Locally compact quantum groups

3. Further aspects of Compact Quantum Groups

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Fields, May 2014
CQGs: Recap

- Unital $C^*$-algebra $A$ with coproduct $\Delta$, satisfying “cancellation”:
  \[
  \overline{\text{lin}}\{(a \otimes 1)\Delta(b) : a, b \in A\} = \overline{\text{lin}}\{(1 \otimes a)\Delta(b) : a, b \in A\} = A \otimes A.
  \]

- There exists an invariant Haar state $\varphi$ with GNS $(L^2(\mathbb{G}), \pi_\varphi, \xi_\varphi)$.

- Formed “left-regular corepresentation” $U \in M(A \otimes B_0(L^2(\mathbb{G})))$:
  \[
  U^*(\xi \otimes \pi_\varphi(a)\xi_\varphi) = (\pi \otimes \pi_\varphi)(\Delta(a))(\xi \otimes \xi_\varphi)
  \]

- Studied category of corepresentations.

- $U$ decomposes as direct sum of all the irreducibles.

- $A_0 \subseteq A$ algebra of matrix coefficients.
Is $A_0$ a $\ast$-algebra?

- Typical element $V_{ij} \in A_0$; so is $V_{ij}^* \in A_0$?
- Motivates looking at $\overline{V} := (V_{ij}^*)$. Still a corepresentation:

$$\Delta(V_{ij}^*) = \Delta(V_{ij})^* = \left( \sum_k V_{ik} \otimes V_{kj} \right)^* = \sum_k V_{ik}^* \otimes V_{kj}^*.$$ 

**Theorem**

Let $V$ be an irreducible corepresentation. Then $\overline{V}$ is equivalent to a unitary corepresentation. In particular, $V_{ij}^* \in A_0$.

**Proof.**

Show that $\overline{V}$ is a sub-corepresentation of $U$. Same game: choose $x \in B(L^2(\mathbb{G}), H_V)$ and set

$$y = (\varphi \otimes \text{id})(\overline{V}^*(1 \otimes x)U),$$

argue that if $y \neq 0$ then $y^*$ implements an isomorphism; if $y = 0$ for all $x$ then derive contradiction.
Let $\text{Irr}(G)$ be the collection of equivalence classes of irreducible representations of $(A, \Delta)$. Choose representatives $u^\alpha$.

**Theorem**

For each $\alpha$ there is a positive, invertible, trace 1 matrix $F^\alpha$ with

$$\varphi((u^\beta_i) u^\alpha_{jq}) = \begin{cases} F^\alpha_{ji} & : \alpha = \beta, p = q, \\ 0 & : \text{otherwise}. \end{cases}$$

**Sketch proof.**

We apply our averaging argument to $x = e_{ij}$ a matrix unit:

$$y = (\varphi \otimes \text{id})((u^\beta_i) (1 \otimes x) u^\alpha) = \cdots = \sum_{p,q} \varphi((u^\beta_i) u^\alpha_{jq}) e_{pq}.$$  

Then $y$ intertwines $u^\alpha, u^\beta$ so is 0 if $\alpha \neq \beta$; otherwise $y = F^\alpha_{ji} 1$. Then . . .
Application: A basis

\[ \varphi((u_{ip}^\beta) * u_{jq}^\alpha) = \delta_{\alpha,\beta} \delta_{p,q} F_{ji}^\alpha. \]

**Theorem**

The set \( \{ u_{ij}^\alpha : \alpha \in \text{Irr}(\mathbb{G}), 1 \leq i, j \leq n_\alpha \} \) is a basis for \( A_0 \).

**Proof.**

By definition this spans \( A_0 \). If \( \sum t_{ij}^\alpha u_{ij}^\alpha = 0 \) for some scalars \( (t_{ij}^\alpha) \) then for any \( \beta, p, q, \)

\[ 0 = \sum_{\alpha, i, j} t_{ij}^\alpha \varphi((u_{pq}^\beta) * u_{ij}^\alpha) = \sum_i F_{ip}^\beta t_{iq}^\beta. \]

As \( F^\beta \) is invertible, this implies that \( t_{iq}^\beta = 0 \) for all \( i, q, \beta, \) as required. \( \square \)
A Hopf $\ast$-algebra

We define $\epsilon : A_0 \to \mathbb{C}$ and $S : A_0 \to A_0$ by

$$\epsilon(u_{ij}^\alpha) = \delta_{i,j}, \quad S(u_{ij}^\alpha) = (u_{ji}^\alpha)^*.$$ 

Or equivalently, for any (finite-dimensional) unitary corepresentation $V$,

$$(S \otimes \text{id})(V) = V^*, \quad (\epsilon \otimes \text{id})(V) = 1.$$ 

Theorem

Then $(A_0, \Delta, \epsilon, S)$ is a Hopf $\ast$-algebra.

This gives a purely algebraic approach to compact quantum groups: the Hopf $\ast$-algebras which can arise are exactly those which are spanned by matrix coefficients of unitary corepresentations.
What happens in the commutative case?

$V$ corresponds to a unitary group representation $\pi : G \rightarrow \mathbb{M}_n$:

$$V \in C(G) \otimes \mathbb{M}_n \cong C(G, \mathbb{M}_n), \quad V = (\pi(s))_{s \in G}.$$  

$$(\text{id} \otimes \omega_{\xi, \eta})(V) = ((\pi(s)\xi|\eta))_{s \in G} \in C(G),$$

$$(\text{id} \otimes \omega_{\xi, \eta})(V^*) = ((\pi(s^{-1})\xi|\eta))_{s \in G} \in C(G).$$

Such continuous functions are linearly dense in $C(G)$.

$$(\epsilon \otimes \text{id})(V) = I \Leftrightarrow \langle \epsilon, (\pi(s)\xi|\eta)_{s \in G} \rangle = \langle \xi|\eta \rangle$$

so we conclude that $\epsilon \in C(G)^*$ is the functional: “evaluate at the group identity”.

$$(S \otimes \text{id})(V) = V^* \Leftrightarrow S((\pi(s)\xi|\eta)_{s \in G}) = (\pi(s^{-1})\xi|\eta)_{s \in G}$$

so $S : C(G) \rightarrow C(G)$ is the $*$-homomorphism induced by the group inverse. In general $\epsilon$ and $S$ are unbounded.
Characters

Theorem

\[ \varphi(u_{ip}^\alpha(u_{jq}^\beta)*) = \delta_{\alpha,\beta} \delta_{i,j} \frac{(F^\alpha)_{qp}^{-1}}{\text{Tr}((F^\alpha)^{-1})}. \]

Set \( t_\alpha = \text{Tr}((F^\alpha)^{-1}) > 0 \) and define a linear map by

\[ f_z : A_0 \rightarrow \mathbb{C}; \quad u_{ij}^\alpha \mapsto ((F^\alpha)^{-z})_{ij} t_\alpha^{-z/2}. \]

Turn \( A_0^* \) into an algebra via \( \langle \mu \ast \lambda, a \rangle = \langle \mu \otimes \lambda, \Delta(a) \rangle \).

Theorem

Each \( f_z \) is a character on \( A_0 \), \( f_0 = \epsilon \), \( f_z(a^*) = f_z(a)^* \) and \( f_z \ast f_w = f_{z+w} \). If we define

\[ \sigma(a) = f_1 \ast a \ast f_1 := (f_1 \otimes \text{id} \otimes f_1) \Delta^2(a) \quad (a \in A_0), \]

then \( \varphi(ab) = \varphi(b \sigma(a)) \). (Note: \( \Delta^2 = (\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta \).

\( \varphi \) is not a trace but it nearly is.
Properties of Haar state on $A$

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<th>Theorem</th>
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<td>( \varphi ) is “faithful” on ( A_0 ) (( \varphi(a^*a) = 0 \implies a = 0 )).</td>
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<th>Proof.</th>
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<td>If ( \varphi(a^*a) = 0 ) then ( \varphi(a^*b) = 0 ) for all ( b \in A_0 ) (Cauchy-Schwarz). Set ( b = u_{pq}^\beta ) and use an F-matrix argument again.</td>
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<td>For ( a \in A ), ( \varphi(a^<em>a) = 0 \iff \varphi(aa^</em>) = 0 ).</td>
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| • Cauchy-Schwarz \( \implies \varphi(a^*b) = 0 \) for all \( b \in A \).  
• Find \( (a_n) \subseteq A_0 \) converging to \( a \) in norm.  
• Recall automorphism \( \sigma \); then \( 0 = \lim_n \varphi(a_n^*\sigma(b)) = \lim_n \varphi(ba_n^*) = \varphi(ba^*) \). |
Further conclusions

**Theorem**

\[ N_\varphi = \{ a \in A : \varphi(a^*a) = 0 \} \text{ is a two-sided ideal in } A. \text{ If} \]
\[ \Lambda : A \to L^2(G); a \mapsto \pi_\varphi(a) \xi_\varphi \text{ is the GNS map, then ker } \Lambda = \ker \pi_\varphi = \ker \varphi = N_\varphi. \]

**Proof.**

- **Standard C*-theory:** \( N_\varphi \) is a left ideal.
- Previous theorem shows \( N_\varphi \) self-adjoint, so an ideal.
- Cauchy-Schwarz shows \( \ker \varphi = \ker N_\varphi \) (\( A \) is unital!)
- By definition \( \ker \Lambda = N_\varphi \) and \( \ker \pi_\varphi \subseteq \ker \Lambda \)
- \( a \in N_\varphi \implies b^*a \in N_\varphi \implies a^*b \in N_\varphi \implies \pi_\varphi(a^*) = 0 \implies \pi_\varphi(a) = 0. \)
- \( \varphi \) really “looks like” it is a trace!
“Reduced” $C^*$-algebras

\[ \ker \Lambda = \ker \pi_\varphi = \ker \varphi = N_\varphi. \]

Let $C(G) = A/N_\varphi$ a $C^*$-algebra; $\varphi$ drops to $C(G)$ and is faithful.

**Theorem**

The GNS space for $\varphi$ on $C(G)$ is isomorphic to $L^2(G)$, and $C(G) \cong \pi_\varphi(A)$.

There is a unital $\ast$-homomorphism $\Delta : C(G) \rightarrow C(G) \otimes C(G)$ turning $C(G)$ into a compact quantum group.

**Proof.**

Form the left-regular representation, but this time use $\pi = \pi_\varphi$ to get $W \in M(\pi_\varphi(A) \otimes B_0(L^2(G))) = M(C(G) \otimes B_0(L^2(G)))$ with

\[ W^*(1 \otimes \pi_\varphi(a))W = (\pi_\varphi \otimes \pi_\varphi)\Delta(a) \quad (a \in A). \]

So define $\Delta$ on $C(G)$ by $\Delta(x) = W^*(1 \otimes x)W$. Density of $A_0$ in $C(G)$ shows that $\Delta$ does map to $C(G) \otimes C(G)$; similarly cancellation holds for $C(G)$. \qed
von Neumann algebra

Let $L^\infty(G) = C(G)''$ in $\mathcal{B}(L^2(G))$. Again define

$$\Delta(x) = W^*(1 \otimes x)W \quad (x \in L^\infty(G)),$$

which by weak*-continuity maps into $L^\infty(G) \bar{\otimes} L^\infty(G)$.

**Theorem**

*The normal extension of $\varphi$ to $L^\infty(G)$ is faithful.*

**Proof.**

- Let $\varphi(x^*x) = 0$ so $x\varphi_\xi = 0$.
- Kaplansky Density: bounded net $(a_i)$ in $C(G)$ with converges strongly to $x$. For $b, c \in A_0$,

$$\langle x\sigma(b)\xi_\varphi | c\xi_\varphi \rangle = \lim_i \varphi(c^*a_i\sigma(b)) = \lim_i \varphi(bc^*a_i) = \lim_i \langle a_i\xi_\varphi | cb^*\xi_\varphi \rangle$$

$$= \langle x\xi_\varphi | cb^*\xi_\varphi \rangle = 0.$$

- Density: $(x\xi|\eta) = 0$ for $\xi, \eta \in L^2(G)$, so $x = 0$. 


Discussion of amenability and $C^*(\Gamma)$

Let $\Gamma$ be a discrete group, so $\hat{\Gamma} := C_r^*(\Gamma)$ is a compact quantum group, 
$\Delta(\lambda(s)) = \lambda(s) \otimes \lambda(s)$

$$\varphi(\lambda(s)) = \delta_{s,e} \implies L^2(\hat{\Gamma}) = l^2(\Gamma).$$

- Could also work with $C^*(\Gamma)$
- Existence of $\Delta$ follows from universal property, as $s \mapsto \lambda(s) \otimes \lambda(s)$ is a unitary representation.
- $\varphi$ is now faithful if and only if $\Gamma$ is amenable.
- $C_r^*(\Gamma) = C^*(\Gamma)$ if and only if $\Gamma$ is amenable.
- $A_0 = \mathbb{C}[\Gamma]$ and $\epsilon : \lambda(s) \mapsto 1$ is bounded on $C^*(\Gamma)$.
- $\epsilon$ bounded on $C_r^*(\Gamma)$ if and only if $\Gamma$ is amenable.
Duality

As $\Delta(\cdot) = W^*(1 \otimes \cdot)W$ and $(\Delta \otimes \text{id})(W) = W_{13}W_{23}$,

$$W_{12}^* W_{23} W_{12} = W_{13} W_{23} \implies W_{23} W_{12} = W_{12} W_{13} W_{23}.$$ 

- This says that $W$ is multiplicative.
- See Baaj–Skandalis, Woronowicz and Sołtan–Woronowicz.
- $\hat{W} := \sigma W^* \sigma$ is also multiplicative.

$$c_0(\hat{G}) = \{(\omega \otimes \text{id})(W)\} \|\cdot\| = \{(\text{id} \otimes \omega)(\hat{W})\} \|\cdot\|, \quad \ell^\infty(\hat{G}) = c_0(\hat{G})''$$

are a $C^*$-algebra and a von Neumann algebra with a coproduct

$$\hat{\Delta}(x) = \hat{W}^*(1 \otimes x)\hat{W} \quad (x \in c_0(G), \ell^\infty(G)).$$

But here $\hat{\Delta} : c_0(\hat{G}) \to M(c_0(\hat{G}) \otimes c_0(\hat{G}))$ is a morphism.

$$W \in L^\infty(G) \otimes \ell^\infty(\hat{G}) \quad W \in M(C(G) \otimes c_0(\hat{G})).$$
Identifying $c_0(\hat{G})$

$$\varphi((u_{\beta}^\alpha)\ast u_{\beta}^\alpha) = \delta_{\alpha,\beta}\delta_{p,q} F_{ji} \quad \implies \quad (u_{jq}^\alpha \xi_\varphi \mid u_{ip}^\beta \xi_\varphi) = \delta_{\alpha,\beta}\delta_{p,q} F_{ji}. $$

- For fixed $\alpha$, $\text{lin}\{u_{jq}^\alpha \xi_\varphi\}$ is isomorphic to $\mathbb{C}^{n_\alpha} \otimes \mathbb{C}^{n_\alpha}$.
- So $L^2(G) \cong \bigoplus_{\alpha \in \text{Irr}(G)} \mathbb{C}^{n_\alpha} \otimes \mathbb{C}^{n_\alpha}$.
- Under this isomorphism,

$$W = \sum_{\alpha} \sum_{i,j} u_{ij}^\alpha \otimes e_{ij}^\alpha$$

where $e_{ij}^\alpha \in M_{n_\alpha}$ acts on the (e.g.) first variable of $\mathbb{C}^{n_\alpha} \otimes \mathbb{C}^{n_\alpha}$.

- Now easy to see that $c_0(\hat{G}) = \{(\omega \otimes \text{id})(W)\} \parallel \parallel$ is isomorphic to $\bigoplus_{\alpha} M_{n_\alpha}$. 
- So as an algebra $c_0(\hat{G})$ is easy; but $\hat{\Delta}$ is complicated (essentially encodes how $u^\alpha \bigoplus u^\beta$ is written as irreducibles.)
Discrete/Compact duality

- $\hat{G}$ is a discrete quantum group. (van Daele: axiomatisation not in terms of compact $G$.)

- There are weights $\hat{\varphi}, \hat{\psi}$ on $\ell^\infty(\hat{G})$

  $$(\text{id} \otimes \hat{\varphi})\hat{\Delta}(x) = \hat{\varphi}(x)1, \quad (\hat{\psi} \otimes \text{id})\hat{\Delta}(x) = \hat{\psi}(x)1.$$ 

- For $x = (x^\alpha) \in \ell^\infty(\hat{G}) = \prod_\alpha \mathbb{M}_{n_\alpha}$,

  $$\hat{\varphi}(x) = \sum_\alpha \Lambda_\alpha^2 \text{Tr}_\alpha(F^\alpha x^\alpha)$$

  where $\Lambda_\alpha^2 = \text{Tr}((F^\alpha)^{-1})$.

- Tomita-Takesaki theory: $\hat{\nabla}$ on $L^2(G)$ implements the modular automorphism group $\hat{\sigma}_t(x) = \hat{\nabla}^{-it} x \hat{\nabla}^{it}$ and conjugation $\ell^\infty(\hat{G}) \to \ell^\infty(\hat{G})'; x \mapsto \hat{J}x^*\hat{J}$.

  (Generalises modular function on $G$ and behaviour of $\text{VN}(G)$).
The map \( x \mapsto \hat{\nabla}^{-it} x \hat{\nabla}^{it} \) also maps \( \mathbb{C}(G) \) into itself, and implements a continuous automorphism group \( (\tau_t) \), the \textit{scaling group}.

On \( A_0 \) we can express this using the characters \( f_{it} \).

Recall the antipode

\[
S((\text{id} \otimes \omega)(W)) = (\text{id} \otimes \omega)(W^*).
\]

Define \( R(x) = \hat{J}x^* \hat{J} \) for \( x \in \mathbb{C}(G) \), which also maps \( \mathbb{C}(G) \) into itself. An anti-*-homomorphism which commutes with \( (\tau_t) \).

We get an (unbounded) analytic extension \( \tau_{-i/2} \) and \( S = R\tau_{-i/2} \).

\( R = S \) iff \( \tau_t = \text{id} \) iff \( \hat{\varphi} = \hat{\psi} \) iff \( \varphi \) is tracial iff \( G \) is a Kac algebra.
Examples/Buzzwords

- Deformations of compact Lie groups: $SU_q(2)$ (Woronowicz). Non-Kac type.
- Quantum permutation groups $S_n^+$ and quantum orthogonal groups $O_n^+$ (Wang).
- “Universal quantum groups”. (Wang, van Daele).
- Liberation of quantum groups; Easy quantum Groups $S_n \subseteq \mathbb{G} \subseteq O_n^+$ (Banica, Speicher).
- Easy quantum groups now well classified (e.g. Curran, Weber, Raum, Freslon).
- Key tool is to study the representation category $\text{Irr} (\mathbb{G})$ and Woronowicz’s generalisation of Tannaka-Krein duality.
- Mostly of Kac type: $L^\infty (\mathbb{G})$ finite von Neumann algebra, lots of work on von Neumann algebra properties of $L^\infty (\mathbb{G})$. (e.g. Brannan, Freslon).
- Next time: what can we say for $L^1 (\mathbb{G})$?
Time allowing: $S_n^+$

Let $(a_{ij})_{i,j=1}^n$ be a matrix of functions on some space $X$ with:

- $a_{ij} = a_{ij}^* = a_{ij}^2$ (so $a_{ij}$ is 0, 1-valued);
- for all $i$, $\sum_j a_{ij} = 1$ and for all $j$, $\sum_i a_{ij} = 1$ (so at each point of $X$, if we evaluate, we get a permutation matrix).

The maximal commutative $C^*$-algebra generated by such matrices is just the collection of all permutation matrices, i.e. $C(S_n)$.

- Let $C(S_n^+)$ be the non-commutative $C^*$-algebra generated by such matrices.
- Universal property: if $A$ any $C^*$-algebra and $\hat{a}_{ij} \in A$ elements with the relations, there is a unique $\ast$-homomorphism $\theta : C(S_n^+) \to A$ with $\theta(a_{ij}) = \hat{a}_{ij}$.
- Apply with $A = C(S_n^+) \otimes C(S_n^+)$ and $\hat{a}_{ij} = \sum_k a_{ik} \otimes a_{kj}$.
- Gives $\Delta : A \to A \otimes A$ coproduct.
- Can manually check the cancellation conditions.