Haagerup Approximation Property and positive cones associated with a von Neumann algebra

Rui OKAYASU

joint with Reiji TOMATSU

May 29, 2014
Workshop on Operator Spaces, Locally Compact Quantum Groups and Amenability
Fields Institute

Supported by JSPS KAKENHI Grant Number 25800065.
Definition (Haagerup 1979)
A locally compact group $G$ has the HAP if there exist positive definite functions $\varphi_n$ on $G$ such that

(a) $\varphi_n \to 1$ uniformly on compact subsets;
(b) $\varphi_n \in C_0(G)$.

Definition (Choda 1983)
A finite v.N. algebra $M$ with a f.n. tracial state has the HAP if there exist c.c.p. normal maps $\tau_n$ on $M$ such that

(A) $\tau_n \to \text{id}_M$ in $\text{-WOT}$;
(B) $\circ \tau_n \leq T \tau_n$ and $T \tau_n \in K(H)$ satisfying $T \tau_n(x) = \tau_n(x)$ for $x \in M$. 
Definition (Haagerup 1979)

A locally compact group $G$ has the HAP if
\[ \exists \text{ positive definite functions } \varphi_n \text{ on } G \text{ such that} \]

(a) $\varphi_n \to 1$ uniformly on compact subsets;
(b) $\varphi_n \in C_0(G)$.

Definition (Choda 1983)

A finite v.N. algebra $M$ with a f.n. tracial state $\tau$ has the HAP if
\[ \exists \text{ c.c.p. normal maps } \Phi_n \text{ on } M \text{ such that} \]

(A) $\Phi_n \to \text{id}_M$ in $\sigma$-WOT;
(B) $\tau \circ \Phi_n \leq \tau$ and $T_n \in \mathbb{K}(H_\tau)$ satisfying
\[ T_n(x\xi_\tau) = \Phi_n(x)\xi_\tau \quad \text{for } x \in M. \]
Standard form

Theorem (Haagerup 1975)

Any v.N. algebra is \(*\)-isomorphic to a v.N. algebra \(M\) on a Hilbert space \(H\) such that there exists a conjugate-linear isometric involution \(J\) on \(H\) and a self-dual positive cone \(P\) in \(H\) with the following properties:

1. \(JMJ = M'\);
2. \(J\xi = \xi\) for any \(\xi \in P\);
3. \(xJxJP \subset P\) for any \(x \in M\);
4. \(JcJ = c^*\) for any \(c \in Z(M) := M \cap M'\).

Such a quadruple \((M, H, J, P)\) is called a standard form.
Theorem (Haagerup 1975)

Any v.N. algebra is $\ast$-isomorphic to a v.N. algebra $M$ on a Hilbert space $H$ such that there exists a conjugate-linear isometric involution $J$ on $H$ and a self-dual positive cone $P$ in $H$ with the following properties:

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Such a quadruple $(M, H, J, P)$ is called a standard form.

Theorem (Ando-Haagerup 2012)

The condition (4) can be dropped.
Let $\varphi$ be a f.n.s. weight on a v.N. algebra $M$. 
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- $n_\varphi := \{ x \in M \mid \varphi(x^*x) < \infty \}$. 

$H'$ is the completion of $n'$ with respect to $kxk^2$.

$n' \rightarrow H'$ is the canonical injection.

$A' = n' \cap n^* \rangle$ is the associated left Hilbert algebra with $\langle x \rangle \cdot \langle y \rangle = \langle xy \rangle$.

$\varphi'$ is the corresponding representation of $M$ on $H'$.

$S'$ is the closure of the conjugate-linear operator $T \rightarrow T'$ on $H'$, which has the polar decomposition $S' = J'1 = 2'$.

Where $J'$ is the modular involution and $\mathcal{P}'$ is the modular operator.

$P' = \{ (J'x) \mid x \in A' \}$ is the self-dual positive cone.

Then the quadruple $(n'_\varphi(M); H'; J'; P')$ is a standard form.
Let $\varphi$ be a f.n.s. weight on a v.N. algebra $M$.

- $n_\varphi := \{ x \in M \mid \varphi(x^*x) < \infty \}$.
- $H_\varphi$ is the completion of $n_\varphi$ with respect to $\|x\|_\varphi^2 := \varphi(x^*x)$.
Let $\varphi$ be a f.n.s. weight on a v.N. algebra $M$.

- $n_{\varphi} := \{x \in M \mid \varphi(x^*x) < \infty\}$.
- $H_{\varphi}$ is the completion of $n_{\varphi}$ with respect to $||x||_{\varphi}^2 := \varphi(x^*x)$.
- $\Lambda_{\varphi} : n_{\varphi} \to H_{\varphi}$ is the canonical injection.
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- $\Lambda_\varphi : n_\varphi \to H_\varphi$ is the canonical injection.
- $\mathcal{A}_\varphi := \Lambda_\varphi(n_\varphi \cap n_\varphi^*)$ is the associated left Hilbert algebra with
  \[
  \Lambda_\varphi(x) \cdot \Lambda_\varphi(y) := \Lambda_\varphi(xy), \quad \Lambda_\varphi(x)^\# := \Lambda_\varphi(x^*).
  \]
Let $\varphi$ be a f.n.s. weight on a v.N. algebra $M$.

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- $\pi_\varphi$ is the corresponding representation of $M$ on $H_\varphi$. 

$S_\varphi$ is the closure of the conjugate-linear operator $7 \mapsto \Lambda_\varphi(7)$ on $H_\varphi$, which has the polar decomposition $S_\varphi = J_\varphi 1_{2^\varphi}$; where $J_\varphi$ is the modular involution and $1_{2^\varphi}$ is the modular operator.

$P_\varphi := \{J_\varphi(\pi_\varphi(x)) \mid x \in A_\varphi\}$ is the self-dual positive cone.

Then the quadruple $(\Lambda_\varphi(M) ; H_\varphi ; J_\varphi ; P_\varphi)$ is a standard form.
Let $\varphi$ be a f.n.s. weight on a v.N. algebra $M$.

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- $S_\varphi$ is the closure of the conjugate-linear operator $\xi \mapsto \xi^\#$ on $H_\varphi$, which has the polar decomposition
  \[ S_\varphi = J_\varphi \Delta_\varphi^{1/2}, \]
  where $J_\varphi$ is the modular involution and $\Delta_\varphi$ is the modular operator.
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- $n_\varphi := \{x \in M \mid \varphi(x^*x) < \infty\}$.
- $H_\varphi$ is the completion of $n_\varphi$ with respect to $\|x\|^2_\varphi := \varphi(x^*x)$.
- $\Lambda_\varphi : n_\varphi \to H_\varphi$ is the canonical injection.
- $A_\varphi := \Lambda_\varphi(n_\varphi \cap n^{\ast}_\varphi)$ is the associated left Hilbert algebra with
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- \( P_\varphi := \{ \xi(J_\varphi \xi) \mid \xi \in A_\varphi \} \) is the self-dual positive cone.

Then the quadruple \( (\pi_\varphi(M), H_\varphi, J_\varphi, P_\varphi) \) is a standard form.
Let \((M, H, J, P)\) and \((\mathbb{M}_n, \mathbb{M}_n, J_{\text{tr}_n}, \mathbb{M}_n^+)\) be standard forms.
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**Definition**

- \([\xi_{ij}] \in \mathbb{M}_n(H)\) is \(n\)-positive if

\[
\sum_{i,j=1}^{n} x_i J x_j J \xi_{ij} \in P \quad \text{for any } x_1, \ldots, x_n \in M.
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- \(P^{(n)} := \{[\xi_{ij}] \in \mathbb{M}_n(H) : n\text{-positive}\}\).
Let \((M, H, J, P)\) and \((\mathbb{M}_n, \mathbb{M}_n, J_{\text{tr}_n}, \mathbb{M}_n^+)\) be standard forms.

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  \]

- \(P^{(n)} := \{[\xi_{ij}] \in \mathbb{M}_n(H) : \text{\(n\)-positive}\}\).

**Theorem (Schmitt-Wittstock 1982, Miura-Tomiyama 1984)**

\((\mathbb{M}_n(M), \mathbb{M}_n(H), J \otimes J_{\text{tr}_n}, P^{(n)})\) is a standard form.
Definition

Let \((M, H, J, P)\) be a standard form. A bounded linear operator \(T: H \rightarrow H\) is completely positive \((\text{c.p.})\) if

\[(T \otimes \text{id}_n)P^{(n)} \subset P^{(n)} \quad \text{for any } n \geq 1.\]
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Definition (O-Tomatsu 2013)

A v.N. algebra \(M\) has the HAP if

\(\exists\) standard form \((M, H, J, P)\);

\(\exists\) c.c.p. \(T_n \in \mathcal{K}(H)\) such that \(T_n \to 1_H\) in SOT.
Definition
Let \((M, H, J, P)\) be a standard form.
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Definition (O-Tomatsu 2013)
A v.N. algebra \(M\) has the \textbf{HAP} if
\[\exists\ \text{standard form } (M, H, J, P);\]
\[\exists \text{ c.c.p. } T_n \in \mathbb{K}(H) \text{ such that } T_n \to 1_H \text{ in SOT}.\]

Remark
The HAP does not depend on the choice of \((M, H, J, P)\).
Theorem (Torpe 1981, Junge-Ruan-Xu 2005)

A v.N. algebra $M$ is injective

$\iff \exists$ finite rank c.c.p. $T_n$ on $H$ such that $T_n \to 1_H$ in SOT.
Theorem (O-Tomatsu 2013)

If $p_n \in M$ are projections with $p_n \nearrow 1_M$, then $M$ has the HAP $\iff p_nMp_n$ has the HAP for all $n$;
Theorem (O-Tomatsu 2013)

- If $p_n \in M$ are projections with $p_n \uparrow 1_M$, then $M$ has the HAP $\iff p_n M p_n$ has the HAP for all $n$;
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Theorem (O-Tomatsu 2013)

- If $p_n \in M$ are projections with $p_n \uparrow 1_M$, then $M$ has the HAP $\iff p_nMp_n$ has the HAP for all $n$;
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- $M_1 \bar{\otimes} M_2$ has the HAP $\iff M_1, M_2$ have the HAP;
Theorem (O-Tomatsu 2013)

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- $M_1 \bar{\otimes} M_2$ has the HAP $\iff M_1, M_2$ have the HAP;
- $M$ has the HAP $\iff M'$ has the HAP.
Theorem (O-Tomatsu 2013)

Let $\alpha$ be an action of a locally compact group $G$ on a v.N. algebra $M$. If $M^G$ has the HAP, then $M$ has the HAP; if $G$ is amenable and $M$ has the HAP, then $M^G$ has the HAP.

Corollary (O-Tomatsu 2013)

A v.N. algebra $M$ has the HAP if and only if so does its core $eM = M^R$. 
Theorem (O-Tomatsu 2013)

Let $\alpha$ be an action of a locally compact group $G$ on a v.N. algebra $M$.

- If $M \rtimes_{\alpha} G$ has the HAP, then $M$ has the HAP;

Corollary (O-Tomatsu 2013)

A v.N. algebra $M$ has the HAP if and only if so does its core $eM = M \cap R$. 
Theorem (O-Tomatsu 2013)

Let $\alpha$ be an action of a locally compact group $G$ on a v.N. algebra $M$.

- If $M \rtimes_\alpha G$ has the HAP, then $M$ has the HAP;
- If $G$ is amenable and $M$ has the HAP, then $M \rtimes_\alpha G$ has the HAP.
Theorem (O-Tomatsu 2013)
Let $\alpha$ be an action of a locally compact group $G$ on a v.N. algebra $M$.

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Corollary (O-Tomatsu 2013)
A v.N. algebra $M$ has the HAP if and only if so does its core $\widetilde{M} := M \rtimes_{\sigma} \mathbb{R}$. 
Theorem (O-Tomatsu 2013)

If $E : M \to N$ is a (not necessarily normal) conditional expectation and $M$ has the HAP, then $N$ has the HAP.
Let \( \varphi \) be a f.n. state on a \( \sigma \)-finite v.N. algebra \( M \).
Let \( (H_\varphi, \xi_\varphi) \) be the GNS representation and \( \Delta_\varphi \) be the modular operator.
Let $\varphi$ be a f.n. state on a $\sigma$-finite v.N. algebra $M$.
Let $(H_\varphi, \xi_\varphi)$ be the GNS representation and $\Delta_\varphi$ be the modular operator.

**Theorem (O-Tomatsu 2013)**

A $\sigma$-finite v.N. algebra $M$ has the HAP if and only if

- $\exists$ c.p. compact contractions $T_n$ on $H_\varphi$ such that $T_n \to 1_{H_\varphi}$ in SOT;
- $\exists$ c.c.p. normal maps $\Phi_n$ on $M$ such that $\varphi \circ \Phi_n \leq \varphi$ and

$$T_n(\Delta_\varphi^{1/4} x \xi_\varphi) = \Delta_\varphi^{1/4} \Phi_n(x) \xi_\varphi \quad \text{for } x \in M.$$
Let $\varphi$ be a f.n. state on a $\sigma$-finite v.N. algebra $M$. Let $(H_\varphi, \xi_\varphi)$ be the GNS representation and $\Delta_\varphi$ be the modular operator.

**Theorem (O-Tomatsu 2013)**

A $\sigma$-finite v.N. algebra $M$ has the HAP if and only if

1. There exist c.p. compact contractions $T_n$ on $H_\varphi$ such that $T_n \to 1_{H_\varphi}$ in SOT;
2. There exist c.c.p. normal maps $\Phi_n$ on $M$ such that $\varphi \circ \Phi_n \leq \varphi$ and

$$T_n(\Delta_\varphi^{1/4} x \xi_\varphi) = \Delta_\varphi^{1/4} \Phi_n(x) \xi_\varphi \quad \text{for } x \in M.$$

**Remark**

Our HAP is equivalent to the original definition when $M$ is finite.
Definition (Caspers-Skalski 2013)

A (\(\sigma\)-finite) v.N. algebra \(M\) (with a f.n. state \(\varphi\)) has the **CS-HAP** if

- \(\exists\) compact contractions \(T_n\) on \(H_\varphi\) such that \(T_n \to 1_{H_\varphi}\) in SOT;
- \(\exists\) c.p. normal maps \(\Phi_n\) on \(M\) such that \(\varphi \circ \Phi_n \leq \varphi\) and

\[
T_n(x\xi_\varphi) = \Phi_n(x)\xi_\varphi \quad \text{for} \ x \in M.
\]
Definition (Caspers-Skalski 2013)

A (σ-finite) v.N. algebra $M$ (with a f.n. state $\varphi$) has the CS-HAP if

- ∃ compact contractions $T_n$ on $H_\varphi$ such that $T_n \to 1_{H_\varphi}$ in SOT;
- ∃ c.p. normal maps $\Phi_n$ on $M$ such that $\varphi \circ \Phi_n \leq \varphi$ and

\[ T_n(x\xi_\varphi) = \Phi_n(x)\xi_\varphi \quad \text{for} \ x \in M. \]

Remark

The OT-HAP is equivalent to the CS-HAP.
Proof

Let $M$ be a v.N. algebra.
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$M$ has the HAP if and only if its core $\tilde{M} := M \rtimes_\sigma \mathbb{R}$ has the HAP.
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So we may assume that $M = N \otimes \mathcal{B}(H)$, where $N$ is finite.
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So we may assume that $M = N \otimes \mathbb{B}(H)$, where $N$ is finite.

Then $M$ has the HAP if and only if $N$ has the HAP.
Proof

Let $M$ be a v.N. algebra.

$M$ has the HAP if and only if its core $\widetilde{M} := M \rtimes_\sigma \mathbb{R}$ has the HAP.

So we may assume that $M = N \otimes \mathcal{B}(H)$, where $N$ is finite.

Then $M$ has the HAP if and only if $N$ has the HAP.

In the case of finite v.N. algebras, CS-HAP and OT-HAP are equivalent to the original one.
Let $M$ be a $\sigma$-finite v.N. algebra $M$ with a f.n. state $\varphi$.

**OT-HAP**

- $\exists$ c.p. compact contractions $T_n$ on $H_\varphi$ such that $T_n \to 1_{H_\varphi}$ in SOT;
- $\exists$ c.c.p. normal maps $\Phi_n$ on $M$ such that $\varphi \circ \Phi_n \leq \varphi$ and

$$T_n(\Delta^{1/4}_\varphi x \xi_\varphi) = \Delta^{1/4}_\varphi \Phi_n(x) \xi_\varphi \quad \text{for} \ x \in M.$$ 

**CS-HAP**

- $\exists$ compact contractions $T_n$ on $H_\varphi$ such that $T_n \to 1_{H_\varphi}$ in SOT;
- $\exists$ c.p. normal maps $\Phi_n$ on $M$ such that $\varphi \circ \Phi_n \leq \varphi$ and

$$T_n(x \xi_\varphi) = \Phi_n(x) \xi_\varphi \quad \text{for} \ x \in M.$$
Let $M$ be a $\sigma$-finite v.N. algebra $M$ with a f.n. state $\varphi$.

**OT-HAP**

- $\exists$ c.p. compact contractions $T_n$ on $H_\varphi$ such that $T_n \to 1_{H_\varphi}$ in SOT;
- $\exists$ c.c.p. normal maps $\Phi_n$ on $M$ such that $\varphi \circ \Phi_n \leq \varphi$ and

$$T_n(\Delta^{1/4}_{\varphi} x \xi_\varphi) = \Delta^{1/4}_{\varphi} \Phi_n(x) \xi_\varphi \quad \text{for } x \in M.$$ 

**CS-HAP**

- $\exists$ compact contractions $T_n$ on $H_\varphi$ such that $T_n \to 1_{H_\varphi}$ in SOT;
- $\exists$ c.p. normal maps $\Phi_n$ on $M$ such that $\varphi \circ \Phi_n \leq \varphi$ and

$$T_n(x \xi_\varphi) = \Phi_n(x) \xi_\varphi \quad \text{for } x \in M.$$
Let $\mathcal{M}$ be a v.N. algebra $\mathcal{M}$ with a f.n.s. weight $\varphi$. 

$\mathcal{A}$ is the associated left Hilbert algebra.
Araki’s positive cones

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$$P_{\varphi}^{\#} = \{ \xi \xi^* | \xi \in \mathcal{A}_\varphi \} \quad \text{and} \quad P_{\varphi} = P_{\varphi}^{\#} = \{ \xi(J_\varphi \xi) | \xi \in \mathcal{A}_\varphi \}.$$
Let $\mathcal{M}$ be a v.N. algebra $\mathcal{M}$ with a f.n.s. weight $\varphi$. Recall that $\mathcal{A}_\varphi$ is the associated left Hilbert algebra. Consider the following positive cone:

$$P^\#_\varphi = \{ \xi\xi^* \mid \xi \in \mathcal{A}_\varphi \} \quad \text{and} \quad P_\varphi = P_\varphi^\# = \{ \xi(J_\varphi \xi) \mid \xi \in \mathcal{A}_\varphi \}.$$

**Definition (Araki 1974)**

$$P_\varphi^\alpha := \Delta_\varphi^{\alpha} P^\#_\varphi \quad \text{for} \ 0 \leq \alpha \leq 1/2.$$
Let $M$ be a v.N. algebra $M$ with a f.n.s. weight $\varphi$. Recall that $A_\varphi$ is the associated left Hilbert algebra. Consider the following positive cone:

$$P^\#_\varphi = \{ \xi \xi^\# | \xi \in A_\varphi \} \quad \text{and} \quad P_\varphi = P^{\downarrow}_\varphi = \{ \xi (J_\varphi \xi) | \xi \in A_\varphi \}. $$

**Definition (Araki 1974)**

$$P_\varphi^\alpha := \Delta_\varphi^\alpha P^\#_\varphi \quad \text{for} \quad 0 \leq \alpha \leq 1/2.$$ 

- $P^0_\varphi = P^\#_\varphi$ and $P^{1/4}_\varphi = P_\varphi = P^{\downarrow}_\varphi$. 


Let $M$ be a v.N. algebra $M$ with a f.n.s. weight $\varphi$. Recall that $A_\varphi$ is the associated left Hilbert algebra. Consider the following positive cone:

$$P^\#_\varphi = \{ \xi \xi^\# \mid \xi \in A_\varphi \} \quad \text{and} \quad P_\varphi = P^\dagger_\varphi = \{ \xi (J_\varphi \xi) \mid \xi \in A_\varphi \}.$$  

**Definition (Araki 1974)**

$$P^\alpha_\varphi := \Delta^\alpha_\varphi P^\#_\varphi \quad \text{for} \ 0 \leq \alpha \leq 1/2.$$  

- $P^0_\varphi = P^\#_\varphi$ and $P^{1/4}_\varphi = P_\varphi = P^\dagger_\varphi$;

- $J_\varphi P^\alpha_\varphi = P^{1/2-\alpha}_\varphi$.  


Let $M$ be a v.N. algebra $M$ with a f.n.s. weight $\varphi$. Recall that $A_{\varphi}$ is the associated left Hilbert algebra. Consider the following positive cone:

$$P_{\varphi}^\# = \{ \xi\xi^\# | \xi \in A_{\varphi} \} \quad \text{and} \quad P_{\varphi} = P_{\varphi}^h = \{ \xi(J_{\varphi}\xi) | \xi \in A_{\varphi} \}.$$ 

**Definition (Araki 1974)**

$$P_{\varphi}^\alpha := \Delta_{\varphi}^\alpha P_{\varphi}^\# \quad \text{for} \ 0 \leq \alpha \leq 1/2.$$ 

- $P_{\varphi}^0 = P_{\varphi}^\#$ and $P_{\varphi}^{1/4} = P_{\varphi} = P_{\varphi}^h$;
- $J_{\varphi}P_{\varphi}^\alpha = P_{\varphi}^{1/2-\alpha}$;
- $P_{\varphi}^{1/2-\alpha} = \{ \eta \in H_{\varphi} : \langle \eta, \xi \rangle \geq 0 \ \text{for} \ \xi \in P_{\varphi}^\alpha \}$. 

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**Araki’s positive cones**
Let $0 \leq \alpha \leq 1/2$. Let $M$ be a v.N. algebra with a f.n.s weight $\varphi$.

**Definition (O-Tomatsu 2014)**

A v.N. algebra $M$ has the $\alpha$-HAP if

- there exist contractions $T_n \in \mathbb{K}(H_\varphi)$ such that $T_n \to 1_{H_\varphi}$ in SOT;
- $T_n$ is completely positive with respect to $P_\varphi^\alpha$. 

**Remark.** It can be proved that the $\alpha$-HAP does not depend on the choice of $\varphi$.
Let $0 \leq \alpha \leq 1/2$. Let $M$ be a v.N. algebra with a f.n.s weight $\varphi$.

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It can be proved that the $\alpha$-HAP does not depend on the choice of $\varphi$. 
Let $M$ be a von Neumann algebra.

**Theorem (O-Tomatsu 2014)**

The following are equivalent:

1. $M$ has the OT-HAP, i.e., $1/4$-HAP;
2. $M$ has the CS-HAP;
3. $M$ has the 0-HAP;
4. $M$ has the $\alpha$-HAP for some/all $\alpha$;
5. For any f.n.s. weight $\phi$, $\exists$ c.c.p. normal maps $\Phi_n$ on $M$ such that
   - $\phi \circ \Phi_n \leq \phi$;
   - $\Phi_n \to \text{id}_M$ in $\sigma$-WOT;
   - for any $0 \leq \alpha \leq 1/2$, the associated c.c.p. operators $T_n^\alpha$ are compact and $T_n^\alpha \to 1_{H_\phi}$, where
     $$T_n^\alpha(\Delta_\phi \Lambda_\phi(x)) = \Delta_\phi \Lambda_\phi(\Phi_n(x)) \quad \text{for } x \in n_\phi.$$
Proof of $(1) \implies (2)$

We may assume that $M$ is $\sigma$-finite with a f.n. state $\varphi$. 
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T_n(\Delta^{1/4}_\varphi x\xi_\varphi) := \Delta^{1/4}_\varphi \Phi_n(x)\xi_\varphi.
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Let \(g_\beta(t) := \sqrt{\beta/\pi} \exp(-\beta t^2)\) for \(\beta > 0\).
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Then $T_{n, \beta, \gamma}^0 (x\xi_\varphi) := \Phi_{n, \beta, \gamma}(x)\xi_\varphi$ such that

$$T_{n, \beta, \gamma}^0 = (U_\beta \Delta^{-1/4}_\varphi) T_n(\Delta^{1/4}_\varphi U_\gamma) \in \mathbb{K}(H_\varphi),$$

because $U_\beta \Delta^{-1/4}_\varphi, \Delta^{1/4}_\varphi U_\gamma \in \mathbb{B}(H_\varphi)$. 

Independency on the choice of positive cones

Let $M$ be a von Neumann algebra.

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Proof of $\text{(3) } \implies \text{(4)}$

Lemma (O-Tomatsu 2014)

Let $\alpha \in [0, 1/4]$ and $T \in \mathcal{B}(H\varphi)$ be completely positive with respect to $P^\alpha_\varphi$. Then for $\beta \in [\alpha, 1/2 - \alpha],$

- $\Delta^{\beta - \alpha}_\varphi T \Delta^{\alpha - \beta}_\varphi$ can be extended to a bounded operator on $H\varphi$ with the norm $||T||$, which is completely positive with respect to $P^\beta_\varphi$.
- If $T$ is compact, then so does $\Delta^{\beta - \alpha}_\varphi T \Delta^{\alpha - \beta}_\varphi$. 
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- If $T$ is compact, then so does $\Delta^\beta_{\varphi} T \Delta^{\alpha-\beta}_{\varphi}$.

**Proof** For simplicity, assume that $\alpha = 0$. 

Proof of (3) $\Rightarrow$ (4)

Lemma (O-Tomatsu 2014)

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- If $T$ is compact, then so does $\Delta_{\varphi}^{\beta-\alpha} T \Delta_{\varphi}^{\alpha-\beta}$.

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For simplicity, assume that $\alpha = 0$.

Now suppose that $T$ is bounded (or compact) at the endpoint 0.
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Rui OKAYASU (OKU)  
HAP and positive cones  
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Apply the three lines Theorem.
Let $1 < p < \infty$ and $L^p(M)$ be Haagerup’s non-commutative $L^p$-space.
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A v.N. algebra $M$ has the $L^p$-HAP if there exist compact contractiions $T_n$ on $L^p(M)$ such that

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Note that $(M, L^2(M), *, L^2(M)^+)$ is a standard form.

**Theorem (O-Tomatsu 2014)**

A v.N. algebra $M$ has the HAP, i.e., $L^2$-HAP

$\iff M$ has the $L^p$-HAP for some/all $1 < p < \infty$. 

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