocycles $B^n(A) = \text{Im}\delta$, are the $n$-cochains $H^n(A, Y) = Z^n(A, Y)/B^n(A, Y)$, is the $n$-cohomology.
5.2 Higher Cohomology – a quick definition

- We define higher cohomology, $\mathcal{H}^n(A, Y)$ by generalising the 2-cocycle formula to the complex

$$
\rightarrow L^{n-1}(A, Y) \rightarrow L^n(A, Y) \rightarrow L^{n+1}(A, Y) \rightarrow
$$

with coboundary maps

$$
\delta(T)(a_1, a_2, \ldots, a_n) = + a_1 T(a_2, a_3, \ldots, a_n)
$$

$$
+ \sum_{j=1}^{n-1} (-1)^j T(a_1, \ldots, a_j a_{j+1}, \ldots)
$$

$$
+ (-1)^n T(a_1, a_2, \ldots, a_{n-1}) a_n
$$
5.2 Higher Cohomology – a quick definition

- We define higher cohomology, $\mathcal{H}^n(A, Y)$, by generalising the 2-cocycle formula to the complex

$$\rightarrow L^{n-1}(A, Y) \rightarrow L^n(A, Y) \rightarrow L^{n+1}(A, Y) \rightarrow$$

- with coboundary maps

$$\delta(T)(a_1, a_2, \ldots, a_n) = + a_1 T(a_2, a_3, \ldots, a_n)$$

$$+ \sum_{j=1}^{n-1} (-1)^j T(a_1, \ldots, a_j a_{j+1}, \ldots)$$

$$+ (-1)^n T(a_1, a_2, \ldots, a_{n-1}) a_n$$

- It is easy to check $\delta^2 = 0$ [Ex], which defines a chain complex
5.2 Higher Cohomology – a quick definition

- We define higher cohomology, $\mathcal{H}^n(A, Y)$, by generalising the 2-cocycle formula to the complex

$$\rightarrow L^{n-1}(A, Y) \rightarrow L^n(A, Y) \rightarrow L^{n+1}(A, Y) \rightarrow$$

- with coboundary maps

$$\delta(T)(a_1, a_2, \ldots, a_n) = + a_1 T(a_2, a_3, \ldots, a_n)$$

$$+ \sum_{j=1}^{n-1} (-1)^j T(a_1, \ldots, a_j a_{j+1}, \ldots)$$

$$+ (-1)^n T(a_1, a_2, \ldots, a_{n-1}) a_n$$

- It is easy to check $\delta^2 = 0$ [Ex], which defines a chain complex

- $Z^n(A) = \text{Ker } \delta$, are the $n$-cocycles
We define higher cohomology, $H^n(A, Y)$, by generalising the 2-cocycle formula to the complex

$$\rightarrow L^{n-1}(A, Y) \rightarrow L^n(A, Y) \rightarrow L^{n+1}(A, Y) \rightarrow$$

with coboundary maps

$$\delta(T)(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n-1} (-1)^j T(a_1, \ldots, a_j a_{j+1}, \ldots) + (-1)^n T(a_1, a_2, \ldots, a_{n-1}) a_n$$

It is easy to check $\delta^2 = 0$ [Ex], which defines a chain complex

$\mathcal{Z}^n(A) = \text{Ker} \, \delta$, are the $n$-cocycles

$\mathcal{B}^n(A) = \text{Im} \, \delta$, are the $n$-cochains
5.2 Higher Cohomology – a quick definition

- We define higher cohomology, $\mathcal{H}^n(A, Y)$, by generalising the 2-cocycle formula to the complex

\[ \rightarrow L^{n-1}(A, Y) \rightarrow L^n(A, Y) \rightarrow L^{n+1}(A, Y) \rightarrow \]

- with coboundary maps

\[
\delta(T)(a_1, a_2, \ldots, a_n) = + a_1 T(a_2, a_3, \ldots, a_n) \\
+ \sum_{j=1}^{n-1} (-1)^j T(a_1, \ldots, a_j a_{j+1}, \ldots) \\
+ (-1)^n T(a_1, a_2, \ldots, a_{n-1}) a_n
\]

- It is easy to check $\delta^2 = 0$ [Ex], which defines a chain complex.

- $\mathcal{Z}^n(A) = \text{Ker} \, \delta$, are the $n$-cocycles.

- $\mathcal{B}^n(A) = \text{Im} \, \delta$, are the $n$-cochains.

- $\mathcal{H}^n(A, Y) = \mathcal{Z}^n(A, Y)/\mathcal{B}^n(A, Y)$, is the $n$-cohomology.
5.3 Using simpler cocycles

It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions.
5.3 Using simpler cocycles

- It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions.
- e.g. that they vanish if any argument is a scalar multiple of $1 \in A$.

$P_n = A^\hat{\otimes} \otimes A$ and $d: P_{n+1} \to P_n$ by

$$d(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{j=1}^{n+1} (-1)^{j+1} a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n$$

is called the Bar resolution of $A$. It is an exact complex of biprojective modules.

Notice that $A \otimes A(P_n, Y) = L_n(A, Y)$, and the induced maps are the same as the $\delta$ above.

In fact we could take another resolution by biprojective $A$-modules and get the same answer, i.e.

If we set $H_n(A, Y) = \text{Ker} \delta \left[ A \otimes A(P_n, Y) \to A \otimes A(P_{n-1}, Y) \right] / \text{Im} \delta$

then $H_n(A, Y) = H_n(A, Y)$. . . This is surprising!
5.3 Using simpler cocycles

- It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions.
- e.g. that they vanish if any argument is a scalar multiple of $1 \in A$
- This can be proved by induction, but there is a very flexible tool available to prove such results.
5.3 Using simpler cocycles

- It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions.
- e.g. that they vanish if any argument is a scalar multiple of \(1 \in A\)
- This can be proved by induction, but there is a very flexible tool available to prove such results.
- Define \(P_n = A \hat{\otimes}^n A \hat{\otimes} A\) and \(d : P_{n+1} \to P_n\) by
It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions, e.g. that they vanish if any argument is a scalar multiple of $1 \in A$. This can be proved by induction, but there is a very flexible tool available to prove such results.

Define $P_n = \mathbb{A} \widehat{\otimes} \mathbb{A} \widehat{\otimes} \mathbb{A}$ and $d : P_{n+1} \rightarrow P_n$ by

$$d(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{j=1}^{n} (-1)^{j+1} a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n$$
5.3 Using simpler cocycles

- It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions.
- e.g. that they vanish if any argument is a scalar multiple of \(1 \in A\).
- This can be proved by induction, but there is a very flexible tool available to prove such results.

Define \(P_n = A \widehat{\otimes}^n A \widehat{\otimes} A\) and \(d : P_{n+1} \to P_n\) by

\[
d(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{j=1}^{n} (-1)^{j+1} a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n
\]

0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots
5.3 Using simpler cocycles

- It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions.
- e.g. that they vanish if any argument is a scalar multiple of $1 \in A$.
- This can be proved by induction, but there is a very flexible tool available to prove such results.

Define $P_n = A \hat{\otimes}^n A \hat{\otimes} A$ and $d : P_{n+1} \to P_n$ by

\[ d(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{j=1}^{n} (-1)^{j+1} a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n \]

- $0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$
5.3 Using simpler cocycles

- It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions.
- e.g. that they vanish if any argument is a scalar multiple of $1 \in A$
- This can be proved by induction, but there is a very flexible tool available to prove such results.
- Define $P_n = A \hat{\otimes} \otimes^n A \hat{\otimes} A$ and $d : P_{n+1} \rightarrow P_n$ by
  \[d(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{j=1}^{n} (-1)^{j+1} a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n\]
- $0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$ is called the Bar resolution of $A$. 

\[\text{It is an exact complex of biprojective modules.}\]
\[\text{Notice that } A_h(A(P_n, Y)) = L_n(A, Y), \text{ and the induced maps are the same as the } \delta \text{ above.}\]
\[\text{In fact we could take another resolution by biprojective } A \text{ modules and get the same answer, i.e.}\]
\[\text{If we set } H_n(A, Y) = \text{Ker } \delta \text{ then } H_n(A, Y) = H_n(A, Y).\]

This is surprising!
5.3 Using simpler cocycles

- It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions.
- e.g. that they vanish if any argument is a scalar multiple of $1 \in A$
- This can be proved by induction, but there is a very flexible tool available to prove such results.
- Define $P_n = A \hat{\otimes} \cdots \hat{\otimes} A \hat{\otimes} A$ and $d : P_{n+1} \to P_n$ by
  $$d(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{j=1}^n (-1)^{j+1} a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n$$
- $0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$ is called the Bar resolution of $A$.
- It is an exact complex of biprojective modules.
5.3 Using simpler cocycles

- It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions. e.g. that they vanish if any argument is a scalar multiple of $1 \in A$
- This can be proved by induction, but there is a very flexible tool available to prove such results.
- Define $P_n = A \hat{\otimes} \otimes^n A \hat{\otimes} A$ and $d : P_{n+1} \rightarrow P_n$ by
  \[
d(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{j=1}^{n} (-1)^{j+1} a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n
  \]
- $0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$ is called the $Bar$ resolution of $A$.
- It is an exact complex of biprojective modules.
- Notice that $A^h_A(P_n, Y) = L^n(A, Y)$, and the induced maps are the same as the $\delta$ above [Ex]
5.3 Using simpler cocycles

- It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions.
- e.g. that they vanish if any argument is a scalar multiple of $1 \in A$.
- This can be proved by induction, but there is a very flexible tool available to prove such results.
- Define $P_n = A \hat{\otimes} A \hat{\otimes} A$ and $d : P_{n+1} \to P_n$ by
  $d(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{j=1}^n (-1)^{j+1} a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n$
- $0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$ is called the Bar resolution of $A$.
- It is an exact complex of biprojective modules.
- Notice that $A h_A(P_n, Y) = L^n(A, Y)$, and the induced maps are the same as the $\delta$ above [Ex].
- In fact we could take another resolution by biprojective $A$ modules and get the same answer, i.e.
5.3 Using simpler cocycles

- It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions.
- e.g. that they vanish if any argument is a scalar multiple of $1 \in A$.
- This can be proved by induction, but there is a very flexible tool available to prove such results.
- Define $P_n = A \hat{\otimes} \cdots \hat{\otimes} A$ and $d : P_{n+1} \to P_n$ by
  \[
  d(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{j=1}^{n} (-1)^{j+1} a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n
  \]
- $0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$ is called the $Bar$ resolution of $A$.
- It is an exact complex of biprojective modules.
- Notice that $A h_A(P_n, Y) = L^n(A, Y)$, and the induced maps are the same as the $\delta$ above [Ex]
- In fact we could take another resolution by biprojective $A$ modules and get the same answer, i.e.
- If we set $H^n(A, Y) = \text{Ker} \delta_{[A h_A(P_n, Y) \to A h_A(P_{n-1}, Y)]} / \text{Im} \delta$
5.3 Using simpler cocycles

- It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions.
- e.g. that they vanish if any argument is a scalar multiple of \(1 \in A\)
- This can be proved by induction, but there is a very flexible tool available to prove such results.
- Define \(P_n = \hat{A} \otimes \cdots \otimes \hat{A} \otimes A\) and \(d : P_{n+1} \to P_n\) by
- \[d(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{j=1}^{n} (-1)^{j+1} a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n\]
- \(0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots\) is called the Bar resolution of \(A\).
- It is an exact complex of biprojective modules.
- Notice that \(A h_A(P_n, Y) = L^n(A, Y)\), and the induced maps are the same as the \(\delta\) above [Ex]
- In fact we could take another resolution by biprojective \(A\) modules and get the same answer, i.e.
- If we set \(H^n(A, Y) = \text{Ker} \delta_{[A h_A(P_n, Y) \to A h_A(P_{n-1}, Y)]}/\text{Im} \delta\)
- then \(H^n(A, Y) = \mathcal{H}^n(A, Y)\)
5.3 Using simpler cocycles

- It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions.
- E.g., that they vanish if any argument is a scalar multiple of \(1 \in A\).
- This can be proved by induction, but there is a very flexible tool available to prove such results.

Define \(P_n = \hat{A} \otimes \hat{A} \otimes \cdots \otimes \hat{A} \) and \(d : P_{n+1} \to P_n\) by
\[
d(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{j=1}^{n} (-1)^{j+1} a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n
\]

0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots is called the Bar resolution of \(A\).

It is an exact complex of biprojective modules.

Notice that \(A h_A(P_n, Y) = L^n(A, Y)\), and the induced maps are the same as the \(\delta\) above [Ex].

In fact, we could take another resolution by biprojective \(A\) modules and get the same answer, i.e.

If we set \(H^n(A, Y) = \text{Ker} \delta_{[A h_A(P_n, Y) \to A h_A(P_{n-1}, Y)]} / \text{Im} \delta\)

then \(H^n(A, Y) = \mathcal{H}^n(A, Y)\).
5.3 Using simpler cocycles

- It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions.
- e.g. that they vanish if any argument is a scalar multiple of $1 \in A$.
- This can be proved by induction, but there is a very flexible tool available to prove such results.
- Define $P_n = A \widehat{\otimes} A \widehat{\otimes} A$ and $d : P_{n+1} \to P_n$ by
  $$d(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{j=1}^{n} (-1)^{j+1} a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n$$
- $0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$ is called the Bar resolution of $A$.
- It is an exact complex of biprojective modules.
- Notice that $A h_A(P_n, Y) = L^n(A, Y)$, and the induced maps are the same as the $\delta$ above [Ex].
- In fact we could take another resolution by biprojective $A$ modules and get the same answer, i.e.
- If we set $H^n(A, Y) = \text{Ker } \delta_{[A h_A(P_n, Y) \to A h_A(P_{n-1}, Y)]} / \text{Im } \delta$
- then $H^n(A, Y) = \mathcal{H}^n(A, Y)$ … This is surprising!
5.3 Using simpler cocycles

- It is often convenient to be able to assume that both your cocycles and coboundaries satisfy additional conditions.
- e.g. that they vanish if any argument is a scalar multiple of 1 ∈ A.
- This can be proved by induction, but there is a very flexible tool available to prove such results.

Define $P_n = A^{\hat{\otimes}^n} A^{\hat{\otimes} A}$ and $d : P_{n+1} \to P_n$ by

$$d(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{j=1}^{n} (-1)^{j+1} a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n$$

$0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$ is called the Bar resolution of $A$.

- It is an exact complex of biprojective modules.
- Notice that $A h_A(P_n, Y) = L^n(A, Y)$, and the induced maps are the same as the $\delta$ above [Ex].
- In fact we could take another resolution by biprojective $A$ modules and get the same answer, i.e.

If we set $H^n(A, Y) = \text{Ker} \delta_{[A h_A(P_n, Y) \to A h_A(P_{n-1}, Y)]]} / \text{Im} \delta$

then $H^n(A, Y) = \mathcal{H}^n(A, Y)$ . . . This is surprising!
5.4 Comparing Projective Resolutions

The proof that $H^n(A, Y) = \mathcal{H}(A, Y)$ does to depend on the resolution involves several steps.

In general one compares two resolutions:

\[ 0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots \]

\[ 0 \leftarrow A \leftarrow Q_0 \leftarrow Q_1 \leftarrow Q_2 \leftarrow \cdots \]

Using the fact that the $Q_n$ are projective to build the comparison, this gives maps:

\[ A_h A(P_0, Y) \rightarrow A_h A(P_1, Y) \rightarrow A_h A(P_2, Y) \cdots \]

\[ A_h A(Q_0, Y) \rightarrow A_h A(Q_1, Y) \rightarrow A_h A(Q_2, Y) \cdots \]

which gives maps:

\[ Z^n P(A, Y) \rightarrow Z^n Q(A, Y) \]

\[ B^n P(A, Y) \rightarrow B^n Q(A, Y) \]

\[ H^n P(A, Y) \rightarrow H^n Q(A, Y) \]

To check the last is an isomorphism one begins with the case $Q_n = P_n$.

Beware: the maps $\uparrow$ may not have been chosen to be isomorphisms.
The proof that $H^n(A, Y) = \mathcal{H}(A, Y)$ does to depend on the resolution involves several steps.

In general one compares two resolutions:

\[
\begin{align*}
0 & \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots \\
\uparrow & \uparrow \uparrow \uparrow \uparrow \\
0 & \leftarrow A \leftarrow Q_0 \leftarrow Q_1 \leftarrow Q_2 \leftarrow \cdots
\end{align*}
\]

Using the fact that the $Q_n$ are projective to build the comparison. This gives maps:

\[
\begin{align*}
A & \rightarrow A \rightarrow A \\
\downarrow & \downarrow & \downarrow \\
A & \rightarrow A \rightarrow A
\end{align*}
\]

which gives maps:

\[
\begin{align*}
Z^n P(A, Y) & \rightarrow Z^n Q(A, Y) \\
B^n P(A, Y) & \rightarrow B^n Q(A, Y) \\
H^n P(A, Y) & \rightarrow H^n Q(A, Y)
\end{align*}
\]

To check the last is an isomorphism one begins with the case $Q_n = P_n$.

Beware: the maps $\uparrow$, may not have been chosen to be isomorphisms.
5.4 Comparing Projective Resolutions

The proof that $H^n(A, Y) = \mathcal{H}(A, Y)$ does to depend on the resolution involves several steps.

In general one compares two resolutions:

\[ 0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots \]

\[ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \]

\[ 0 \leftarrow A \leftarrow Q_0 \leftarrow Q_1 \leftarrow Q_2 \leftarrow \cdots \]

Using the fact that the $Q_n$ are projective to build the comparison.
The proof that $H^n(A, Y) = \mathcal{H}(A, Y)$ does to depend on the resolution involves several steps.

In general one compares two resolutions:

$$
\begin{align*}
0 & \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots \\
\uparrow & \uparrow \uparrow \uparrow \uparrow \uparrow \\
0 & \leftarrow A \leftarrow Q_0 \leftarrow Q_1 \leftarrow Q_2 \leftarrow \cdots 
\end{align*}
$$

Using the fact that the $Q_n$ are projective to build the comparison.

This gives maps

$$
\begin{align*}
A h_A(P_0, Y) & \rightarrow A h_A(P_1, Y) \rightarrow A h_A(P_2, Y) \cdots \\
\downarrow & \downarrow \downarrow \\
A h_A(Q_0, Y) & \rightarrow A h_A(Q_1, Y) \rightarrow A h_A(Q_2, Y) \cdots 
\end{align*}
$$
5.4 Comparing Projective Resolutions

- The proof that $H^n(A, Y) = \mathcal{H}(A, Y)$ does to depend on the resolution involves several steps.

- In general one compares two resolutions:
  
  $0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$

  $\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

  $0 \leftarrow A \leftarrow Q_0 \leftarrow Q_1 \leftarrow Q_2 \leftarrow \cdots$

- Using the fact that the $Q_n$ are projective to build the comparison.

- This gives maps
  
  $A h_A(P_0, Y) \rightarrow A h_A(P_1, Y) \rightarrow A h_A(P_2, Y) \cdots$

  $\downarrow \quad \downarrow \quad \downarrow$

  $A h_A(Q_0, Y) \rightarrow A h_A(Q_1, Y) \rightarrow A h_A(Q_2, Y) \cdots$

- which gives maps:
The proof that $H^n(A, Y) = \mathcal{H}(A, Y)$ does to depend on the resolution involves several steps.

In general one compares two resolutions:

\[
\begin{array}{cccccc}
0 & \leftarrow & A & \leftarrow & P_0 & \leftarrow & P_1 & \leftarrow & P_2 & \leftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 & \leftarrow & A & \leftarrow & Q_0 & \leftarrow & Q_1 & \leftarrow & Q_2 & \leftarrow & \cdots 
\end{array}
\]

Using the fact that the $Q_n$ are projective to build the comparison.

This gives maps

\[
\begin{align*}
A h_A(P_0, Y) & \rightarrow A h_A(P_1, Y) \rightarrow A h_A(P_2, Y) \cdots \\
\downarrow & \downarrow & \downarrow \\
A h_A(Q_0, Y) & \rightarrow A h_A(Q_1, Y) \rightarrow A h_A(Q_2, Y) \cdots 
\end{align*}
\]

which gives maps:

\[
\mathcal{Z}^n_P(A, Y) \rightarrow \mathcal{Z}^n_Q(A, Y)
\]
The proof that $H^n(A, Y) = \mathcal{H}(A, Y)$ does to depend on the resolution involves several steps.

In general one compares two resolutions:

\[
\begin{align*}
0 & \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots \\
& \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\
0 & \leftarrow A \leftarrow Q_0 \leftarrow Q_1 \leftarrow Q_2 \leftarrow \cdots
\end{align*}
\]

Using the fact that the $Q_n$ are projective to build the comparison.

This gives maps

\[
A h_A(P_0, Y) \rightarrow A h_A(P_1, Y) \rightarrow A h_A(P_2, Y) \cdots
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

\[
A h_A(Q_0, Y) \rightarrow A h_A(Q_1, Y) \rightarrow A h_A(Q_2, Y) \cdots
\]

which gives maps:

- $\mathcal{Z}_n^P(A, Y) \rightarrow \mathcal{Z}_n^Q(A, Y)$
- $\mathcal{B}_n^P(A, Y) \rightarrow \mathcal{B}_n^Q(A, Y)$
The proof that $H^n(A, Y) = \mathcal{H}(A, Y)$ does to depend on the resolution involves several steps.

In general one compares two resolutions:

\[
\begin{array}{ccccccc}
0 & \leftarrow & A & \leftarrow & P_0 & \leftarrow & P_1 & \leftarrow & P_2 & \leftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \leftarrow & A & \leftarrow & Q_0 & \leftarrow & Q_1 & \leftarrow & Q_2 & \leftarrow & \cdots \\
\end{array}
\]

Using the fact that the $Q_n$ are projective to build the comparison.

This gives maps:

\[
\begin{align*}
A h_A(P_0, Y) & \rightarrow A h_A(P_1, Y) \rightarrow A h_A(P_2, Y) \cdots \\
\downarrow & \downarrow & \downarrow \\
A h_A(Q_0, Y) & \rightarrow A h_A(Q_1, Y) \rightarrow A h_A(Q_2, Y) \cdots \\
\end{align*}
\]

which gives maps:

- $\mathcal{Z}_P^n(A, Y) \rightarrow \mathcal{Z}_Q^n(A, Y)$
- $\mathcal{B}_P^n(A, Y) \rightarrow \mathcal{B}_Q^n(A, Y)$
- $\mathcal{H}_P^n(A, Y) \rightarrow \mathcal{H}_Q^n(A, Y)$
5.4 Comparing Projective Resolutions

- The proof that $H^n(A, Y) = \mathcal{H}(A, Y)$ does to depend on the resolution involves several steps.
- In general one compares two resolutions:
  \[
  0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots \\
  \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\
  0 \leftarrow A \leftarrow Q_0 \leftarrow Q_1 \leftarrow Q_2 \leftarrow \cdots
  \]
- Using the fact that the $Q_n$ are projective to build the comparison.
- This gives maps:
  \[
  A h_A(P_0, Y) \rightarrow A h_A(P_1, Y) \rightarrow A h_A(P_2, Y) \cdots \\
  \downarrow \quad \downarrow \quad \downarrow \\
  A h_A(Q_0, Y) \rightarrow A h_A(Q_1, Y) \rightarrow A h_A(Q_2, Y) \cdots
  \]
- which gives maps:
  - $Z^n_P(A, Y) \rightarrow Z^n_Q(A, Y)$
  - $B^n_P(A, Y) \rightarrow B^n_Q(A, Y)$
  - $\mathcal{H}^n_P(A, Y) \rightarrow \mathcal{H}^n_Q(A, Y)$
- To check the last is an isomorphism one begins with the case $Q_n = P_n$.
5.4 Comparing Projective Resolutions

- The proof that $H^n(A, Y) = \mathcal{H}(A, Y)$ does to depend on the resolution involves several steps.
- In general one compares two resolutions:

\[
\begin{array}{ccccccc}
0 & \leftarrow & A & \leftarrow & P_0 & \leftarrow & P_1 & \leftarrow & P_2 & \leftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \leftarrow & A & \leftarrow & Q_0 & \leftarrow & Q_1 & \leftarrow & Q_2 & \leftarrow & \cdots
\end{array}
\]

- Using the fact that the $Q_n$ are projective to build the comparison.
- This gives maps

\[
\begin{align*}
A h_A(P_0, Y) & \rightarrow A h_A(P_1, Y) \rightarrow A h_A(P_2, Y) \rightarrow \cdots \\
A h_A(Q_0, Y) & \rightarrow A h_A(Q_1, Y) \rightarrow A h_A(Q_2, Y) \rightarrow \cdots
\end{align*}
\]

- which gives maps:
  - $Z^n_P(A, Y) \rightarrow Z^n_Q(A, Y)$
  - $B^n_P(A, Y) \rightarrow B^n_Q(A, Y)$
  - $\mathcal{H}_P^n(A, Y) \rightarrow \mathcal{H}_Q^n(A, Y)$
- To check the last is an isomorphism one begins with the case $Q_n = P_n$
- Beware: the maps $\uparrow$, may not have been chosen to be isomorphisms.

Michael C. White (Newcastle University)  Cohomology of Banach Algebras  15 - 16 May, 2014  21 / 35
5.5 Using injective resolutions

We are also allowed to use biinjective resolutions of the bimodule
\[ 0 \to Y \to I_0 \to I_1 \to I_2 \to \cdots \]
5.5 Using injective resolutions

We are also allowed to use biinjective resolutions of the bimodule

$$0 \rightarrow Y \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots$$

and then the cohomology is given by

$$H^n(A, Y) = \text{Ker} \delta : A_hA(A, I_n) \rightarrow A_hA(A, I_{n+1}) / \text{Im} \delta$$

Theorem

$A$ is amenable iff $A'$ is a bimodule direct summand of $(A \hat{\otimes} A)'$ iff $A'$ is biinjective.

If the bimodule is already biinjective, then it has a short resolution

$$0 \rightarrow I \rightarrow I \rightarrow 0 \rightarrow \cdots$$

It is clear that all the $H^n(A, Y)$ defined above are trivial, for $n > 0$

So it is not just 'trivial' derivations which biinjective modules have

All of the higher cohomology vanishes
5.5 Using injective resolutions

We are also allowed to use biinjective resolutions of the bimodule

\[ 0 \to Y \to I_0 \to I_1 \to I_2 \to \cdots \]

and then the cohomology is given by

\[ H^n(A, Y) = \text{Ker} \delta : A h_A(A, I_n) \to A h_A(A, I_{n+1}) / \text{Im} \delta \]
We are also allowed to use biinjective resolutions of the bimodule
\[ 0 \to Y \to I_0 \to I_1 \to I_2 \to \cdots \]
and then the cohomology if given by
\[ H^n(A, Y) = \ker \delta : A h_A(A, I_n) \to A h_A(A, I_{n+1}) / \text{Im} \delta \]

**Theorem** $A$ is amenable iff
$A'$ is a bimodule direct summand of $(A \hat{\otimes} A)'$
5.5 Using injective resolutions

We are also allowed to use biinjective resolutions of the bimodule
\[ 0 \rightarrow Y \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots \]
and then the cohomology if given by
\[ H^n(A, Y) = \ker \delta : A h_A(A, I_n) \rightarrow A h_A(A, I_{n+1}) / \text{Im} \delta \]

**Theorem** $A$ is amenable iff $A'$ is a bimodule direct summand of $(A \hat{\otimes} A)'$ iff $A'$ is biinjective.
5.5 Using injective resolutions

We are also allowed to use biinjective resolutions of the bimodule
\[ 0 \to Y \to I_0 \to I_1 \to I_2 \to \cdots \]
and then the cohomology is given by
\[ H^n(A, Y) = \text{Ker} \delta : A h_A(A, I_n) \to A h_A(A, I_{n+1}) / \text{Im} \delta \]

**Theorem** \( A \) is amenable iff \( A' \) is a bimodule direct summand of \((A \hat{\otimes} A)'\) iff \( A' \) is biinjective.

\[ H^n(A, I) = 0, \text{ for biinjective modules} \]
5.5 Using injective resolutions

We are also allowed to use biinjective resolutions of the bimodule

\[ 0 \to Y \to I_0 \to I_1 \to I_2 \to \cdots \]

and then the cohomology if given by

\[ H^n(A, Y) = \text{Ker} \delta : A h_A(A, I_n) \to A h_A(A, I_{n+1}) / \text{Im} \delta \]

**Theorem** $A$ is amenable iff

$A'$ is a bimodule direct summand of $(A \hat{\otimes} A)'$ iff $A'$ is biinjective.

\[ H^n(A, I) = 0, \text{ for biinjective modules} \]

- If the bimodule is already biinjective, then it has a short resolution
5.5 Using injective resolutions

We are also allowed to use biinjective resolutions of the bimodule

\[ 0 \rightarrow Y \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots \]

and then the cohomology if given by

\[ H^n(A, Y) = \text{Ker} \delta : A\, h_A(A, I_n) \rightarrow A\, h_A(A, I_{n+1}) / \text{Im} \delta \]

**Theorem** \( A \) is amenable iff

\( A' \) is a bimodule direct summand of \((A \hat{\otimes} A)'\) iff \( A' \) is biinjective.

\( H^n(A, I) = 0 \), for biinjective modules

- If the bimodule is already biinjective, then it has a short resolution
  \[ 0 \rightarrow I \rightarrow I \rightarrow 0 \rightarrow \cdots \]
5.5 Using injective resolutions

We are also allowed to use biinjective resolutions of the bimodule

\[ 0 \rightarrow Y \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots \]

and then the cohomology if given by

\[ H^n(A, Y) = \text{Ker } \delta : A h_A(A, I_n) \rightarrow A h_A(A, I_{n+1}) / \text{Im } \delta \]

**Theorem** \( A \) is amenable iff

\( A' \) is a bimodule direct summand of \( (A \hat{\otimes} A)' \) iff \( A' \) is biinjective.

\[ \mathcal{H}^n(A, I) = 0, \text{ for biinjective modules} \]

- If the bimodule is already biinjective, then it has a short resolution
  \[ 0 \rightarrow I \rightarrow I \rightarrow 0 \rightarrow \cdots \]

- It is clear that all the \( H^n(A, Y) \) defined above are trivial, for \( n > 0 \)
5.5 Using injective resolutions

We are also allowed to use biinjective resolutions of the bimodule

\[ 0 \to Y \to I_0 \to I_1 \to I_2 \to \cdots \]

and then the cohomology is given by

\[ H^n(A, Y) = \text{Ker} \, \delta : A^h(A, I_n) \to A^h(A, I_{n+1}) / \text{Im} \, \delta \]

**Theorem** $A$ is amenable iff

$A'$ is a bimodule direct summand of $(A \hat{\otimes} A)'$ iff $A'$ is biinjective.

$H^n(A, I) = 0$, for biinjective modules

- If the bimodule is already biinjective, then it has a short resolution

  \[ 0 \to I \to I \to 0 \to \cdots \]

- It is clear that all the $H^n(A, Y)$ defined above are trivial, for $n > 0$

- So it is not just ‘trivial’ derivations which biinjective modules have
We are also allowed to use biinjective resolutions of the bimodule
\[ 0 \to Y \to I_0 \to I_1 \to I_2 \to \cdots \]
and then the cohomology if given by
\[ H^n(A, Y) = \text{Ker} \delta : A h_A(A, I_n) \to A h_A(A, I_{n+1}) / \text{Im} \delta \]

**Theorem** $A$ is amenable iff $A'$ is a bimodule direct summand of $(A \hat{\otimes} A)'$ iff $A'$ is biinjective.

$H^n(A, I) = 0$, for biinjective modules

- If the bimodule is already biinjective, then it has a short resolution
  \[ 0 \to I \to I \to 0 \to \cdots \]
- It is clear that all the $H^n(A, Y)$ defined above are trivial, for $n > 0$
- So it is not just ‘trivial’ derivations which biinjective modules have
- All of the higher cohomology vanishes
5.6 Dimension Reduction

We have already seen $H^1(A, X)$ is related to $H^2(A, Z)$. 
We have already seen $\mathcal{H}^1(A, X)$ is related to $\mathcal{H}^2(A, Z)$. More Generally . . .
We have already seen $\mathcal{H}^1(A, X)$ is related to $\mathcal{H}^2(A, Z)$. More Generally . . .

**Long Exact Sequences**
We have already seen $\mathcal{H}^1(A, X)$ is related to $\mathcal{H}^2(A, Z)$. More Generally . . .

**Long Exact Sequences**

- **Theorem** Given an *admissible* short exact sequence

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

There is a long exact sequence

$$\cdots \rightarrow \mathcal{H}^n(A, Y) \rightarrow \mathcal{H}^n(A, Z) \rightarrow \mathcal{H}^{n+1}(A, X) \rightarrow \mathcal{H}^{n+1}(A, Y) \rightarrow \cdots$$

If we select a bimodule $Y$ so that $\mathcal{H}^n(A, Y) = 0$ then

$$\mathcal{H}^n(A, Z) \cong \mathcal{H}^{n+1}(A, X)$$

We select $Y = L(A \hat{\otimes} A, X)$, which is bijective

Which gives $Z = L(A \hat{\otimes} A, X)/X$

So we have that $\mathcal{H}^{n+1}(A, X) \cong \mathcal{H}^n(A, Z)$

Hence it is isomorphic to some $\mathcal{H}^1(A, W)$.
5.6 Dimension Reduction

We have already seen $\mathcal{H}^1(A, X)$ is related to $\mathcal{H}^2(A, Z)$. More Generally . . .

Long Exact Sequences

- **Theorem** Given an *admissible* short exact sequence

  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$
5.6 Dimension Reduction

We have already seen $H^1(A, X)$ is related to $H^2(A, Z)$. More Generally . . .

**Long Exact Sequences**

- **Theorem** Given an *admissible* short exact sequence
  
  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$

  There is a long exact sequence
5.6 Dimension Reduction

We have already seen $\mathcal{H}^1(A, X)$ is related to $\mathcal{H}^2(A, Z)$. More Generally . . .

**Long Exact Sequences**

- **Theorem** Given an *admissible* short exact sequence
  
  $0 \rightarrow X \hookrightarrow Y \rightarrow Z \rightarrow 0$

  There is a long exact sequence

  $\cdots \rightarrow \mathcal{H}^n(A, Y) \rightarrow \mathcal{H}^n(A, Z) \rightarrow \mathcal{H}^{n+1}(A, X) \rightarrow \mathcal{H}^{n+1}(A, Y) \rightarrow \cdots$
We have already seen $\mathcal{H}^1(A, X)$ is related to $\mathcal{H}^2(A, Z)$. More Generally . . .

**Long Exact Sequences**

- **Theorem** Given an **admissible** short exact sequence
  
  $$0 \to X \to Y \to Z \to 0$$

- There is a long exact sequence
  
  $$\cdots \to \mathcal{H}^n(A, Y) \to \mathcal{H}^n(A, Z) \to \mathcal{H}^{n+1}(A, X) \to \mathcal{H}^{n+1}(A, Y) \to \cdots$$

- If we select a bimodule $Y$ so that $\mathcal{H}^n(A, Y) = 0$ then
5.6 Dimension Reduction

We have already seen $\mathcal{H}^1(A, X)$ is related to $\mathcal{H}^2(A, Z)$. More Generally . . .

**Long Exact Sequences**

- **Theorem** Given an *admissible* short exact sequence
  
  $$0 \rightarrow X \hookrightarrow Y \rightarrow Z \rightarrow 0$$

  There is a long exact sequence

  $$\cdots \rightarrow \mathcal{H}^n(A, Y) \rightarrow \mathcal{H}^n(A, Z) \rightarrow \mathcal{H}^{n+1}(A, X) \rightarrow \mathcal{H}^{n+1}(A, Y) \rightarrow \cdots$$

  If we select a bimodule $Y$ so that $\mathcal{H}^n(A, Y) = 0$ then

  $\mathcal{H}^n(A, Z) \cong \mathcal{H}^{n+1}(A, X)$
5.6 Dimension Reduction

We have already seen $\mathcal{H}^1(A, X)$ is related to $\mathcal{H}^2(A, Z)$. More Generally . . .

### Long Exact Sequences

- **Theorem** Given an *admissible* short exact sequence
  
  $0 \to X \hookrightarrow Y \to Z \to 0$

  There is a long exact sequence
  
  $\cdots \to \mathcal{H}^n(A, Y) \to \mathcal{H}^n(A, Z) \to \mathcal{H}^{n+1}(A, X) \to \mathcal{H}^{n+1}(A, Y) \to \cdots$

  If we select a bimodule $Y$ so that $\mathcal{H}^n(A, Y) = 0$ then
  
  $\mathcal{H}^n(A, Z) \cong \mathcal{H}^{n+1}(A, X)$

  We select $Y = L(A \hat{\otimes} A, X)$, are it is bijective
5.6 Dimension Reduction

We have already seen $\mathcal{H}^1(A, X)$ is related to $\mathcal{H}^2(A, Z)$. More Generally . . .

Long Exact Sequences

- **Theorem** Given an *admissible* short exact sequence
  
  \[ 0 \to X \hookrightarrow Y \to Z \to 0 \]

  There is a long exact sequence

  \[ \cdots \to \mathcal{H}^n(A, Y) \to \mathcal{H}^n(A, Z) \to \mathcal{H}^{n+1}(A, X) \to \mathcal{H}^{n+1}(A, Y) \to \cdots \]

  If we select a bimodule $Y$ so that $\mathcal{H}^n(A, Y) = 0$ then

  $\mathcal{H}^n(A, Z) \cong \mathcal{H}^{n+1}(A, X)$

  We select $Y = L(A \hat{\otimes} A, X)$, are it is biinjective

  Which gives $Z = L(A \hat{\otimes} A, X)/X$
5.6 Dimension Reduction

We have already seen $\mathcal{H}^1(A, X)$ is related to $\mathcal{H}^2(A, Z)$. More Generally . . .

**Long Exact Sequences**

- **Theorem** Given an *admissible* short exact sequence
  
  $0 \to X \to Y \to Z \to 0$

  There is a long exact sequence

  $\cdots \to \mathcal{H}^n(A, Y) \to \mathcal{H}^n(A, Z) \to \mathcal{H}^{n+1}(A, X) \to \mathcal{H}^{n+1}(A, Y) \to \cdots$

  If we select a bimodule $Y$ so that $\mathcal{H}^n(A, Y) = 0$ then

  $\mathcal{H}^n(A, Z) \cong \mathcal{H}^{n+1}(A, X)$

  We select $Y = L(A \hat{\otimes} A, X)$, are it is biinjective

  Which gives $Z = L(A \hat{\otimes} A, X)/X$

  So we have that
5.6 Dimension Reduction

We have already seen $\mathcal{H}^1(A, X)$ is related to $\mathcal{H}^2(A, Z)$. More Generally . . .

**Long Exact Sequences**

- **Theorem** Given an admissible short exact sequence
  
  $$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

  There is a long exact sequence

  $$\cdots \rightarrow \mathcal{H}^n(A, Y) \rightarrow \mathcal{H}^n(A, Z) \rightarrow \mathcal{H}^{n+1}(A, X) \rightarrow \mathcal{H}^{n+1}(A, Y) \rightarrow \cdots$$

  If we select a bimodule $Y$ so that $\mathcal{H}^n(A, Y) = 0$ then

  $\mathcal{H}^n(A, Z) \cong \mathcal{H}^{n+1}(A, X)$

  We select $Y = L(A \hat{\otimes} A, X)$, are it is biinjective

  Which gives $Z = L(A \hat{\otimes} A, X)/X$

  So we have that

  $\mathcal{H}^{n+1}(A, X) \cong \mathcal{H}^n(A, Z)$
5.6 Dimension Reduction

We have already seen $\mathcal{H}^1(A, X)$ is related to $\mathcal{H}^2(A, Z)$. More Generally . . .

**Long Exact Sequences**

- **Theorem** Given an *admissible* short exact sequence
  
  \[ 0 \to X \to Y \to Z \to 0 \]

  There is a long exact sequence

  \[ \cdots \to \mathcal{H}^n(A, Y) \to \mathcal{H}^n(A, Z) \to \mathcal{H}^{n+1}(A, X) \to \mathcal{H}^{n+1}(A, Y) \to \cdots \]

- If we select a bimodule $Y$ so that $\mathcal{H}^n(A, Y) = 0$ then
  \[ \mathcal{H}^n(A, Z) \cong \mathcal{H}^{n+1}(A, X) \]

- We select $Y = L(A \hat{\otimes} A, X)$, are it is biinjective

- Which gives $Z = L(A \hat{\otimes} A, X)/X$

- So we have that
  \[ \mathcal{H}^{n+1}(A, X) \cong \mathcal{H}^n(A, Z) \]

- Hence it is isomorphic to some $\mathcal{H}^1(A, W)$
5.7 Other Resolutions - I

We have heard that we can use other biprojective resolutions to compute cohomology:

**Unit normalised resolution**

\[ P_n = A \hat{\otimes} \cdots \hat{\otimes} A / \mathbb{C} \hat{\otimes} A \]

We set \( P_n \) and use the \( d : P_{n+1} \to P_n \) as above. It is clear that these modules are of the correct form to be biprojective.

To see that the complex is exact, we define \( s : P_n \to P_{n+1} \) by

\[ s(\omega) = s(a_1 \otimes \cdots \otimes a_n) = 1 \otimes \omega \]

Observe \( ds(\omega) = d(1 \otimes \omega) = 1.\omega - 1 \otimes d(\omega) \).

We usually write:

\[ ds + sd = 1, \]

and call \( s \) a contracting homotopy.

**Exactness:**

If \( \eta \) is in \( \ker d \), then \( \eta = \eta + sd(\eta) = ds(\eta) \in \text{im } d \).

Hence \( H^n(A, Y) \) is the (usual) unit normalised cohomology.
5.7 Other Resolutions - I

We have heard that we can use other biprojective resolutions to compute cohomology:

**Unit normalised resolution**

- We set \( P_n = A \hat{\otimes} \hat{\otimes}^{n} (A/1\mathbb{C}) \hat{\otimes} A \) and use the \( d : P_{n+1} \rightarrow P_n \) as above.
5.7 Other Resolutions - I

We have heard that we can use other biprojective resolutions to compute cohomology:

**Unit normalised resolution**

- We set \( P_n = \hat{\bigotimes}^n (A/1\mathbf{C}) \hat{\bigotimes} A \) and use the \( d : P_{n+1} \to P_n \) as above.
- It is clear that these modules are of the correct form to be biprojective.
5.7 Other Resolutions - I

We have heard that we can use other biprojective resolutions to compute cohomology:

**Unit normalised resolution**

- We set $P_n = A \hat{\otimes} \otimes^n (A/1C) \hat{\otimes} A$ and use the $d : P_{n+1} \to P_n$ as above.
- It is clear that these modules are of the correct form to be biprojective.
- To see that the complex is exact, we define $s : P_n \to P_{n+1}$ by
5.7 Other Resolutions - I

We have heard that we can use other biprojective resolutions to compute cohomology:

**Unit normalised resolution**

- We set $P_n = A \hat{\otimes} \hat{\otimes}^n (A/\mathbf{C}) \hat{\otimes} A$ and use the $d : P_{n+1} \rightarrow P_n$ as above.
- It is clear that these modules are of the correct form to be biprojective.
- To see that the complex is exact, we define $s : P_n \rightarrow P_{n+1}$ by
  - $s(\omega) = s(a_1 \otimes \cdots \otimes a_n) = 1 \otimes \omega$
5.7 Other Resolutions - I

We have heard that we can use other biprojective resolutions to compute cohomology:

**Unit normalised resolution**

- We set $P_n = A^\hat{\hat{\otimes}}^n (A/1\mathbf{C})\hat{\otimes}A$ and use the $d : P_{n+1} \rightarrow P_n$ as above.
- It is clear that these modules are of the correct form to be biprojective.
- To see that the complex is exact, we define $s : P_n \rightarrow P_{n+1}$ by
  - $s(\omega) = s(a_1 \otimes \cdots \otimes a_n) = 1 \otimes \omega$
  - Observe $ds(\omega) = d(1 \otimes \omega) = 1.\omega - 1 \otimes d(\omega)$,
5.7 Other Resolutions - I

We have heard that the we can use other biprojective resolutions to compute cohomology:

### Unit normalised resolution

- We set $P_n = A \hat{\otimes} \bigotimes^n (A/1C) \hat{\otimes} A$ and use the $d : P_{n+1} \to P_n$ as above.
- It is clear that these modules are of the correct form to be biprojective.
- To see that the complex is exact, we define $s : P_n \to P_{n+1}$ by

  $s(\omega) = s(a_1 \otimes \cdots \otimes a_n) = 1 \otimes \omega$

- Observe $ds(\omega) = d(1 \otimes \omega) = 1. \omega - 1 \otimes d(\omega)$,
- We usually write: $ds + sd = 1$, and call $s$ a contracting homotopy.
5.7 Other Resolutions - I

We have heard that we can use other biprojective resolutions to compute cohomology:

**Unit normalised resolution**

- We set \( P_n = A \widehat{\otimes} \otimes^n (A/1\mathcal{C}) \otimes A \) and use the \( d : P_{n+1} \to P_n \) as above.
- It is clear that these modules are of the correct form to be biprojective.
- To see that the complex is exact, we define \( s : P_n \to P_{n+1} \) by
  \[
s(\omega) = s(a_1 \otimes \cdots \otimes a_n) = 1 \otimes \omega
  \]
- Observe \( ds(\omega) = d(1 \otimes \omega) = 1. \omega - 1 \otimes d(\omega) \),
- We usually write: \( ds + sd = 1 \), and call \( s \) a *contracting homotopy*.
- Exactness:
5.7 Other Resolutions - I

We have heard that we can use other biprojective resolutions to compute cohomology:

**Unit normalised resolution**

- We set \( P_n = A^\bigotimes \overset{n}{\bigotimes} (A/1C)^\bigotimes A \) and use the \( d : P_{n+1} \to P_n \) as above.
- It is clear that these modules are of the correct form to be biprojective.
- To see that the complex is exact, we define \( s : P_n \to P_{n+1} \) by \( s(\omega) = s(a_1 \otimes \cdots \otimes a_n) = 1 \otimes \omega \).
- Observe \( ds(\omega) = d(1 \otimes \omega) = 1 \cdot \omega - 1 \otimes d(\omega) \).
- We usually write: \( ds + sd = 1 \), and call \( s \) a **contracting homotopy**.
- Exactness:
5.7 Other Resolutions - I

We have heard that we can use other biprojective resolutions to compute cohomology:

**Unit normalised resolution**

- We set $P_n = A \hat{\otimes} \otimes^n (A/1\mathbb{C}) \hat{\otimes} A$ and use the $d : P_{n+1} \to P_n$ as above.
- It is clear that these modules are of the correct form to be biprojective.
- To see that the complex is exact, we define $s : P_n \to P_{n+1}$ by
  - $s(\omega) = s(a_1 \otimes \cdots \otimes a_n) = 1 \otimes \omega$
  - Observe $ds(\omega) = d(1 \otimes \omega) = 1.\omega - 1 \otimes d(\omega)$,
  - We usually write: $ds + sd = 1$, and call $s$ a **contracting homotopy**
  - Exactness: if $\eta$ is in Ker $d$, then $\eta = \eta + sd(\eta) = ds(\eta) \in \text{Im} d$
5.7 Other Resolutions - I

We have heard that we can use other biprojective resolutions to compute cohomology:

**Unit normalised resolution**

- We set $P_n = A \hat{\otimes} \hat{n}(A/1C) \hat{\otimes} A$ and use the $d : P_{n+1} \to P_n$ as above.
- It is clear that these modules are of the correct form to be biprojective.
- To see that the complex is exact, we define $s : P_n \to P_{n+1}$ by $s(\omega) = s(a_1 \otimes \cdots \otimes a_n) = 1 \otimes \omega$.
- Observe $ds(\omega) = d(1 \otimes \omega) = 1.\omega - 1 \otimes d(\omega)$.
- We usually write: $ds + sd = 1$, and call $s$ a contracting homotopy.
- Exactness: if $\eta$ is in $\text{Ker } d$, then $\eta = \eta + sd(\eta) = ds(\eta) \in \text{Im } d$.
- Now note that $A h_A(P_n, Y)$ is exactly the unit normalised maps [Ex].
We have heard that we can use other biprojective resolutions to compute cohomology:

**Unit normalised resolution**

- We set \( P_n = A \widehat{\otimes} \widehat{\otimes}^n (A/1_C) \widehat{\otimes} A \) and use the \( d : P_{n+1} \to P_n \) as above.

- It is clear that these modules are of the correct form to be biprojective.

- To see that the complex is exact, we define \( s : P_n \to P_{n+1} \) by
  \[
s(\omega) = s(a_1 \otimes \cdots \otimes a_n) = 1 \otimes \omega
  \]

- Observe \( ds(\omega) = d(1 \otimes \omega) = 1.\omega - 1 \otimes d(\omega) \).

- We usually write: \( ds + sd = 1 \), and call \( s \) a *contracting homotopy*.

- **Exactness:** if \( \eta \) is in \( \text{Ker} \ d \), then \( \eta = \eta + sd(\eta) = ds(\eta) \in \text{Im} \ d \).

- Now note that \( \mathcal{A}h_A(P_n, Y) \) is exactly the unit normalised maps \([\text{Ex}]\).

- Hence \( H^n(A, Y) \) is the (usual) unit normalised cohomology.
In fact for dual modules we can use even more resolutions.
In fact for dual modules we can use even more resolutions we only need to have a bi-injective resolution of $A'$. 

Note the $I_n = P_n'$ would be such a bi-injective resolution.

Let $B$ be an amenable subalgebra of $A$. We set $I_n = L_n + 2B(A, C)$ such that for $c \in B$ and $a_i \in A$, we have $T(\cdots, a_j c, a_j+1, \cdots) = T(\cdots, a_j, ca_j+1, \cdots)$, $\delta$ as above.

Note these are indeed bimodules, which are submodules of $L_n + 2B(A, C)$. We need to check that the modules are all bi-injective. We give a bimodule projection from $L_n + 2B(A, C)$ onto $L_n + 2B(A, C)$.

Set $\bar{T}(\cdots, a_j, a_j+1, \cdots) = \text{lim}_{\lambda} \sum_{i=1}^{\infty} T(\cdots, a_j, a(\lambda i) a_j+1, \cdots)$. This is a bimodule map, and $\bar{T}$ is normal between $a_j$ and $a_j+1$. Now repeat in each place to make fully $B$-normal.
In fact for dual modules we can use even more resolutions we only need to have a bi-injective resolution of $A'$. Note the $I_n = P_n'$ would be such a bi-injective resolution.

**Normalisation w.r.t. and amenable subalgebra**
5.8 Other Resolutions - II

In fact for dual modules we can use even more resolutions we only need to have a bi-injective resolution of $A'$. Note the $I_n = P'_n$ would be such a bi-injective resolution.

Normalisation w.r.t. and amenable subalgebra

Let $B$ be an amenable subalgebra of $A$
- We set $I_n = L_B^{n+2}(A, C)$ such that for $c \in B$ and $a_i \in A$
In fact for dual modules we can use even more resolutions we only need to have a bi-injective resolution of $A'$. Note the $I_n = P'_n$ would be such a bi-injective resolution.

**Normalisation w.r.t. and amenable subalgebra**

Let $B$ be an amenable subalgebra of $A$

- We set $I_n = L_B^{n+2}(A, C)$ such that for $c \in B$ and $a_i \in A$
- $T(\cdots, a_j c, a_{j+1}, \cdots) = T(\cdots, a_j, c a_{j+1}, \cdots)$,
5.8 Other Resolutions - II

In fact for dual modules we can use even more resolutions we only need to have a bi-injective resolution of $A'$. Note the $I_n = P'_n$ would be such a bi-injective resolution

**Normalisation w.r.t. and amenable subalgebra**

Let $B$ be an amenable subalgebra of $A$

- We set $I_n = L_{\mathcal{B}}^{n+2}(A, C)$ such that for $c \in B$ and $a_i \in A$
- $T(\cdots, a_j c, a_{j+1}, \cdots) = T(\cdots, a_j, ca_{j+1}, \cdots)$,
5.8 Other Resolutions - II

In fact for dual modules we can use even more resolutions we only need to have a bi-injective resolution of $A'$. Note the $I_n = P'_n$ would be such a bi-injective resolution

**Normalisation w.r.t. and amenable subalgebra**

Let $B$ be an amenable subalgebra of $A$

- We set $I_n = L_{B}^{n+2}(A, C)$ such that for $c \in B$ and $a_i \in A$
- $T(\cdots, a_j c, a_{j+1}, \cdots) = T(\cdots, a_j, ca_{j+1}, \cdots)$, $\delta$ as above
5.8 Other Resolutions - II

In fact for dual modules we can use even more resolutions we only need to have a bi-injective resolution of $A'$. Note the $I_n = P'_n$ would be such a bi-injective resolution

### Normalisation w.r.t. and amenable subalgebra

Let $B$ be an amenable subalgebra of $A$

- We set $I_n = L_B^{n+2}(A, C)$ such that for $c \in B$ and $a_i \in A$
- $T(\cdots, ajc, aj+1, \cdots) = T(\cdots, aj, caj+1, \cdots)$, $\delta$ as above
- Note these are indeed bimodules, which are submodules of $L_B^{n+2}(A, C)$
5.8 Other Resolutions - II

In fact for dual modules we can use even more resolutions we only need to have a bi-injective resolution of $A'$. Note the $I_n = P'_n$ would be such a bi-injective resolution.

**Normalisation w.r.t. and amenable subalgebra**

Let $B$ be an amenable subalgebra of $A$

- We set $I_n = L_B^{n+2}(A, C)$ such that for $c \in B$ and $a_i \in A$
- $T(\cdots, a_jc, a_{j+1}, \cdots) = T(\cdots, a_j, ca_{j+1}, \cdots)$, $\delta$ as above
- Note these are indeed bimodules, which are submodules of $L_B^{n+2}(A, C)$
- We need to check that the modules are all bi-injective.
In fact for dual modules we can use even more resolutions we only need to have a bi-injective resolution of $A'$.

Note the $I_n = P'_n$ would be such a bi-injective resolution

**Normalisation w.r.t. and amenable subalgebra**

Let $B$ be an amenable subalgebra of $A$

- We set $I_n = L_{B}^{n+2}(A, C)$ such that for $c \in B$ and $a_i \in A$
- $T(\cdots, ajc, aj+1, \cdots) = T(\cdots, aj, caj+1, \cdots)$, $\delta$ as above
- Note these are indeed bimodules, which are submodules of $L^{n+2}(A, C)$
- We need to check that the modules are all bi-injective.
- We give a bimodule projection from $L^{n+2}(A, C)$ onto $L_{B}^{n+2}(A, C)$
In fact for dual modules we can use even more resolutions we only need to have a bi-injective resolution of $A'$. Note the $I_n = P'_n$ would be such a bi-injective resolution.

Normalisation w.r.t. and amenable subalgebra

Let $B$ be an amenable subalgebra of $A$

- We set $I_n = L^{n+2}_B(A, C)$ such that for $c \in B$ and $a_i \in A$
- $T(\cdots, a_jc, a_{j+1}, \cdots) = T(\cdots, a_j, ca_{j+1}, \cdots)$, $\delta$ as above
- Note these are indeed bimodules, which are submodules of $L^{n+2}(A, C)$
- We need to check that the modules are all bi-injective.
- We give a bimodule projection from $L^{n+2}(A, C)$ onto $L^{n+2}_B(A, C)$
- Set $\bar{T}(\cdots, a_j, a_{j+1}, \cdots) = \operatorname{LIM}\lambda \sum_{i=1}^{\infty} T(\cdots, a_ja_i^\lambda, b_i^\lambda a_{j+1}, \cdots)$
In fact for dual modules we can use even more resolutions we only need to have a bi-injective resolution of $A'$. Note the $I_n = P'_n$ would be such a bi-injective resolution.

**Normalisation w.r.t. and amenable subalgebra**

Let $B$ be an amenable subalgebra of $A$

- We set $I_n = L^{n+2}_B(A, C)$ such that for $c \in B$ and $a \in A$
- $T(\cdots, a_j c, a_{j+1}, \cdots) = T(\cdots, a_j, c a_{j+1}, \cdots)$, $\delta$ as above
- Note these are indeed bimodules, which are submodules of $L^{n+2}(A, C)$
- We need to check that the modules are all bi-injective.
- We give a bimodule projection from $L^{n+2}(A, C)$ onto $L^{n+2}_B(A, C)$
- Set $\bar{T}(\cdots, a_j, a_{j+1}, \cdots) = \text{LIM}_{\lambda} \sum_{i=1}^{\infty} T(\cdots, a_j a_i^\lambda, b_i^\lambda a_{j+1}, \cdots)$
- This is a bimodule map, and $\bar{T}$ is normal between $a_j$ and $a_{j+1}$
5.8 Other Resolutions - II

In fact for dual modules we can use even more resolutions we only need to have a bi-injective resolution of $A'$.
Note the $I_n = P'_n$ would be such a bi-injective resolution

Normalisation w.r.t. and amenable subalgebra

Let $B$ be an amenable subalgebra of $A$

- We set $I_n = L_B^{n+2}(A, C)$ such that for $c \in B$ and $a_i \in A$
- $T(\cdots, a_jc, a_{j+1}, \cdots) = T(\cdots, a_j, ca_{j+1}, \cdots)$, $\delta$ as above
- Note these are indeed bimodules, which are submodules of $L_B^{n+2}(A, C)$
- We need to check that the modules are all bi-injective.
- We give a bimodule projection from $L^{n+2}(A, C)$ onto $L_B^{n+2}(A, C)$
- Set $\bar{T}(\cdots, a_j, a_{j+1}, \cdots) = \lim_{\lambda} \sum_{i=1}^{\infty} T(\cdots, a_ja_i^\lambda, b_i^\lambda a_{j+1}, \cdots)$
- This is a bimodule map, and $\bar{T}$ is normal between $a_j$ and $a_{j+1}$
- Now repeat in each place to make fully $B$-normal
6.1 Extensions and Ext

Extensions give Derivations

Given an admissible short exact sequence, of left $A$-modules

\[ 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \]

This is an extension on $X$ by $Z$. We can write the left module action on $Y$ as

\[ \theta_Y(a) = (\theta_Z(a) + D(a), \theta_X(a)) \]

where $D$ is a derivation into $L(X, Z)$, [Ex]

This is inner iff $Y \cong X \oplus Z$ as an $A$-module, [Ex]
Extensions give Derivations

- Given an admissible short exact sequence, of left $A$-modules

\[ 0 \leftarrow X \leftarrow Y \leftarrow Z \leftarrow 0 \]

This is an extension on $X$ by $Z$

We can write the left module action on $Y$ as

\[ \theta_Y(a) = (\theta_Z(a)D(a) + \theta_X(a)) \]

where $D$ is a derivation into $L(X, Z)$.

This is inner iff $Y \sim = X \oplus Z$ as an $A$-module.
6.1 Extensions and Ext

Extensions give Derivations

- Given an admissible short exact sequence, of left $A$-modules
- $0 \leftarrow X \leftarrow Y \leftarrow Z \leftarrow 0$

This is an extension on $X$ by $Z$. We can write the left module action on $Y$ as

$$\theta_Y(a) = (\theta_Z(a) D(a) \theta_X(a))$$

where $D$ is a derivation into $\mathcal{L}(X, Z)$. [Ex]

This is inner if $Y \cong X \oplus Z$ as an $A$-module. [Ex]

Michael C. White (Newcastle University)  Cohomology of Banach Algebras  15 - 16 May, 2014  26 / 35
6.1 Extensions and Ext

Extensions give Derivations

- Given an admissible short exact sequence, of left $A$-modules
- $0 \leftarrow X \leftarrow Y \leftarrow Z \leftarrow 0$
- This is an extension on $X$ by $Z$
6.1 Extensions and Ext

Extensions give Derivations

- Given an admissible short exact sequence, of left $A$-modules
  
  $0 \leftarrow X \leftarrow Y \leftarrow Z \leftarrow 0$

- This is an extension on $X$ by $Z$

- We can write the left module action on $Y$ as

\[
\theta_Y(a) = \left( \theta_Z(a) D(a) \theta_X(a) \right)
\]

where $D$ is a derivation into $L(X, Z)$. This is inner iff $Y \sim = X \oplus Z$ as an $A$-module.
Extensions give Derivations

- Given an admissible short exact sequence, of left $A$-modules
- $0 \leftarrow X \leftarrow Y \leftarrow Z \leftarrow 0$
- This is an extension on $X$ by $Z$
- We can write the left module action on $Y$ as
- $\theta_Y(a) = \begin{pmatrix} \theta_Z(a) & D(a) \\ 0 & \theta_X(a) \end{pmatrix}$
6.1 Extensions and Ext

Extensions give Derivations

- Given an admissible short exact sequence, of left $A$-modules
  
  $0 \leftarrow X \leftarrow Y \leftarrow Z \leftarrow 0$

- This is an extension on $X$ by $Z$

- We can write the left module action on $Y$ as
  
  $\theta_Y(a) = \begin{pmatrix} \theta_Z(a) & D(a) \\ 0 & \theta_X(a) \end{pmatrix}$

- where $D$ is a derivation into $L(X, Z)$, [Ex]
6.1 Extensions and Ext

Extensions give Derivations

- Given an admissible short exact sequence, of left $A$-modules
  
  $0 \leftarrow X \leftarrow Y \leftarrow Z \leftarrow 0$

- This is an extension on $X$ by $Z$

- We can write the left module action on $Y$ as
  
  $\theta_Y(a) = \begin{pmatrix} \theta_Z(a) & D(a) \\ 0 & \theta_X(a) \end{pmatrix}$

- where $D$ is a derivation into $L(X, Z)$, [Ex]

- This is inner iff $Y \cong X \oplus Z$ as an $A$-module [Ex]
6.1 Extensions and Ext

Extensions give Derivations

- Given an admissible short exact sequence, of left $A$-modules:
  $$0 \leftarrow X \leftarrow Y \leftarrow Z \leftarrow 0$$
- This is an extension on $X$ by $Z$
- We can write the left module action on $Y$ as:
  $$\theta_Y(a) = \begin{pmatrix} \theta_Z(a) & D(a) \\ 0 & \theta_X(a) \end{pmatrix}$$
- where $D$ is a derivation into $L(X, Z)$, [Ex]
- This is inner iff $Y \cong X \oplus Z$ as an $A$-module [Ex]
6.1 Extensions and Ext

Extensions give Derivations

- Given an admissible short exact sequence, of left $A$-modules
  
  $$0 \leftarrow X \leftarrow Y \leftarrow Z \leftarrow 0$$

- This is an extension on $X$ by $Z$

- We can write the left module action on $Y$ as
  
  $$\theta_Y(a) = \begin{pmatrix} \theta_Z(a) & D(a) \\ 0 & \theta_X(a) \end{pmatrix}$$

- where $D$ is a derivation into $L(X, Z)$, [Ex]

- This is inner iff $Y \cong X \oplus Z$ as an $A$-module [Ex]
6.2 Two variable cohomology

Ext as 2 variable cohomology

We have seen $H_1(A, L(X, Z))$ classifies extensions of $X$ by $Z$. This leads us to consider this special bimodule's higher cohomology.

We could make the definition $\text{Ext}^n_A(X, Z) := H^n(A, L(X, Z))$.

In fact these cohomology groups compare longer extensions $0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n \rightarrow Z \rightarrow 0$.

But we are interested in another definition of $\text{Ext}^n_A(X, Z)$ which allows us more resolutions to compute $H^n(A, L(X, Z))$. 

Michael C. White (Newcastle University) Cohomology of Banach Algebras 15 - 16 May, 2014 27 / 35
6.2 Two variable cohomology

Ext as 2 variable cohomology

- We have seen $H^1(A, L(X, Z))$ classifies extensions of $X$ by $Z$
6.2 Two variable cohomology

Ext as 2 variable cohomology

- We have seen $\mathcal{H}^1(A, L(X, Z))$ classifies extensions of $X$ by $Z$.
- This leads us to consider this special bimodule’s higher cohomology.
We have seen $\mathcal{H}^1(A, L(X, Z))$ classifies extensions of $X$ by $Z$.

This leads us to consider this special bimodule’s higher cohomology.

We could make the definition $\text{Ext}_A^n(X, Z) := \mathcal{H}^n(A, L(X, Z))$. 

In fact these cohomology groups compare longer extensions

$$0 \leftarrow X \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow Z \leftarrow 0$$

But we are interested in another definition of $\text{Ext}_A^n(X, Z)$.

This allows us more resolutions to compute $\mathcal{H}^n(A, L(X, Z))$. 
6.2 Two variable cohomology

**Ext as 2 variable cohomology**

- We have seen $\mathcal{H}^1(A, L(X, Z))$ classifies extensions of $X$ by $Z$
- This leads us to consider this special bimodule’s higher cohomology
- We could make the definition $\text{Ext}_A^n(X, Z) := \mathcal{H}^n(A, L(X, Z))$
- In fact these cohomology groups compare longer extensions
6.2 Two variable cohomology

**Ext as 2 variable cohomology**

- We have seen $\mathcal{H}^1(A, L(X, Z))$ classifies extensions of $X$ by $Z$.
- This leads us to consider this special bimodule’s higher cohomology.
- We could make the definition $\text{Ext}^n_A(X, Z) := \mathcal{H}^n(A, L(X, Z))$.
- In fact these cohomology groups compare longer extensions.
- $0 \leftarrow X \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow Z \leftarrow 0$
6.2 Two variable cohomology

Ext as 2 variable cohomology

- We have seen $\mathcal{H}^1(A, L(X, Z))$ classifies extensions of $X$ by $Z$
- This leads us to consider this special bimodule’s higher cohomology
- We could make the definition $\text{Ext}_A^n(X, Z) := \mathcal{H}^n(A, L(X, Z))$
- In fact these cohomology groups compare longer extensions
  $$0 \leftarrow X \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow Z \leftarrow 0$$
- But we are interested in another definition of $\text{Ext}_A^n(X, Z)$
6.2 Two variable cohomology

**Ext as 2 variable cohomology**

- We have seen $\mathcal{H}^1(A, L(X, Z))$ classifies extensions of $X$ by $Z$.
- This leads us to consider this special bimodule’s higher cohomology.
- We could make the definition $\text{Ext}_A^n(X, Z) := \mathcal{H}^n(A, L(X, Z))$.
- In fact these cohomology groups compare longer extensions.
- $0 \leftarrow X \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow Z \leftarrow 0$
- But we are interested in another definition of $\text{Ext}_A^n(X, Z)$.
- This allows us more resolutions to compute $\mathcal{H}^n(A, L(X, Z))$.
6.2 Two variable cohomology

**Ext as 2 variable cohomology**

- We have seen $\mathcal{H}^1(A, L(X, Z))$ classifies extensions of $X$ by $Z$.
- This leads us to consider this special bimodule’s higher cohomology.
- We could make the definition $\text{Ext}_A^n(X, Z) := \mathcal{H}^n(A, L(X, Z))$.
- In fact these cohomology groups compare longer extensions.
- $0 \leftarrow X \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow Z \leftarrow 0$
- But we are interested in another definition of $\text{Ext}_A^n(X, Z)$.
- This allows us more resolutions to compute $\mathcal{H}^n(A, L(X, Z))$. 
6.2 Two variable cohomology

**Ext as 2 variable cohomology**

- We have seen $H^1(A, L(X, Z))$ classifies extensions of $X$ by $Z$
- This leads us to consider this special bimodule’s higher cohomology
- We could make the definition $\text{Ext}_A^n(X, Z) := H^n(A, L(X, Z))$
- In fact these cohomology groups compare longer extensions
  \[ 0 \leftarrow X \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow Z \leftarrow 0 \]
- But we are interested in another definition of $\text{Ext}_A^n(X, Z)$
- This allows us more resolutions to compute $H^n(A, L(X, Z))$
6.3 One sided injective resolutions

Ext via left modules

This allows us more resolutions to compute $H^n(A, \mathcal{L}(X, Z))$.

Theorem

Given any injective resolution of the left module $Y_0 \to Y \to I_0 \to I_1 \to \cdots$, one-sided here!

$A_h(X, I_0) \to A_h(X, I_1) \to A_h(X, I_2) \to \cdots$

Is a complex whose homology is $H^n(A, \mathcal{L}(X, Z))$.

We are also allowed to use a projective resolution of $X_0 \leftarrow X \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$ along with the homology of $A_h(P_0, Z) \to A_h(P_1, Z) \to A_h(P_2, Z) \to \cdots$.

Michael C. White (Newcastle University)
Ext via left modules

- This allows us more resolutions to compute $\mathcal{H}^n(A, L(X, Z))$
6.3 One sided injective resolutions

Ext via left modules

- This allows us more resolutions to compute $\mathcal{H}^n(A, L(X, Z))$
- **Theorem** Given any injective resolution of the left module $Y$
6.3 One sided injective resolutions

Ext via left modules

This allows us more resolutions to compute $\mathcal{H}^n(A, L(X, Z))$

**Theorem** Given any injective resolution of the left module $Y$

$0 \to Y \to I_0 \to I_1 \to \cdots$, 

We are also allowed to use a projective resolution of 

$0 \leftarrow X \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$
6.3 One sided injective resolutions

Ext via left modules

- This allows us more resolutions to compute $\mathcal{H}^n(A, L(X, Z))$
- **Theorem** Given any injective resolution of the left module $Y$
- $0 \to Y \to I_0 \to I_1 \to \cdots$, 
6.3 One sided injective resolutions

Ext via left modules

- This allows us more resolutions to compute $\mathcal{H}^n(A, L(X, Z))$
- **Theorem** Given any injective resolution of the left module $Y$
  
  $$0 \rightarrow Y \rightarrow l_0 \rightarrow l_1 \rightarrow \cdots,$$
  one-sided here!

  $$\_ah(X, l_0) \rightarrow \_ah(X, l_1) \rightarrow \_ah(X, l_2) \rightarrow$$
6.3 One sided injective resolutions

Ext via left modules

- This allows us more resolutions to compute $H^n(A, L(X, Z))$
- **Theorem** Given any injective resolution of the left module $Y$
- $0 \to Y \to l_0 \to l_1 \to \cdots$, one-sided here!
- $\mathcal{A} h(X, l_0) \to \mathcal{A} h(X, l_1) \to \mathcal{A} h(X, l_2) \to$
- Is a complex whose homology is $H^n(A, L(X, Z))$
6.3 One sided injective resolutions

**Ext via left modules**

- This allows us more resolutions to compute $\mathcal{H}^n(A, L(X, Z))$
- **Theorem** Given any injective resolution of the left module $Y$
  - $0 \to Y \to I_0 \to I_1 \to \cdots$, one-sided here!
  - $A\hom(X, I_0) \to A\hom(X, I_1) \to A\hom(X, I_2) \to$
- Is a complex whose homology is $\mathcal{H}^n(A, L(X, Z))$
- We are also allowed to use a projective resolution of $X$
6.3 One sided injective resolutions

**Ext via left modules**

- This allows us more resolutions to compute $\mathcal{H}^n(A, L(X, Z))$
- **Theorem** Given any injective resolution of the left module $Y$
  - $0 \rightarrow Y \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots$, one-sided here!
  - $A_h(X, I_0) \rightarrow A_h(X, I_1) \rightarrow A_h(X, I_2) \rightarrow$
  - Is a complex whose homology is $\mathcal{H}^n(A, L(X, Z))$
- We are also allowed to use a projective resolution of $X$
  - $0 \leftarrow X \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$
6.3 One sided injective resolutions

Ext via left modules

- This allows us more resolutions to compute $\mathcal{H}^n(A, L(X, Z))$
- **Theorem** Given any injective resolution of the left module $Y$
  
  $0 \rightarrow Y \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots$, one-sided here!

  $$A h(X, I_0) \rightarrow A h(X, I_1) \rightarrow A h(X, I_2) \rightarrow$$

- Is a complex whose homology is $\mathcal{H}^n(A, L(X, Z))$
- We are also allowed to use a projective resolution of $X$
  
  $0 \leftarrow X \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$

  along with the homology of
6.3 One sided injective resolutions

**Ext via left modules**

- This allows us more resolutions to compute $\mathcal{H}^n(A, L(X, Z))$
- **Theorem** Given any injective resolution of the left module $Y$
  $$0 \to Y \to l_0 \to l_1 \to \cdots$$, one-sided here!
  $$A h(X, l_0) \to A h(X, l_1) \to A h(X, l_2) \to \cdots$$
  Is a complex whose homology is $\mathcal{H}^n(A, L(X, Z))$
- We are also allowed to use a projective resolution of $X$
  $$0 \leftarrow X \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$$
  along with the homology of
  $$A h(P_0, Z) \to A h(P_1, Z) \to A h(P_2, Z) \to \cdots$$
6.3 One sided injective resolutions

**Ext via left modules**

- This allows us more resolutions to compute $\mathcal{H}^n(A, L(X, Z))$
- **Theorem** Given any injective resolution of the left module $Y$
  
  $0 \to Y \to I_0 \to I_1 \to \cdots$, one-sided here!
  
  $A\,h(X, I_0) \to A\,h(X, I_1) \to A\,h(X, I_2) \to$

  Is a complex whose homology is $\mathcal{H}^n(A, L(X, Z))$

  We are also allowed to use a projective resolution of $X$

  $0 \leftarrow X \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$

  along with the homology of

  $A\,h(P_0, Z) \to A\,h(P_1, Z) \to A\,h(P_2, Z) \to$
6.3 One sided injective resolutions

**Ext via left modules**

- This allows us more resolutions to compute $\mathcal{H}^n(A, L(X, Z))$
- **Theorem** Given any injective resolution of the left module $Y$
  \[ 0 \rightarrow Y \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots, \text{ one-sided here!} \]
  \[ A_h(X, I_0) \rightarrow A_h(X, I_1) \rightarrow A_h(X, I_2) \rightarrow \]
  Is a complex whose homology is $\mathcal{H}^n(A, L(X, Z))$
- We are also allowed to use a projective resolution of $X$
  \[ 0 \leftarrow X \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \]
  along with the homology of
  \[ A_h(P_0, Z) \rightarrow A_h(P_1, Z) \rightarrow A_h(P_2, Z) \rightarrow \]
6.4 Special bimodules

Application 1

- **Theorem** $\mathcal{H}^1(A, L(A, Y)) = 0$ [Ex]
6.4 Special bimodules

Application 1

- **Theorem** $\mathcal{H}^1(A, L(A, Y)) = 0$ [Ex]
- **Proof** $A$ has a short projective resolution: $0 \leftarrow A \leftarrow A \leftarrow 0 \leftarrow \cdots$
6.4 Special bimodules

Application 1

**Theorem** $\mathcal{H}^1(A, L(A, Y)) = 0$ [Ex]

**Proof** $A$ has a short projective resolution: $0 \leftarrow A \leftarrow A \leftarrow 0 \leftarrow \cdots$

We are allowed to compute $\mathcal{H}^n(A, L(A, Y))$ using
6.4 Special bimodules

Application 1

- **Theorem** \( \mathcal{H}^1(A, L(A, Y)) = 0 \) [Ex]
- **Proof** \( A \) has a short projective resolution: \( 0 \leftarrow A \leftarrow A \leftarrow 0 \leftarrow \cdots \)
  
  We are allowed to compute \( \mathcal{H}^n(A, L(A, Y)) \) using

\[
A h(A, Y) \rightarrow A h(0, Y) \rightarrow A h(0, Y) \rightarrow \]

Application 2

If \( \psi \) is a character where \( \text{Ker} \ \psi \) has a b.a.i., then \( \mathcal{H}^1(A, L(X, C\psi)) = 0 \)

**Proof** \( C\psi \) has a short injective resolution: \( 0 \rightarrow C\psi \rightarrow C\psi \rightarrow 0 \rightarrow \cdots \)

We are allowed to compute \( \mathcal{H}^n(A, L(X, C\psi)) \) using

\[
A h(X, C\psi) \rightarrow A h(X, 0) \rightarrow A h(X, 0) \rightarrow \]
### Application 1

**Theorem** \( \mathcal{H}^1(A, L(A, Y)) = 0 \) [Ex]

**Proof** \( A \) has a short projective resolution: \( 0 \leftarrow A \leftarrow A \leftarrow 0 \leftarrow \cdots \)

We are allowed to compute \( \mathcal{H}^n(A, L(A, Y)) \) using

\[
A h(A, Y) \to A h(0, Y) \to A h(0, Y) \to
\]

This complex is almost all 0’s, so \( \mathcal{H}^n(A, L(A, Y)) = 0 \)
**6.4 Special bimodules**

### Application 1

- **Theorem** $\mathcal{H}^1(A, L(A, Y)) = 0$ [Ex]
- **Proof** $A$ has a short projective resolution: $0 \leftarrow A \leftarrow A \leftarrow 0 \leftarrow \cdots$
  
  We are allowed to compute $\mathcal{H}^n(A, L(A, Y))$ using
  
  $A h(A, Y) \rightarrow A h(0, Y) \rightarrow A h(0, Y) \rightarrow$

  This complex is almost all 0’s, so $\mathcal{H}^n(A, L(A, Y)) = 0$
6.4 Special bimodules

**Application 1**

- **Theorem** $\mathcal{H}^1(A, L(A, Y)) = 0$ [Ex]
- **Proof** $A$ has a short projective resolution: $0 \leftarrow A \leftarrow A \leftarrow 0 \leftarrow \cdots$
- We are allowed to compute $\mathcal{H}^n(A, L(A, Y))$ using
- $A h(A, Y) \rightarrow A h(0, Y) \rightarrow A h(0, Y) \rightarrow$
- This complex is almost all 0’s, so $\mathcal{H}^n(A, L(A, Y)) = 0$

**Application 2**

- If $\psi$ is a character where $\text{Ker} \psi$ has a b.a.i., then $\mathcal{H}^1(A, L(X, C_\psi)) = 0$
6.4 Special bimodules

Application 1

- **Theorem** \( \mathcal{H}^1(A, L(A, Y)) = 0 \) [Ex]
- **Proof** \( A \) has a short projective resolution: \( 0 \leftarrow A \leftarrow A \leftarrow 0 \leftarrow \cdots \)
- We are allowed to compute \( \mathcal{H}^n(A, L(A, Y)) \) using
  \[
  _A h(A, Y) \rightarrow _A h(0, Y) \rightarrow _A h(0, Y) \rightarrow 
  \]
- This complex is almost all 0’s, so \( \mathcal{H}^n(A, L(A, Y)) = 0 \)

Application 2

- If \( \psi \) is a character where \( \text{Ker} \ \psi \) has a b.a.i., then \( \mathcal{H}^1(A, L(X, C_\psi)) = 0 \)
- **Proof** \( C_\psi \) has a short injective resolution: \( 0 \rightarrow C_\psi \rightarrow C_\psi \rightarrow 0 \rightarrow \cdots \)
6.4 Special bimodules

Application 1

**Theorem** $\mathcal{H}^1(A, L(A, Y)) = 0$ [Ex]

**Proof**
- $A$ has a short projective resolution: $0 \leftarrow A \leftarrow A \leftarrow 0 \leftarrow \cdots$
- We are allowed to compute $\mathcal{H}^n(A, L(A, Y))$ using $A h(A, Y) \rightarrow A h(0, Y) \rightarrow A h(0, Y) \rightarrow$
- This complex is almost all 0's, so $\mathcal{H}^n(A, L(A, Y)) = 0$

Application 2

If $\psi$ is a character where $\text{Ker} \psi$ has a b.a.i., then $\mathcal{H}^1(A, L(X, C_\psi)) = 0$

**Proof**
- $C_\psi$ has a short injective resolution: $0 \rightarrow C_\psi \rightarrow C_\psi \rightarrow 0 \rightarrow \cdots$
- We are allowed to compute $\mathcal{H}^n(A, L(X, C_\psi))$ using
6.4 Special bimodules

Application 1

- **Theorem** $\mathcal{H}^1(A, L(A, Y)) = 0$ [Ex]
- **Proof** $A$ has a short projective resolution: $0 \leftarrow A \leftarrow A \leftarrow 0 \leftarrow \cdots$
- We are allowed to compute $\mathcal{H}^n(A, L(A, Y))$ using
- $Ah(A, Y) \rightarrow Ah(0, Y) \rightarrow Ah(0, Y) \rightarrow$
- This complex is almost all 0’s, so $\mathcal{H}^n(A, L(A, Y)) = 0$

Application 2

- If $\psi$ is a character where Ker $\psi$ has a b.a.i., then $\mathcal{H}^1(A, L(X, C_\psi)) = 0$
- **Proof** $C_\psi$ has a short injective resolution: $0 \rightarrow C_\psi \rightarrow C_\psi \rightarrow 0 \rightarrow \cdots$
- We are allowed to compute $\mathcal{H}^n(A, L(X, C_\psi))$ using
- $Ah(X, C_\psi) \rightarrow Ah(X, 0) \rightarrow Ah(X, 0) \rightarrow$
Application 1

**Theorem** $\mathcal{H}^1(A, L(A, Y)) = 0$ [Ex]

**Proof** $A$ has a short projective resolution: $0 \leftarrow A \leftarrow A \leftarrow 0 \leftarrow \cdots$

We are allowed to compute $\mathcal{H}^n(A, L(A, Y))$ using

\[ A^h(A, Y) \rightarrow A^h(0, Y) \rightarrow A^h(0, Y) \rightarrow \]

This complex is almost all 0’s, so $\mathcal{H}^n(A, L(A, Y)) = 0$

Application 2

If $\psi$ is a character where $\text{Ker} \psi$ has a b.a.i., then $\mathcal{H}^1(A, L(X, C_\psi)) = 0$

**Proof** $C_\psi$ has a short injective resolution: $0 \rightarrow C_\psi \rightarrow C_\psi \rightarrow 0 \rightarrow \cdots$

We are allowed to compute $\mathcal{H}^n(A, L(X, C_\psi))$ using

\[ A^h(X, C_\psi) \rightarrow A^h(X, 0) \rightarrow A^h(X, 0) \rightarrow \]

This complex is almost all 0’s, so $\mathcal{H}^n(A, L(X, C_\psi)) = 0$
6.4 Special bimodules

**Application 1**
- **Theorem** $\mathcal{H}^1(A, L(A, Y)) = 0$ [Ex]
- **Proof** $A$ has a short projective resolution: $0 \leftarrow A \leftarrow A \leftarrow 0 \leftarrow \cdots$
- We are allowed to compute $\mathcal{H}^n(A, L(A, Y))$ using
  \[ A h(A, Y) \rightarrow A h(0, Y) \rightarrow A h(0, Y) \rightarrow \]
- This complex is almost all 0’s, so $\mathcal{H}^n(A, L(A, Y)) = 0$

**Application 2**
- If $\psi$ is a character where $\text{Ker} \psi$ has a b.a.i., then $\mathcal{H}^1(A, L(X, C_\psi)) = 0$
- **Proof** $C_\psi$ has a short injective resolution: $0 \rightarrow C_\psi \rightarrow C_\psi \rightarrow 0 \rightarrow \cdots$
- We are allowed to compute $\mathcal{H}^n(A, L(X, C_\psi))$ using
  \[ A h(X, C_\psi) \rightarrow A h(X, 0) \rightarrow A h(X, 0) \rightarrow \]
- This complex is almost all 0’s, so $\mathcal{H}^n(A, L(X, C_\psi)) = 0$
6.4 Special bimodules

Application 1

- **Theorem** $\mathcal{H}^1(A, L(A, Y)) = 0$ [Ex]
- **Proof** $A$ has a short projective resolution: $0 \leftarrow A \leftarrow A \leftarrow 0 \leftarrow \cdots$
- We are allowed to compute $\mathcal{H}^n(A, L(A, Y))$ using
  
  $A h(A, Y) \to A h(0, Y) \to A h(0, Y) \to$

  This complex is almost all 0’s, so $\mathcal{H}^n(A, L(A, Y)) = 0$

Application 2

- If $\psi$ is a character where $\text{Ker} \psi$ has a b.a.i., then $\mathcal{H}^1(A, L(X, C_\psi)) = 0$
- **Proof** $C_\psi$ has a short injective resolution: $0 \to C_\psi \to C_\psi \to 0 \to \cdots$
- We are allowed to compute $\mathcal{H}^n(A, L(X, C_\psi))$ using
  
  $A h(X, C_\psi) \to A h(X, 0) \to A h(X, 0) \to$

  This complex is almost all 0’s, so $\mathcal{H}^n(A, L(X, C_\psi)) = 0$
6.5 Tensor bimodules

Homology groups and Tor

If $E$ is a left module, and $F$ a right module then $L(E, F') = (\hat{\otimes} F)'$ is a bimodule of the sort considered above and $\text{Ext}^n(E, F')$ is easier to compute. In fact we can define homology groups $H_n(A, M)$ for bimodules. This is the homology of the predual of the complex for $H_n(A, M')$. Recall $L(\hat{\otimes} A, M') \cong (\hat{\otimes} A \hat{\otimes} M)'$. Then define $\text{Tor}^A_n(E, F) = H_n(A, E \hat{\otimes} F)$.

Often $H_n(A, M)$ is the predual of $H_n(A, M')$, which makes it natural for dual modules. However, sometimes neither is even a Banach Space.
6.5 Tensor bimodules

Homology groups and Tor

- If $E$ is a left module, and $F$ a right module
Homology groups and Tor

- If $E$ is a left module, and $F$ a right module
- then $L(E, F') = (E \hat{\otimes} F)'$ is a bimodule of the sort considered above
6.5 Tensor bimodules

Homology groups and Tor

- If $E$ is a left module, and $F$ a right module
- then $L(E, F') = (E \hat{\otimes} F)'$ is a bimodule of the sort considered above
- and $\text{Ext}^n(E, F')$ is easier to compute
6.5 Tensor bimodules

Homology groups and Tor

- If $E$ is a left module, and $F$ a right module
- then $L(E, F') = (E \hat{\otimes} F)'$ is a bimodule of the sort considered above
- and $\text{Ext}^n(E, F')$ is easier to compute
- In fact we can define homology groups $\mathcal{H}_n(A, M)$ for bimodules
6.5 Tensor bimodules

Homology groups and Tor

- If $E$ is a left module, and $F$ a right module
- then $L(E, F') = (E \hat{\otimes} F)'$ is a bimodule of the sort considered above
- and $\Ext^n(E, F')$ is easier to compute
- In fact we can define homology groups $\mathcal{H}_n(A, M)$ for bimodules
- This is the homology of the predual of the complex for $\mathcal{H}_n(A, M')$
6.5 Tensor bimodules

Homology groups and Tor

- If $E$ is a left module, and $F$ a right module
- then $L(E, F') = (E \hat{\otimes} F)'$ is a bimodule of the sort considered above
- and $\text{Ext}^n(E, F')$ is easier to compute
- In fact we can define homology groups $\mathcal{H}_n(A, M)$ for bimodules
- This is the homology of the predual of the complex for $\mathcal{H}_n(A, M')$
- Recall $L(\hat{\bigotimes}^n A, M') \cong (\hat{\bigotimes}^n A \hat{\otimes} M)'$
Homology groups and Tor

- If $E$ is a left module, and $F$ a right module
- then $L(E, F') = (E \hat{\otimes} F)'$ is a bimodule of the sort considered above
- and $\text{Ext}^n(E, F')$ is easier to compute
- In fact we can define homology groups $\mathcal{H}_n(A, M)$ for bimodules
- This is the homology of the predual of the complex for $\mathcal{H}_n(A, M')$
- Recall $L(\bigotimes^n A, M') \cong (\bigotimes^n A \hat{\otimes} M)'$
- Then define $\text{Tor}^A_n(E, F) = \mathcal{H}_n(A, E \hat{\otimes} F)$
### 6.5 Tensor bimodules

#### Homology groups and Tor

- If $E$ is a left module, and $F$ a right module
- then $L(E, F') = (E \hat{\otimes} F)'$ is a bimodule of the sort considered above
- and $\text{Ext}^n(E, F')$ is easier to compute
- In fact we can define homology groups $\mathcal{H}_n(A, M)$ for bimodules
- This is the homology of the predual of the complex for $\mathcal{H}_n(A, M')$
- Recall $L(\hat{\otimes}^n A, M') \cong (\hat{\otimes}^n A \hat{\otimes} M)'$
- Then define $\text{Tor}^A_n(E, F) = \mathcal{H}_n(A, E \hat{\otimes} F)$
- Often $\mathcal{H}_n(A, M)$ is the predual of $\mathcal{H}^n(A, M')$,
6.5 Tensor bimodules

**Homology groups and Tor**

- If $E$ is a left module, and $F$ a right module
  - then $L(E, F') = (E \hat{\otimes} F)'$ is a bimodule of the sort considered above
  - and $\text{Ext}^n(E, F')$ is easier to compute
- In fact we can define homology groups $\mathcal{H}_n(A, M)$ for bimodules
- This is the homology of the predual of the complex for $\mathcal{H}_n(A, M')$
- Recall $L(\hat{\bigotimes}^n A, M') \cong (\hat{\bigotimes}^n A \hat{\otimes} M)'$
- Then define $\text{Tor}_n^A(E, F) = \mathcal{H}_n(A, E \hat{\otimes} F)$
- Often $\mathcal{H}_n(A, M)$ is the predual of $\mathcal{H}^n(A, M')$,
- which makes is natural for dual modules
6.5 Tensor bimodules

Homology groups and Tor

- If $E$ is a left module, and $F$ a right module
- then $L(E, F') = (E \hat{\otimes} F)'$ is a bimodule of the sort considered above
- and $\text{Ext}^n(E, F')$ is easier to compute
- In fact we can define homology groups $\mathcal{H}_n(A, M)$ for bimodules
- This is the homology of the predual of the complex for $\mathcal{H}_n(A, M')$
- Recall $L(\hat{\bigotimes}^n A, M') \cong (\hat{\bigotimes}^n A \hat{\otimes} M)'$
- Then define $\text{Tor}_n^A(E, F) = \mathcal{H}_n(A, E \hat{\otimes} F)$
- Often $\mathcal{H}_n(A, M)$ is the predual of $\mathcal{H}_n(A, M')$,
- which makes is natural for dual modules
- However, sometimes neither is even a Banach Space
6.5 Tensor bimodules

Homology groups and Tor

- If $E$ is a left module, and $F$ a right module
- then $L(E, F') = (E \hat{\otimes} F)'$ is a bimodule of the sort considered above
- and $\text{Ext}^n(E, F')$ is easier to compute
- In fact we can define homology groups $\mathcal{H}_n(A, M)$ for bimodules
- This is the homology of the predual of the complex for $\mathcal{H}_n(A, M')$
- Recall $L(\hat{\bigotimes}^n A, M') \cong (\hat{\bigotimes}^n A \hat{\otimes} M)'$
- Then define $\text{Tor}^A_n(E, F) = \mathcal{H}_n(A, E \hat{\otimes} F)$
- Often $\mathcal{H}_n(A, M)$ is the predual of $\mathcal{H}^n(A, M')$, which makes is natural for dual modules
- However, sometimes neither is even a Banach Space
6.5 Tensor bimodules

Homology groups and Tor

- If $E$ is a left module, and $F$ a right module
- then $L(E, F') = (E \hat{\otimes} F)'$ is a bimodule of the sort considered above
- and $\text{Ext}^n(E, F')$ is easier to compute
- In fact we can define homology groups $\mathcal{H}_n(A, M)$ for bimodules
- This is the homology of the predual of the complex for $\mathcal{H}_n(A, M')$
- Recall $L(\hat{\bigotimes}^n A, M') \cong (\hat{\bigotimes}^n A \hat{\otimes} M)'$
- Then define $\text{Tor}^A_n(E, F) = \mathcal{H}_n(A, E \hat{\otimes} F)$
- Often $\mathcal{H}_n(A, M)$ is the predual of $\mathcal{H}^n(A, M')$
- which makes is natural for dual modules
- However, sometimes neither is even a Banach Space
Recall

- We compute the simplicial cohomology, $\mathcal{H}H^n(A)$ using the complex

$$
\rightarrow L^{n-1}(A; A') \rightarrow L^n(A; A') \rightarrow L^{n+1}(A; A') \rightarrow
$$
Cyclic Cohomology – a quick review

Recall

- We compute the simplicial cohomology, $\mathcal{HH}^n(A)$ using the complex

$$\rightarrow L^{n-1}(A; A') \rightarrow L^n(A; A') \rightarrow L^{n+1}(A; A') \rightarrow$$

- which is actually just

$$\rightarrow (\hat{\otimes}^n A)' \rightarrow (\hat{\otimes}^{n+1} A)' \rightarrow (\hat{\otimes}^{n+2} A)' \rightarrow$$
Cyclic Cohomology – a quick review

Recall

- We compute the simplicial cohomology, $HH^n(A)$ using the complex

$$\rightarrow L^{n-1}(A; A') \rightarrow L^n(A; A') \rightarrow L^{n+1}(A; A') \rightarrow$$

- which is actually just

$$\rightarrow (\widehat{\bigotimes}^n A)' \rightarrow (\widehat{\bigotimes}^{n+1} A)' \rightarrow (\widehat{\bigotimes}^{n+2} A)' \rightarrow$$

- this extra symmetry allows us to impose an extra condition on our multilinear maps. We say a map $T$ is cyclic if

$$T(a_1, a_2, \ldots, a_n)(a_0) = (-1)^n T(a_2, a_3, \ldots, a_0)(a_1)$$
Recall

- We compute the simplicial cohomology, $\mathcal{HH}^n(A)$ using the complex
  \[ L^{n-1}(A; A') \to L^n(A; A') \to L^{n+1}(A; A') \to \]
  which is actually just
  \[ (\hat{\otimes}^n A)' \to (\hat{\otimes}^{n+1} A)' \to (\hat{\otimes}^{n+2} A)' \to \]

- This extra symmetry allows us to impose an extra condition on our multilinear maps. We say a map $T$ is cyclic if
  \[ T(a_1, a_2, \ldots, a_n)(a_0) = (-1)^n T(a_2, a_3, \ldots, a_0)(a_1) \]

- Surprisingly if $T$ is cyclic then so is $\delta T$, this allows us to make the definitions
Cyclic Cohomology – a quick review

Recall

- We compute the simplicial cohomology, $\mathcal{H}H^n(A)$ using the complex

  $\to L^{n-1}(A; A') \to L^n(A; A') \to L^{n+1}(A; A') \to$

  which is actually just

  $\to (\bigotimes^n A)' \to (\bigotimes^{n+1} A)' \to (\bigotimes^{n+2} A)' \to$

- this extra symmetry allows us to impose an extra condition on our multilinear maps. We say a map $T$ is cyclic if

  $T(a_1, a_2, \ldots, a_n)(a_0) = (-1)^n T(a_2, a_3, \ldots, a_0)(a_1)$

- Surprisingly if $T$ is cyclic then so is $\delta T$, this allows us to make the definitions
  - $\mathcal{Z}C^n(A)$ the space of cocycles which are cyclic;
Cyclic Cohomology – a quick review

Recall

- We compute the simplicial cohomology, $\mathcal{H}\mathcal{H}^n(A)$ using the complex
  \[ \rightarrow L^{n-1}(A; A') \rightarrow L^n(A; A') \rightarrow L^{n+1}(A; A') \rightarrow \]

  which is actually just
  \[ \rightarrow (\hat{\otimes}^n A)' \rightarrow (\hat{\otimes}^{n+1} A)' \rightarrow (\hat{\otimes}^{n+2} A)' \rightarrow \]

- this extra symmetry allows us to impose an extra condition on our multilinear maps. We say a map $T$ is cyclic if
  \[ T(a_1, a_2, \ldots, a_n)(a_0) = (-1)^n T(a_2, a_3, \ldots, a_0)(a_1) \]

- Surprisingly if $T$ is cyclic then so is $\delta T$, this allows us to make the definitions
  - $\mathcal{Z}C^n(A)$ the space of cocycles which are cyclic;
  - $\mathcal{B}C^n(A)$ the space of boundaries of cyclic cochains; (not coboundaries which are cyclic)
Cyclic Cohomology – a quick review

Recall

- We compute the simplicial cohomology, $\mathcal{H}\mathcal{H}^n(A)$ using the complex

\[ L^{n-1}(A; A') \to L^n(A; A') \to L^{n+1}(A; A') \to \]

- which is actually just

\[ \hat{\otimes}^n A' \to (\hat{\otimes} A')' \to (\hat{\otimes}^{n+1} A')' \to (\hat{\otimes}^{n+2} A')' \to \]

- this extra symmetry allows us to impose an extra condition on our multilinear maps. We say a map $T$ is cyclic if

\[ T(a_1, a_2, \ldots, a_n)(a_0) = (-1)^n T(a_2, a_3, \ldots, a_0)(a_1) \]

- Surprisingly if $T$ is cyclic then so is $\delta T$, this allows us to make the definitions

- $\mathcal{Z}C^n(A)$ the space of cocycles which are cyclic;
- $\mathcal{B}C^n(A)$ the space of boundaries of cyclic cochains; (not coboundaries which are cyclic);
- $\mathcal{H}C^n(A) = \mathcal{Z}C^n(A)/\mathcal{B}C^n(A)$. 
7.1 Connes-Tzygan

- The Simplicial and the Cyclic cohomology groups are connected by the Connes-Tzygan long exact sequences.

\[ 0 \to \mathcal{H}H^1(A) \to \mathcal{H}C^0(A) \to \mathcal{H}C^2(A) \to \mathcal{H}H^2(A) \to \mathcal{H}C^1(A) \to \cdots \]
\[ \to \mathcal{H}H^n(A) \to \mathcal{H}C^{n-1}(A) \to \mathcal{H}C^{n+1}(A) \to \mathcal{H}H^{n+1}(A) \to \cdots \]
The Simplicial and the Cyclic cohomology groups are connected by the Connes-Tzygan long exact sequences.

\[0 \rightarrow \mathcal{HH}^1(A) \rightarrow \mathcal{HC}^0(A) \rightarrow \mathcal{HC}^2(A) \rightarrow \mathcal{HH}^2(A) \rightarrow \mathcal{HC}^1(A) \rightarrow \cdots \]
\[\rightarrow \mathcal{HH}^n(A) \rightarrow \mathcal{HC}^{n-1}(A) \rightarrow \mathcal{HC}^{n+1}(A) \rightarrow \mathcal{HH}^{n+1}(A) \rightarrow \cdots\]

2. Observations
The Simplicial and the Cyclic cohomology groups are connected by the Connes-Tzygan long exact sequences.

$$0 \to \mathcal{H}\mathcal{H}^1(A) \to \mathcal{H}\mathcal{C}^0(A) \to \mathcal{H}\mathcal{C}^2(A) \to \mathcal{H}\mathcal{H}^2(A) \to \mathcal{H}\mathcal{C}^1(A) \to \cdots$$
$$\to \mathcal{H}\mathcal{H}^n(A) \to \mathcal{H}\mathcal{C}^{n-1}(A) \to \mathcal{H}\mathcal{C}^{n+1}(A) \to \mathcal{H}\mathcal{H}^{n+1}(A) \to \cdots$$

2 Observations

- If, for large \(n\), \(\mathcal{H}\mathcal{H}^n(A) = 0\) then \(\mathcal{H}\mathcal{C}^{n-1}(A) \cong \mathcal{H}\mathcal{C}^{n+1}(A)\) and so we only have \(\mathcal{H}\mathcal{C}^{\text{odd}}(A)\) and \(\mathcal{H}\mathcal{C}^{\text{even}}(A)\);
7.1 Connes-Tzygan

- The Simplicial and the Cyclic cohomology groups are connected by the Connes-Tzygan long exact sequences.

\[ 0 \to \mathcal{H}H^1(A) \to \mathcal{H}C^0(A) \to \mathcal{H}C^2(A) \to \mathcal{H}H^2(A) \to \mathcal{H}C^1(A) \to \cdots \]
\[ \to \mathcal{H}H^n(A) \to \mathcal{H}C^{n-1}(A) \to \mathcal{H}C^{n+1}(A) \to \mathcal{H}H^{n+1}(A) \to \cdots \]

- 2 Observations
  - If, for large \( n \), \( \mathcal{H}H^n(A) = 0 \) then \( \mathcal{H}C^{n-1}(A) \cong \mathcal{H}C^{n+1}(A) \) and so we only have \( \mathcal{H}C^{\text{odd}}(A) \) and \( \mathcal{H}C^{\text{even}}(A) \);
  - If, for large \( n \), \( \mathcal{H}C^n(A) = 0 \) then \( \mathcal{H}H^n(A) = 0 \);
7.1 Connes-Tzygan

- The Simplicial and the Cyclic cohomology groups are connected by the Connes-Tzygan long exact sequences.

\[ 0 \to \mathcal{H}H^1(A) \to \mathcal{H}C^0(A) \to \mathcal{H}C^2(A) \to \mathcal{H}H^2(A) \to \mathcal{H}C^1(A) \to \cdots \]
\[ \to \mathcal{H}H^n(A) \to \mathcal{H}C^{n-1}(A) \to \mathcal{H}C^{n+1}(A) \to \mathcal{H}H^{n+1}(A) \to \cdots \]

- 2 Observations
  - If, for large \( n \), \( \mathcal{H}H^n(A) = 0 \) then \( \mathcal{H}C^{n-1}(A) \cong \mathcal{H}C^{n+1}(A) \) and so we only have \( \mathcal{H}C^{\text{odd}}(A) \) and \( \mathcal{H}C^{\text{even}}(A) \);
  - If, for large \( n \), \( \mathcal{H}C^n(A) = 0 \) then \( \mathcal{H}H^n(A) = 0 \);
  - In fact it rarely happens like this as \( \mathcal{H}C^{\text{odd}}(C) = 0 \) and \( \mathcal{H}C^{\text{even}}(C) = 0 \),
7.1 Connes-Tzygan

- The Simplicial and the Cyclic cohomology groups are connected by the Connes-Tzygan long exact sequences.

\[ 0 \to \mathcal{H}\mathcal{H}^1(A) \to \mathcal{H}\mathcal{C}^0(A) \to \mathcal{H}\mathcal{C}^2(A) \to \mathcal{H}\mathcal{H}^2(A) \to \mathcal{H}\mathcal{C}^1(A) \to \cdots \]
\[ \to \mathcal{H}\mathcal{H}^n(A) \to \mathcal{H}\mathcal{C}^{n-1}(A) \to \mathcal{H}\mathcal{C}^{n+1}(A) \to \mathcal{H}\mathcal{H}^{n+1}(A) \to \cdots \]

- 2 Observations
  - If, for large \( n \), \( \mathcal{H}\mathcal{H}^n(A) = 0 \) then \( \mathcal{H}\mathcal{C}^{n-1}(A) \cong \mathcal{H}\mathcal{C}^{n+1}(A) \) and so we only have \( \mathcal{H}\mathcal{C}^{\text{odd}}(A) \) and \( \mathcal{H}\mathcal{C}^{\text{even}}(A) \);
  - If, for large \( n \), \( \mathcal{H}\mathcal{C}^n(A) = 0 \) then \( \mathcal{H}\mathcal{H}^n(A) = 0 \);
  - In fact it rarely happens like this as \( \mathcal{H}\mathcal{C}^{\text{odd}}(C) = 0 \) and \( \mathcal{H}\mathcal{C}^{\text{even}}(C) = 0 \).
The Simplicial and the Cyclic cohomology groups are connected by the Connes-Tzygan long exact sequences.

\[ 0 \to \mathcal{HH}^1(A) \to \mathcal{HC}^0(A) \to \mathcal{HC}^2(A) \to \mathcal{HH}^2(A) \to \mathcal{HC}^1(A) \to \cdots \]
\[ \to \mathcal{HH}^n(A) \to \mathcal{HC}^{n-1}(A) \to \mathcal{HC}^{n+1}(A) \to \mathcal{HH}^{n+1}(A) \to \cdots \]

2 Observations

- If, for large \( n \), \( \mathcal{HH}^n(A) = 0 \) then \( \mathcal{HC}^{n-1}(A) \cong \mathcal{HC}^{n+1}(A) \) and so we only have \( \mathcal{HC}^{odd}(A) \) and \( \mathcal{HC}^{even}(A) \);
- If, for large \( n \), \( \mathcal{HC}^n(A) = 0 \) then \( \mathcal{HH}^n(A) = 0 \);
- In fact it rarely happens like this as \( \mathcal{HC}^{odd}(C) = 0 \) and \( \mathcal{HC}^{even}(C) = 0 \), but \( \mathcal{HC}^{n-1}(A) \cong \mathcal{HC}^{n+1}(A) \) is often enough to deduce the triviality of the higher simplicial cohomology groups.
7.2 Example of Cyclic Cohomology

- e.g. 1: The algebras $\ell^1(\mathbb{Z}_+, +)$ has simplicial derivations, namely

$$D(z^n)(z^m) = nD(z^1)(z^{n+m-1}) = \frac{n}{n+m} D(z^{n+m})(1) = \tau_D(z^{n+m})$$

where $\tau_D$ is any element of $A'$ (trace), which vanishes on 1.
7.2 Example of Cyclic Cohomology

- e.g. 1: The algebras $\ell^1(\mathbb{Z}_+, +)$ has simplicial derivations, namely

$$D(z^n)(z^m) = nD(z^1)(z^{n+m-1}) = \frac{n}{n + m}D(z^{n+m})(1) = \tau_D(z^{n+m})$$

where $\tau_D$ is any element of $A'$ (trace), which vanishes on 1.

- However the following simple computation shows that $\ell^1(\mathbb{Z}_+, +)$ has no simplicial derivations:

$$D(z^n)(z^m) = \frac{n}{n + m}D(z^{n+m})(1) = \frac{n}{n + m}D(1)(z^{n+m}) = 0.$$
e.g. 1: The algebras $\ell^1(\mathbb{Z}_+, +)$ has simplicial derivations, namely

$$D(z^n)(z^m) = nD(z^1)(z^{n+m-1}) = \frac{n}{n+m}D(z^{n+m})(1) = \tau_D(z^{n+m})$$

where $\tau_D$ is any element of $A'$ (trace), which vanishes on 1.

However the following simple computation shows that $\ell^1(\mathbb{Z}_+, +)$ has no simplicial derivations:

$$D(z^n)(z^m) = \frac{n}{n+m}D(z^{n+m})(1) = \frac{n}{n+m}D(1)(z^{n+m}) = 0.$$

It then follows from the Connes-Tzygan long exact sequence

$$0 \to \mathcal{H}H^1(A) \to \mathcal{H}C^0(A) \to \mathcal{H}C^2(A) \to 0$$

which gives $\mathcal{H}C^2(A) = \mathbb{C}$. 
Recall a 2-cocycle $\phi$ is called cyclic if $\phi(f, g)(h) = +\phi(g, h)(f)$.

Given any trace $\tau$ we can define a cyclic 2-cocycle by $\phi_\tau(f, g)(h) = \tau(fgh)$. [Check the cocycle identity.]

Moreover, it is difficult for such cyclic 2-cocycles to cobound, for if $\phi = \delta \psi$, then given any idempotent $e \in A$, we have $\phi(e, e)(e) = \psi(e)(e^2) - \psi(e^2)(2) + \psi(e)(e) = \psi(e)(e)$, but as $\tau(e) = \tau(eee) = \phi(e, e)(e) = -\psi(e)(e) = 0$.

Thus we have a non-vanishing class in $HC_2(A)$ whenever we have a trace which does not vanish on some idempotent. However, this is often the only way they can arise. (Recall, we have already seen that $HC_2(\ell_1(\mathbb{Z}^+)) = \mathbb{C}$.)

However, for $C_2$, the Hilbert-Schmidt operators, one can see that $\phi_\tau(f, g)(h) = \tau(fgh)$, but $C_2$ has no non-trivial trace.
Recall a 2-cocycle $\phi$ is called cyclic if $\phi(f, g)(h) = +\phi(g, h)(f)$.

Given any trace $\tau$ we can define a cyclic 2-cocycle by $\phi_\tau(f, g)(h) = \tau(fgh)$. [Check the cocycle identity.]
Recall a 2-cocycle $\phi$ is called cyclic if $\phi(f, g)(h) = +\phi(g, h)(f)$.

Given any trace $\tau$ we can define a cyclic 2-cocycle by $\phi_\tau(f, g)(h) = \tau(fgh)$. [Check the cocycle identity.]

Moreover, it is difficult for such cyclic 2-cocycles to cobound, for if $\phi = \delta \psi$, then given any idempotent $e \in A$, we have

$$\phi(e, e)(e) = \psi(e)(e^2) - \psi(e^2)(2) + \psi(e)(e^2) = \psi(e)(e),$$

but as $\tau(e) = \tau(eee) = \phi(e, e)(e) = -\psi(e)(e) = 0$. 
Recall a 2-cocycle $\phi$ is called cyclic if $\phi(f, g)(h) = +\phi(g, h)(f)$.

Given any trace $\tau$ we can define a cyclic 2-cocycle by $\phi_{\tau}(f, g)(h) = \tau(fgh)$. [Check the cocycle identity.]

Moreover, it is difficult for such cyclic 2-cocycles to cobound, for if $\phi = \delta \psi$, then given any idempotent $e \in A$, we have

$$\phi(e, e)(e) = \psi(e)(e^2) - \psi(e^2)(2) + \psi(e)(e^2) = \psi(e)(e),$$

but as $\tau(e) = \tau(eee) = \phi(e, e)(e) = -\psi(e)(e) = 0$.

Thus we have a non-vanishing class in $HC^2(A)$ whenever we have a trace which does not vanish on some idempotent.
Recall a 2-cocycle \( \phi \) is called cyclic if \( \phi(f, g)(h) = +\phi(g, h)(f) \).

Given any trace \( \tau \) we can define a cyclic 2-cocycle by 
\[
\phi_\tau(f, g)(h) = \tau(fgh). \quad [\text{Check the cocycle identity.}]
\]
Moreover, it is difficult for such cyclic 2-cocycles to cobound, for if 
\( \phi = \delta \psi \), then given any idempotent \( e \in A \), we have

\[
\phi(e, e)(e) = \psi(e)(e^2) - \psi(e^2)(2) + \psi(e)(e^2) = \psi(e)(e),
\]
but as \( \tau(e) = \tau(eee) = \phi(e, e)(e) = -\psi(e)(e) = 0 \).

Thus we have a non-vanishing class in \( \mathcal{HC}^2(A) \) whenever we have a trace which does not vanish on some idempotent.

However, this is often the only way they can arise.
Recall a 2-cocycle $\phi$ is called cyclic if $\phi(f, g)(h) = +\phi(g, h)(f)$.

Given any trace $\tau$ we can define a cyclic 2-cocycle by $\phi_{\tau}(f, g)(h) = \tau(fgh)$. [Check the cocycle identity.]

Moreover, it is difficult for such cyclic 2-cocycles to cobound, for if $\phi = \delta \psi$, then given any idempotent $e \in A$, we have

$$\phi(e, e)(e) = \psi(e)(e^2) - \psi(e^2)(2) + \psi(e)(e^2) = \psi(e)(e),$$

but as $\tau(e) = \tau(eee) = \phi(e, e)(e) = -\psi(e)(e) = 0$.

Thus we have a non-vanishing class in $HC^2(A)$ whenever we have a trace which does not vanish on some idempotent.

However, this is often the only way they can arise.

(Recall, we have already seen that $HC^2(\ell^1(\mathbb{Z}_+, +)) = \mathbb{C}$.)
Recall a 2-cocycle $\phi$ is called cyclic if $\phi(f, g)(h) = +\phi(g, h)(f)$.

Given any trace $\tau$ we can define a cyclic 2-cocycle by $\phi_\tau(f, g)(h) = \tau(fgh)$. [Check the cocycle identity.]

Moreover, it is difficult for such cyclic 2-cocycles to cobound, for if $\phi = \delta \psi$, then given any idempotent $e \in A$, we have

$$\phi(e, e)(e) = \psi(e)(e^2) - \psi(e^2)(2) + \psi(e)(e^2) = \psi(e)(e),$$

but as $\tau(e) = \tau(eee) = \phi(e, e)(e) = -\psi(e)(e) = 0$.

Thus we have a non-vanishing class in $HC^2(A)$ whenever we have a trace which does not vanish on some idempotent.

However, this is often the only way they can arise.

(Recall, we have already seen that $HC^2(\ell^1(Z_+, +)) = \mathbb{C}$.)

However, for $C_2$, the Hilbert-Schmidt operators, one can see that $\phi_\tau(f, g)(h) = \tau(fgh)$,
Recall a 2-cocycle $\phi$ is called cyclic if $\phi(f, g)(h) = +\phi(g, h)(f)$. 

Given any trace $\tau$ we can define a cyclic 2-cocycle by $\phi_\tau(f, g)(h) = \tau(fgh)$. [Check the cocycle identity.]

Moreover, it is difficult for such cyclic 2-cocycles to cobound, for if $\phi = \delta \psi$, then given any idempotent $e \in A$, we have

$$\phi(e, e)(e) = \psi(e)(e^2) - \psi(e^2)(2) + \psi(e)(e^2) = \psi(e)(e),$$

but as $\tau(e) = \tau(eee) = \phi(e, e)(e) = -\psi(e)(e) = 0$.

Thus we have a non-vanishing class in $HC^2(A)$ whenever we have a trace which does not vanish on some idempotent.

However, this is often the only way they can arise.

(Recall, we have already seen that $HC^2(\ell^1(Z_+, +)) = \mathbb{C}$.)

However, for $C_2$, the Hilbert-Schmidt operators, one can see that $\phi_\tau(f, g)(h) = \tau(fgh)$,
7.3 Vanishing Higher Cyclic Cohomology

- Recall a 2-cocycle $\phi$ is called cyclic if $\phi(f, g)(h) = +\phi(g, h)(f)$.
- Given any trace $\tau$ we can define a cyclic 2-cocycle by $\phi_\tau(f, g)(h) = \tau(fgh)$. [Check the cocycle identity.]
- Moreover, it is difficult for such cyclic 2-cocycles to cobound, for if $\phi = \delta \psi$, then given any idempotent $e \in A$, we have

$$
\phi(e, e)(e) = \psi(e)(e^2) - \psi(e^2)(2) + \psi(e)(e^2) = \psi(e)(e),
$$

but as $\tau(e) = \tau(eee) = \phi(e, e)(e) = -\psi(e)(e) = 0$.
- Thus we have a non-vanishing class in $HC^2(A)$ whenever we have a trace which does not vanish on some idempotent.
- However, this is often the only way they can arise.
- (Recall, we have already seen that $HC^2(\ell^1(Z_+, +)) = \mathbb{C}$.)
- However, for $C_2$, the Hilbert-Schmidt operators, one can see that $\phi_\tau(f, g)(h) = \tau(fgh)$, but $C_2$ has no non-trivial trace.
7.4 Idempotents and Cyclic Cohomology

- Let $\tau$ be a trace on $A$. So $\tau \in \mathcal{HC}^0(A)$
7.4 Idempotents and Cyclic Cohomology

- Let $\tau$ be a trace on $A$. So $\tau \in HC^0(A)$
- Imagine $e(t)$ is a differentiable family of idempotents

$\tau(e\dot{e}) = \tau(\dot{e}e + e\dot{e}) = \tau(e\dot{e}) + \tau(\dot{e}e) = 0$

So $\tau(e)$ is constant on components

Show $\phi(e, e)(\dot{e}) = 0$ [Ex]
Let \( \tau \) be a trace on \( A \). So \( \tau \in \mathcal{HC}^0(A) \).

Imagine \( e(t) \) is a differentiable family of idempotents.

Note \( \tau(e)' = \tau(\dot{e}) \)
Let $\tau$ be a trace on $A$. So $\tau \in \mathcal{HC}^0(A)$

Imagine $e(t)$ is a differentiable family of idempotents

Note $\tau(e)' = \tau(\dot{e})$

$e^2 = e$
Let \( \tau \) be a trace on \( A \). So \( \tau \in \mathcal{HC}^0(A) \)

Imagine \( e(t) \) is a differentiable family of idempotents

Note \( \tau(e)' = \tau(\dot{e}) \)

\( e^2 = e \)

\( e\dot{e} + \dot{e}e = \dot{e} \)
7.4 Idempotents and Cyclic Cohomology

- Let $\tau$ be a trace on $A$. So $\tau \in HC^0(A)$
- Imagine $e(t)$ is a differentiable family of idempotents
- Note $\tau(e)' = \tau(\dot{e})$
- $e^2 = e$
- $e\dot{e} + \dot{e}e = \dot{e}$
- $e\dot{e} + e\dot{e}e = e\dot{e}$

Michael C. White (Newcastle University)  Cohomology of Banach Algebras  15 - 16 May, 2014  35 / 35
Let $\tau$ be a trace on $A$. So $\tau \in \mathcal{H}C^0(A)$

Imagine $e(t)$ is a differentiable family of idempotents

Note $\tau(e)' = \tau(\dot{e})$

$e^2 = e$

$e\dot{e} + \dot{e}e = \dot{e}$

$e\dot{e} + e\dot{e}e = e\dot{e}$

$\tau(e\dot{e}) + \tau(e\dot{e}e) = \tau(e\dot{e})$
Let $\tau$ be a trace on $A$. So $\tau \in \mathcal{HC}^0(A)$

Imagine $e(t)$ is a differentiable family of idempotents

Note $\tau(e)' = \tau(\dot{e})$

$e^2 = e$

$e\dot{e} + \dot{e}e = \dot{e}$

$e\dot{e} + e\dot{e}e = e\dot{e}$

$\tau(e\dot{e}) + \tau(e\dot{e}e) = \tau(e\dot{e})$

$\tau(e\dot{e}) + \tau(ee\dot{e}) = \tau(e\dot{e})$
7.4 Idempotents and Cyclic Cohomology

- Let $\tau$ be a trace on $A$. So $\tau \in HC^0(A)$
- Imagine $e(t)$ is a differentiable family of idempotents
- Note $\tau(e)' = \tau(\dot{e})$
- $e^2 = e$
- $e\dot{e} + \dot{e}e = \dot{e}$
- $e\dot{e} + e\dot{e}e = e\dot{e}$
- $\tau(e\dot{e}) + \tau(e\dot{e}e) = \tau(e\dot{e})$
- $\tau(e\dot{e}) + \tau(ee\dot{e}) = \tau(e\dot{e})$
- $0 = \tau(ee\dot{e}) = \tau(e\dot{e}) = \tau(\dot{e}e) = 0$
Let \( \tau \) be a trace on \( A \). So \( \tau \in \mathcal{HC}^0(A) \).

Imagine \( e(t) \) is a differentiable family of idempotents.

Note \( \tau(e)' = \tau(\dot{e}) \).

\[
e^2 = e
\]

\[
e\dot{e} + \dot{e}e = \dot{e}
\]

\[
e\dot{e} + e\dot{e} = e\dot{e}
\]

\[
\tau(e\dot{e}) + \tau(e\dot{e}e) = \tau(e\dot{e})
\]

\[
\tau(e\dot{e}) + \tau(ee\dot{e}) = \tau(e\dot{e})
\]

\[
0 = \tau(ee\dot{e}) = \tau(e\dot{e}) = \tau(\dot{e}e) = 0
\]

\[
\tau(e)' = \tau(\dot{e}) = \tau(e\dot{e} + \dot{e}e) = 0
\]
Let $\tau$ be a trace on $A$. So $\tau \in \mathcal{HC}^0(A)$

Imagine $e(t)$ is a differentiable family of idempotents

Note $\tau(e)' = \tau(\dot{e})$

$e^2 = e$

$e\dot{e} + \dot{e}e = \dot{e}$

$e\dot{e} + e\dot{e}e = e\dot{e}$

$\tau(e\dot{e}) + \tau(e\dot{e}e) = \tau(e\dot{e})$

$\tau(e\dot{e}) + \tau(ee\dot{e}) = \tau(e\dot{e})$

$0 = \tau(ee\dot{e}) = \tau(e\dot{e}) = \tau(\dot{e}e) = 0$

$\tau(e)' = \tau(\dot{e}) = \tau(e\dot{e} + \dot{e}e) = 0$

So $\tau(e)$ is constant on components
Let $\tau$ be a trace on $A$. So $\tau \in HC^0(A)$

Imagine $e(t)$ is a differentiable family of idempotents

Note $\tau(e)' = \tau(\dot{e})$

$e^2 = e$

$e\dot{e} + \dot{e}e = \dot{e}$

$e\dot{e} + e\dot{e}e = e\dot{e}$

$\tau(e\dot{e}) + \tau(e\dot{e}e) = \tau(e\dot{e})$

$\tau(e\dot{e}) + \tau(ee\dot{e}) = \tau(e\dot{e})$

$0 = \tau(ee\dot{e}) = \tau(e\dot{e}) = \tau(\dot{e}e) = 0$

$\tau(e)' = \tau(\dot{e}) = \tau(e\dot{e} + \dot{e}e) = 0$

So $\tau(e)$ is constant on components

Now, given $\phi \in HC^2(A)$
7.4 Idempotents and Cyclic Cohomology

- Let \( \tau \) be a trace on \( A \). So \( \tau \in \mathcal{HC}^0(A) \)
- Imagine \( e(t) \) is a differentiable family of idempotents
- Note \( \tau(e)' = \tau(\dot{e}) \)
- \( e^2 = e \)
- \( e\dot{e} + \dot{e}e = \dot{e} \)
- \( e\dot{e} + e\dot{e}e = e\dot{e} \)
- \( \tau(e\dot{e}) + \tau(e\dot{e}e) = \tau(e\dot{e}) \)
- \( \tau(e\dot{e}) + \tau(ee\dot{e}) = \tau(e\dot{e}) \)
- \( 0 = \tau(ee\dot{e}) = \tau(e\dot{e}) = \tau(\dot{e}e) = 0 \)
- \( \tau(e)' = \tau(\dot{e}) = \tau(e\dot{e} + \dot{e}e) = 0 \)
- So \( \tau(e) \) is constant on components
- Now, given \( \phi \in \mathcal{HC}^2(A) \)
- Show \( \phi(e, e)(e)' = 0 \) [Ex]