Weighted Fourier algebras of non-compact Lie groups and its spectrum

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Weighted convolution algebras

- G: locally compact group $\Rightarrow (L^1(G), *)$ is a Banach algebra that can distinguish G.
- A measurable function $w : G \to (0, \infty)$ is called a **weight** if it is sub-multiplicative i.e.

$$w(st) \leq w(s)w(t), \quad s, t \in G.$$

- For a weight w the weighted space $L^1(G, w)$ equipped with the norm $\|f\|_{L^1(G,w)} = \int_G w(x)\,|f(x)|\,dx$ is still **a Banach algebra w.r.t. the convolution**. $L^1(G, w)$ is called a **Beurling algebra on** G.
- (**Examples**) $G = \mathbb{R}$ or $\mathbb{Z}$, $\alpha \geq 0$, $\rho \geq 1$.
  $w_\alpha(x) = (1 + |x|)^\alpha$ (Polynomial type weights)
  $w_\rho(x) = \rho^{|x|}$ (Exponential type weights).

Reformulation using co-multiplication

- We begin with the co-multiplication (the adjoint of the convolution map)

$$\Gamma : L^\infty(G) \to L^\infty(G \times G)$$

given by $\Gamma(f)(s,t) = f(st)$.

- $(L^1(G,w))^* = L^\infty(G,w^{-1})$ with the norm

$$\|f\|_{L^\infty(G;w^{-1})} := \left\|\frac{f}{w}\right\|_\infty,$$

so that $\Phi : L^\infty(G) \to L^\infty(G,w^{-1}),\ \ f \mapsto fw$ is an isometry.

Reformulation using co-multiplication: continued

- Using the convolution again on $L^1(G, w)$ means we will use the same Γ on $L^\infty(G, w^{-1})$. Then, the isometry Φ gives us the modified co-multiplication

$$\widetilde{\Gamma} : L^\infty(G) \to L^\infty(G \times G), \;\; f \mapsto \Gamma(f)\Gamma(w)(w^{-1} \otimes w^{-1}).$$

- Note that $\Gamma(w)(w^{-1} \otimes w^{-1}) \leq 1$ iff w is a weight.
- We would like to do the same procedure in the dual (i.e. Fourier algebra) setting.

The Fourier algebra $A(G)$

- G: locally compact group.
- The group von Neumann algebra $VN(G)$ is given by

$$\{\lambda(x) : x \in G\}'' \subseteq B(L^2(G)),$$

  where $\lambda(x)$ is the left translation (i.e. $\lambda(x)f(y) = f(x^{-1}y)$).
- $\lambda : G \to B(L^2(G))$ is called the **left regular representation**.
- **(Eymard, '64)**
  $A(G) := VN(G)_* = \{f * \check{g} : f, g \in L^2(G)\} \subseteq C_0(G)$, where $\check{g}(x) = g(x^{-1})$.
- $(A(G), \cdot)$ is known to be a commutative Banach algebra distinguishing G, which we call the **Fourier algebra** on G.
- (Example) $G = \mathbb{R}$
  $(A(\mathbb{R}), \cdot) \cong (L^1(\widehat{\mathbb{R}}), *)$

Weighted Fourier algebra - a refined definition

- Recall that w on G gives us M_w a (unbdd) closed, densely defined, positive, invertible operator affilliated to $L^\infty(G)$ acting on $L^2(G)$.
- For $VN(G) \subseteq B(\mathcal{H})$ we will consider W, a (unbdd) closed, densely defined, positive, invertible operator affilliated to $VN(G)$ acting on $\mathcal{H}$.
- We consider the weighted spaces $VN(G, W^{-1}) := \{AW : A \in VN(G)\}$, $\|AW\|_{VN(G,W^{-1})} = \|A\|_{VN(G)}$ and $A(G, W) := \{W^{-1}\phi : \phi \in A(G)\}$, $\left\|W^{-1}\phi\right\|_{A(G,W)} = \|\phi\|_{A(G)}$.
- $\Phi : VN(G) \to VN(G; W^{-1})$, $A \mapsto AW$ is an (complete) isometry.

Weighted Fourier algebra: continued

- The co-multiplication this time is given by
$$\Gamma : VN(G) \to VN(G \times G), \ \ \lambda(x) \mapsto \lambda(x) \otimes \lambda(x).$$
- If we use *"the same"* Γ on $VN(G, W^{-1})$, then by applying Φ we get a modified co-multiplication
$$\widetilde{\Gamma} : VN(G) \to VN(G \times G), \ \ A \mapsto \Gamma(A)\Gamma(W)(W^{-1} \otimes W^{-1}).$$
- We say W is a **weight** on the dual of G if $\Gamma(W)$ and $W \otimes W$ are **strongly commuting** and
$$\left\|\Gamma(W)(W^{-1} \otimes W^{-1})\right\| \leq 1.$$
- Then, $A(G, W)$ is a commutative Banach algebra (under the pointwise multiplication at least when W^{-1} is bounded).
- (**Def**, Ludwig/Spronk/Turowska '12, L/Samei '12)
We call $A(G, W)$ a **Beurling-Fourier algebra on** G.

Extension of $*$-homomorphism and tensor product

- Let $\Delta : \mathcal{M} \subseteq B(H) \to \mathcal{N} \subseteq B(K)$ be a normal $*$-homorphism between VN-alg's. Let T be a self-adjoint operator on H affiliated to $\mathcal{M}$. Then
$$T = \int_{\mathbb{R}} \lambda \, dE_T(\lambda),$$
where E_T is the spectral measure for T with values in $\mathcal{M}$. We define
$$\Delta(T) := \int_{\mathbb{R}} \lambda \, d(\Delta \circ E_T)(\lambda).$$
- Tensor product of two spectral integrals can be understood as another spectral integral with respect to the tensor product of two spectral measures.
- Two self-adjoint operators are called **strongly commuting** if their spectral measures commute. Strongly commuting operators have well-behaving **joint Borel functional calculus**!

Extension of weights

- One serious problem of $A(G, W)$ is to find a nontrivial weight W.
- **(Extension procedure)**
  $H < G$ an abelian subgroup and $\phi : \widehat{H} \to (0, \infty)$ a weight. Then the operator $W = i(M_\phi)$ is a weight on the dual of G, where i is the embedding

$$i : L^\infty(\widehat{H}) \cong VN(H) \hookrightarrow VN(G),\ \lambda_H(x) \mapsto \lambda_G(x).$$

Spectrum of $A(G)$ and $A(G, w)$

- Recall Spec$A(G) \cong G$, where Spec$A(G)$ is the space of non-zero multiplicative functionals on $A(G)$.
- A natural question: What is Spec$A(G, W)$? Any connection to the structure of G?
- We guess that Spec$A(G, w)$ is actually coming from the points of the **complexification** $G_{\mathbb{C}}$ of G.
- For a (real) Lie group G we can associate its (real) Lie algebra $\mathfrak{g}$. Then the complexified Lie agebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$ might have its associated (simply connected) Lie group $G_{\mathbb{C}}$. In this case, we call $G_{\mathbb{C}}$ the **complexification** of G.
- $\mathbb{R}_{\mathbb{C}} = \mathbb{C}$, $\mathbb{T}_{\mathbb{C}} = \mathbb{C}\backslash\{0\}$, $SU(2)_{\mathbb{C}} = SL(2, \mathbb{C})$.
- (**Ludwig/Spronk/Turowska, '12**)
  Our guess is true for compact groups! Key ingredient is the subalgebra Trig$G \subseteq A(G, W)$ generated by the coefficient functions of irreducible unitary representations.

The case of $\mathbb{R}$

- Let $\varphi : A(\mathbb{R}, w_\rho) \cong L^1(\widehat{\mathbb{R}}, w_\rho) \to \mathbb{C}$ is multiplicative (w.r.t. **ptwise multiplication**).
- We consider a subalgebra $\mathcal{A} = C_c^\infty(\widehat{\mathbb{R}})$, which will play the same role as Trig$G$ and let $\tilde{\varphi} = \varphi|_{\mathcal{A}}$. Then $\tilde{\varphi}$ is a distribution which is multiplicative w.r.t. **convolution**.
- In other words, $\tilde{\varphi}$ is a solution to the (distributional) Cauchy functional equation

$$f(x+y) = f(x)f(y),\ x, y \in \mathbb{R},$$

  which we know that the solution must be of the form $e^{2\pi i c x}$ for some $c \in \mathbb{C}$.
- **(Conclusion)** Spec$A(\mathbb{R}, w_\rho) \cong \{c \in \mathbb{C} : |Imc| \leq \frac{1}{2\pi} \log \rho\}$.
- Note that $\rho = 1$ recovers Spec$A(\mathbb{R}) \cong \mathbb{R}$.

The case of $\mathbb{R}$: continued

- **(Main ingredients)**
  (1) The density of $\mathcal{A}$ in $A(\mathbb{R}, w_\rho)$
  (2) Complex Fourier inversion for the elements in $\mathcal{A}$.
- **(Proof)** For $f \in \mathcal{A}$ we know that

$$(*) \quad \varphi(f) = \int_{\mathbb{R}} e^{2\pi icx} f(x)dx.$$

The Paley-Wiener thm says that $\widehat{f}^{\mathbb{R}}$ extends holomorphically to $\mathbb{C}$, Thus we have

$$\varphi(f) = \widehat{f}^{\mathbb{R}}(-c).$$

Thus $(*)$ can be understood as the **complex Fourier inversion** for $\widehat{f}^{\mathbb{R}}$. For the conclusion we just need to check the norm condition for φ.

The Heisenberg group

- $H_1 = \left\{ (x,y,z) = \begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} : x,y,z \in \mathbb{R} \right\}$ be the Heisenberg group with the Haar measure = the Lebesgue measure on $\mathbb{R}^3$.
- $VN(H_1)$ and $A(H_1)$ can be described concretely using representation theory of H_1.
- For any $r \in \mathbb{R}\backslash\{0\}$ we have the Schrödinger representation
$$\pi^r(x,y,z)\xi(w) = e^{2\pi i r(-wy+z)}\xi(-x+w),\ \xi \in L^2(\mathbb{R}).$$
- The left regular representation decomposes
$$\lambda \cong \int_{\mathbb{R}\backslash\{0\}}^{\oplus} \pi^r |r| dr \ \text{ and consequently}$$
$$VN(H_1) \cong L^\infty(\mathbb{R}\backslash\{0\}, |r|dr; B(L^2(\mathbb{R}))),$$
$$A(H_1) \cong L^1(\mathbb{R}\backslash\{0\}, |r|dr; S^1(L^2(\mathbb{R}))),$$

where $S^1(\mathcal{H})$ is the trace class on $\mathcal{H}$.

The Heisenberg group: continued

- **(Fourier transform on H_1)**
  We define
  $$\mathcal{F}^{H_1} : L^1(H_1) \to VN(H_1)$$
  given by
  $$\mathcal{F}^{H_1}(f)(r) = \int_{H_1} f(x,y,z)\pi^r(x,y,z)dxdydz.$$
- **(Fourier inversion on H_1)**
  We define
  $$(\mathcal{F}^{H_1})^{-1} : A(H_1) \to L^\infty(H_1)$$
  given by for $A = (A(r))_r \in A(H_1)$
  $$(\mathcal{F}^{H_1})^{-1}(A)(x,y,z) = \int_{\mathbb{R}\setminus\{0\}} \mathrm{Tr}(A(r)\pi^r(x,y,z))|r|dr.$$

The Heisenberg group: continued 2

- **(Complexification of H_1)**

$$(H_1)_{\mathbb{C}} = \left\{ \begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} : x, y, z \in \mathbb{C} \right\}.$$

- **(Weights on H_1)**
Let $X = \{(x, 0, 0) : x \in \mathbb{R}\}$. By the extension procedure we get the weight W_X^ρ extended from the subgroup X using the weight function $\phi(t) = \rho^{|t|}$, $\rho \geq 1$ on $\mathbb{R}$. Then

$$W_X^\rho(r)\xi = \widehat{\phi} * \xi$$

in distribution sense. From now on $W = W_X^\rho$.

Determining the Spec$A(H_1, W)$: the strategy

- The same approach as the case of $\mathbb{R}$ by using the Euclidean structure behind H_1. First we consider $\mathcal{A} = \mathcal{F}^{\mathbb{R}^3}(C_c^\infty(\mathbb{R}^3))$.
- If $(*)$ $\mathcal{A} \hookrightarrow A(H_1, W)$ continuously with dense range, then $u \in \mathrm{Spec}A(H_1, W)$ is uniquely determined by a point $(\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{C}^3 \cong (H_1)_{\mathbb{C}}$ using distributional Cauchy functional equation on $\mathbb{R}^3$.
- If $(**)$ $\mathcal{A}$ has enough elements allowing complex Fourier inversion of $\mathcal{F}^{H_1}$, then we have...
- **(Ghandehari/L./Samei/Spronk, in progress)** Let $u = u_{(\tilde{x}, \tilde{y}, \tilde{z})}$ is the character on $\mathcal{A}$ coming from $(\tilde{x}, \tilde{y}, \tilde{z}) \in (H_1)_{\mathbb{C}}$. Then u is bounded on $A(H_1, W_X^\rho)$ iff (1) $|\mathrm{Im}\tilde{x}| \leq \frac{1}{2\pi}\log\rho$ and (2) $\mathrm{Im}\tilde{y} = \mathrm{Im}\tilde{z} = 0$.
- The conditions $(*)$ is ok, but we were not able to check $(**)$. Instead we found an intermediate space that fills the gap!!

Entire vectors for unitary representations

- G: a simply connected solvable Lie group with the Lie alg. $\mathfrak{g}$.
  $G_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{C}}$: the complexifications.
  $exp : \mathfrak{g} \to G$ holomorphically exdends to $exp : \mathfrak{g}_{\mathbb{C}} \to G_{\mathbb{C}}$.
- (**Goodman '69**)
  $\pi : G \to B(\mathcal{H}(\pi))$: a strongly conti. unitary representation.
  $v \in \mathcal{H}(\pi)$ is called an **entire vector** for π if
  $G \to \mathcal{H}(\pi)$, $g \mapsto \pi(g)v$ extends holomorphically to $G_{\mathbb{C}}$ (i.e.
  $\mathfrak{g} \xrightarrow{\exp} G \xrightarrow{\pi} B(\mathcal{H}(\pi))$ extends holomorphically to $\mathfrak{g}_{\mathbb{C}}$).
  We call this extension by π_ω and $\mathcal{H}^\omega_\infty(\pi)$ refers to the linear space of all entire vectors for π.

Entire vectors for π^r

- **(A characterization of $\mathcal{H}^{\omega}_{\infty}(\pi^r)$, Goodman '69)**
  A function $f \in L^2(\mathbb{R})$ is an entire vector for π^r if and only if f extends to an entire function on $\mathbb{C}$ and satisfies

$$\sup_{|\mathrm{Im}\ z|<t} e^{t|z|}|f(z)| < \infty$$

  for any $t > 0$. Note that the above condition is independent of the parameter r. Moreover, the n-th Hermite functions φ_n are entire vectors for π^r.
- We use **the same formula for the holomorphic extension** π^r_ω due to the uniqueness of analytic continuation.

Entire vectors for λ on H_1

- **(A criterion for $\mathcal{H}^{\omega}_{\infty}(\lambda)$, Goodman '71)** $f \in L^2(H_1)$ is an entire vector for λ iff (a) Range$(\widehat{f}^{H_1}(r)) \subseteq \mathcal{H}^{\omega}_{\infty}(\pi^r_\omega)$ a.e. and (b) for any $M > 0$

$$\int_{\mathbb{R}\setminus\{0\}} \sup_{|\tilde{x}|,|\tilde{y}|,|\tilde{z}|<M} ||\pi^r_\omega(\tilde{x},\tilde{y},\tilde{z})\widehat{f}^{H_1}(r)||_2^2\,|r|dr < \infty,$$

  where the sup is taken over $(\tilde{x},\tilde{y},\tilde{z}) \in (H_1)_{\mathbb{C}} \cong \mathbb{C}^3$.
- **(Complex Fourier inversion, Goodman '71)**
  If $f \in \mathcal{H}^{\omega}_{\infty}(\lambda)$, then f has the analytic continuation f_ω to $(H_1)_{\mathbb{C}}$ given by the absolutely convergent integral

$$f_\omega(\tilde{x},\tilde{y},\tilde{z}) = \int_{\mathbb{R}\setminus\{0\}} \mathrm{Tr}(\pi^r(\tilde{x},\tilde{y},\tilde{z})\widehat{f}^{H_1}(r))|r|dr$$

  for $(\tilde{x},\tilde{y},\tilde{z}) \in (H_1)_{\mathbb{C}} \cong \mathbb{C}^3$.

Determining the Spec$A(H_1, W)$: the space $\mathcal{K}$ and $\mathcal{B}$

- We define

  $$\mathcal{K} := \{f : e^{t(|x|+|y|+|z|)}(\partial^\alpha f)(x,y,z) \in L^2(\mathbb{R}^3),\ \forall t > 0,\ |\alpha| \le 8\},$$

  where ∂^α refers the partial derivative in the weak sense with the multi-index α, and $\mathcal{B} := \mathcal{F}^{\mathbb{R}^3}(\mathcal{K})$.
- The space $\mathcal{K}$ is a Fréchet space with a canonical family of semi-norms and the embedding $C_c^\infty(\mathbb{R}^3) \hookrightarrow \mathcal{K}$ is continous with dense range. Moreover, $\mathcal{A} \subseteq \mathcal{B}(\subseteq A(G,W))$.
- The space $\mathcal{K}$ is somewhat unusual, but there is a similar space.

  $$\widetilde{\mathcal{K}} = \{g : e^{t|x|}g(x) \in L^2(\mathbb{R}) \text{ for any } t > 0\}.$$

  By a thm of Paley/Wiener $g \in \widetilde{\mathcal{K}}$ iff $\widehat{g}^{\mathbb{R}}$ extends entirely and satisfies $\sup_{|y|\le t} \int_{\mathbb{R}} |\widehat{g}^{\mathbb{R}}(x+iy)|^2 dx < \infty$ for any $t > 0$. Note that the n-th Hermite functions φ_n belong to $\widetilde{\mathcal{K}}$.

Why the space $\mathcal{K}$ and $\mathcal{B}$?

- First, the term $e^{t(|x|+|y|+|z|)}$ allows us to **absorb** the exponential functions coming from the weight W_X^{ρ}.
- Indeed, $\mathcal{F}^G(\widehat{f}^{\mathbb{R}^3})(r)$, $r \neq 0$ is an integral operator with the kernel $K(w,x) = \widehat{f}_1^{\mathbb{R}}(w-x,-rw,r)$, where $\widehat{f}_1^{\mathbb{R}}$ is taking Fourier transform on the 1st variable.
- Moreover $W\mathcal{F}^G(\widehat{f}^{\mathbb{R}^3})(r) = \mathcal{F}^G(\widehat{g}^{\mathbb{R}^3})(r)$ with

$$g(s,t,u) = \rho^{|s|} f(s,t,u).$$

It is clear to see that $g \in \mathcal{K}$.

Why the space $\mathcal{K}$ and $\mathcal{B}$?: continued

- Secondly, $\mathcal{K}$ is big enough to include "nice" functions.
- Let P_{mn} be the rank 1 operator s.t. $P_{mn}\xi = \langle \xi, \varphi_m \rangle \varphi_n$. Then $\{h \otimes P_{mn} : h \in C_c^\infty(\mathbb{R}\backslash\{0\})\}$ is dense in $A(H_1)$.
- Now we consider the function f satisfying

$$K(w,x) = \widehat{f_1}^{\mathbb{R}}(w-x, -rw, r) = \varphi_m(w)\varphi_n(x)h(r)$$

for a fixed $h \in C_c^\infty(\mathbb{R}\backslash\{0\})$ and $m, n \geq 0$. Then f is actually given by

$$f(x,y,z) = i^n e^{2\pi i \frac{xy}{z}} \varphi_m(-\frac{y}{z})\varphi_n(-x)h(z),\ z \neq 0.$$

Then we readily check that $f \in \mathcal{K}$.

- Note that the above $f \notin C_c^\infty(\mathbb{R}^3)$.

Determining the Spec$A(H_1, W)$: continued

- $(*)$ The space $\mathcal{B}$ is a subspace of $A(H_1, W)$ and the map
$$\mathcal{F}^{\mathbb{R}^3} : \mathcal{K}(\mathbb{R}^3) \to A(H_1, W)$$
is continuous.
- The proof of the above depends on the following Fourier algebra norm estimate.
- (**Geller '77**, modified) There is a constant $C > 0$ and linear partial differential operators L_k of order $\leq 2k$ with polynomial coefficients of degree $\leq 2k + 2$ such that
$$||\mathcal{F}^{H_1}(F)||_{A(H_1)} \leq C \sum_{k=0}^{3} ||L_k F||_{L^2(G)}.$$
- Geller actually uses the sublaplacian $\mathcal{L}$ on H_1 and its power $\mathcal{L}^k$, $0 \leq k \leq 3$ with L^1-norm estimate , but we can easily transfer it to L^2-estimate by putting additional weight.

Determining the Spec$A(H_1, W)$: continued 2

- $(**)$ The space $\mathcal{B} \cap \mathcal{H}^{\omega}_{\infty}(\lambda)$ is a dense subspace of $A(H_1, W)$.
- The proof of the above depends on the fact that $h \otimes P_{mn}$, $h \in C_c^{\infty}(\mathbb{R}\backslash\{0\})$ corresponds to a function in $\mathcal{K} \cap \mathcal{H}^{\omega}_{\infty}(\lambda)$ by the criterion of Goodman.

Determining the Spec$A(H_1, W)$: checking norm conditions

- ($\Leftarrow$) We use complex Fourier inversion

$$f_\omega(\tilde{x}, \tilde{y}, \tilde{z}) = \int_{\mathbb{R}\setminus\{0\}} \mathrm{Tr}(\pi^r(\tilde{x}, \tilde{y}, \tilde{z})\widehat{f}^{H_1}(r))|r|dr$$

  and check the uniform boundedness of the operators $\pi^r_\omega(\tilde{x}, \tilde{y}, \tilde{z})W^{-1}(r)$ directly.
- ($\Rightarrow$) Use gaussian functions!

Other non-compact Lie groups

- The case of the Euclidean motion group $E(2)$ can be done similarly, but easier!
- The case of $ax + b$ group is still open due to the absence of enough elements allowing complex Fourier inversions, i.e. entire vectors for λ.