Hypergroups

and their amenability notions

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May 30, 2014
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Amenability of Hypergroup algebra
A locally compact space $H$ is a hypergroup if there exists a convolution $\star : M(H) \times M(H) \rightarrow M(H)$ such that:

$\forall x, y \in H$, $\delta_x \star \delta_y$ is a positive measure with compact support and $\|\delta_x \star \delta_y\|_{M(H)} = 1$. 
HYPERGROUPS

A locally compact space $H$ is a hypergroup if
$\exists * : M(H) \times M(H) \rightarrow M(H)$ called convolution:

- $\forall x, y \in H, \delta_x \ast \delta_y$ is a positive measure with compact support and $\|\delta_x \ast \delta_y\|_{M(H)} = 1$.
- $(x, y) \mapsto \delta_x \ast \delta_y$ is a continuous map from $H \times H$ into $M(H)$ equipped with the weak* topology.
- $(x, y) \rightarrow \text{supp}(\delta_x \ast \delta_y)$ is a continuous mapping from $H \times H$ into $\mathcal{K}(H)$ equipped with the Michael topology.
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- $\exists e \in H, \delta_e$ is the identity of $M(H)$. 
**Hypergroups**

A locally compact space $H$ is a hypergroup if there exists a convolution $\ast : M(H) \times M(H) \to M(H)$ such that:

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- $(x, y) \mapsto \text{supp}(\delta_x \ast \delta_y)$ is a continuous mapping from $H \times H$ into $\mathcal{K}(H)$ equipped with the Michael topology.

- $\exists e \in H$, $\delta_e$ is the identity of $M(H)$.
- $\exists$ a homeomorphism $x \to \check{x}$ of $H$ called involution such that $(\delta_x \ast \delta_y) = \delta_y \ast \delta_{\check{x}}$.
- $e \in \text{supp}(\delta_x \ast \delta_y)$ if and only if $y = \check{x}$. 
HAAR MEASURE

Let $f \in C_c(H)$,

$$L_x f(y) = \delta_{\tilde{x}} * \delta_y(f) =: f(\delta_{\tilde{x}} * \delta_y).$$

A positive non-zero Borel measure $h$ is called a Haar measure if

$$h(L_x f) = h(f), \quad \forall f \in C_c(H), x \in H.$$

For a commutative and/or compact and/or discrete hypergroup, the existence of a Haar measure can be proven.
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For a commutative and/or compact and/or discrete hypergroup, the existence of a Haar measure can be proven.

For $A, B \subseteq H, A \ast B \subseteq H$ where

$$A \ast B := \bigcup_{x \in A, y \in B} \text{supp}(\delta_x \ast \delta_y).$$
Hypergroup algebra

For every $f, g \in L^1(H, h)$,

$$f * g = \int_H f(y) L_y g \, dh(y), \quad f^*(y) = \overline{f(\tilde{y})}.$$  

$L^1(H)(= L^1(H, h))$ forms a *-algebra called hypergroup algebra.
**Fourier Space of Hypergroups**

[Muruganandam, 07] defined Fourier Stieltjes space of hypergroups, similar to group case, and consequently Fourier space of $H$, $A(H)$.

$$A(H)^* = VN(H) = \lambda(L^1(H))'' \subseteq \mathcal{B}(L^2(H)).$$

---

**Proposition.** [A.’14]

For a hypergroup $H$,

$$A(H) := \{ f \ast \tilde{g} : f, g \in L^2(H) \}.$$

And $\|u\|_{A(H)} = \inf\{ \|f\|_2 \|g\|_2 \}$ for all $f, g \in L^2(H)$ s.t. $u = f \ast \tilde{g}$.
Fourier Space of Hypergroups

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The hypergroup $H$ is called **regular Fourier hypergroup** if $A(H)$ is a Banach algebra with respect to pointwise multiplication.
COMMUTATIVE HYPERGROUPS

Let $H$ be a **commutative** hypergroup,

$$\hat{H} := \{ \alpha \in C_b(H) : \alpha(\delta_x \ast \delta_y) = \alpha(x)\alpha(y), \alpha(\bar{x}) = \overline{\alpha(x)}, \text{ and } \alpha \neq 0 \}. $$

$\hat{H}$ is the **Gelfand spectrum** of $L^1(H)$. $\hat{H}$ is called the **dual** of $H$.

$\hat{H}$ is **not** necessarily a hypergroup any more!
COMMUTATIVE HYPERGROUPS

Let $H$ be a commutative hypergroup,

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$\hat{H}$ is not necessarily a hypergroup any more!

Fourier-Stieltjes transform and Fourier transform defined:

$$\mathcal{F} : M(H) \rightarrow C_b(H) \text{ where } \mathcal{F}(\mu)(\alpha) := \int_H \overline{\alpha(x)}d\mu(x).$$

$$\mathcal{F} : L^1(H) \rightarrow C_0(H) \text{ where } \mathcal{F}(f)(\alpha) := \int_H f(x)\overline{\alpha(x)}dh(x)$$
Theorem.
Let $H$ be a **commutative hypergroup**. Then there exists a **non-negative measure** $\pi$ on $\hat{H}$, called **Plancherel measure** of $\hat{H}$ such that

$$
\int_{H} |f(x)|^2 dx = \int_{\hat{H}} |\hat{f}(\alpha)|^2 d\pi(\alpha)
$$

for all $f \in L^1(H) \cap L^2(H)$. 
**Plancherel Measure**

Theorem. Let $H$ be a commutative hypergroup. Then there exists a non-negative measure $\pi$ on $\hat{H}$, called Plancherel measure of $\hat{H}$ such that

$$\int_{H} |f(x)|^2 dx = \int_{\hat{H}} |\hat{f}(\alpha)|^2 d\pi(\alpha)$$

for all $f \in L^1(H) \cap L^2(H)$.

Note that for an arbitrary hypergroup $H$ (unlike group case) the support of the Plancherel measure,

$$\text{supp}(\pi) \neq \hat{H}.$$
**Example 0.**

**Locally compact groups**

Every **locally compact group** $G$, it is a regular Fourier hypergroup.
Example 1. Representations of compact groups

Let $G$ be a compact (quantum) group and $\hat{G}$ the set of all irreducible unitary (co-)representations of $G$.

For each $\pi_1, \pi_2 \in \hat{G}$, $\pi_1 \otimes \pi_2 \cong \sigma_1 \oplus \cdots \oplus \sigma_n$ for $\sigma_1, \cdots, \sigma_n \in \hat{G}$.

Define a convolution on $\ell^1(\hat{G})$ and make $\hat{G}$ into a commutative discrete hypergroup which is called the fusion algebra of $G$. 
**Example 1.**

**Representations of compact groups**

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Define a convolution on $\ell^1(\hat{G})$ and make $\hat{G}$ into a commutative discrete hypergroup which is called the **fusion algebra** of $G$.

$\ell^1(\hat{G})$ is isometrically isomorphic to

$ZA(G) = \{ f \in A(G) : f(yxy^{-1}) = f(y) \text{ for all } x, y \in G \}$.

[A. '13]:

$\hat{G}$ is a regular Fourier hypergroup and $A(\hat{G}) \cong ZL^1(G)$. 
**Example 2.**

**Conjugacy class of $[FC]^B$ groups**

The space of all orbits in a locally compact group $G$ for some relatively compact subgroup $B$ of automorphisms of $G$ including inner ones denoted by $\text{Conj}_B(G)$.

$\text{Conj}_B(G)$ forms a **commutative hypergroup**. $L^1(\text{Conj}_B(G))$ is isometrically isomorphic to $Z_B L^1(G) = \{f \in L^1(G) : f \circ \beta = f \text{ for all } \beta \in B\}$. 
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Conj$_B(G)$ is a **regular Fourier hypergroup**. (Muruganandam ’07)
**Example 2.**

**Conjugacy class of \([\text{FC}]^B\) groups**

The space of all orbits in a locally compact group \(G\) for some relatively compact subgroup \(B\) of automorphisms of \(G\) including inner ones denoted by \(\text{Conj}_B(G)\).

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\(\text{Conj}_B(G)\) is a **regular Fourier hypergroup**. (Muruganandam '07)

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When \(B\) is the set of all inner automorphisms, we use \(\text{Conj}(G)\). Then \(A(\text{Conj}(G)) \cong ZA(G)\).

- \(\text{Conj}(G)\) is a compact hypergroup if \(G\) is compact.
- \(\text{Conj}(G)\) is a discrete hypergroup if \(G\) is discrete.
**EXAMPLE 3.**

**Double coset hypergroups**

Let $G$ be a locally compact group and $K$ be a compact subgroup of $G$.

$$G//K := \{KxK : x \in G\}.$$

forms a hypergroup.

$$L^1(G//K) \cong \{f \in L^1(G) : f \text{ is constant on double cosets of } K\}.$$
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**Double coset hypergroups**

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[Muruganandam ’08]: $G//K$ is a regular Fourier hypergroup and

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Example 4.

Polynomial Hypergroups

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $(a_n)_{n \in \mathbb{N}_0}$ and $(c_n)_{n \in \mathbb{N}_0}$ be sequences of non-zero real numbers and $(b_n)_{n \in \mathbb{N}_0}$ be a sequence of real numbers with the property

\begin{align*}
a_0 + b_0 &= 1 \\
a_n + b_n + c_n &= 1, \quad n \geq 1.
\end{align*}

If $(R_n)_{n \in \mathbb{N}_0}$ is a sequence of polynomials defined by

\begin{align*}
R_0(x) &= 1, \\
R_1(x) &= \frac{1}{a_0} (x - b_0), \\
R_1(x)R_n(x) &= a_nR_{n+1}(x) + b_nR_n(x) + c_nR_{n-1}(x), \quad n \geq 1,
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**Example 4.**

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$$a_0 + b_0 = 1$$
$$a_n + b_n + c_n = 1, \quad n \geq 1.$$

If $(R_n)_{n \in \mathbb{N}_0}$ is a sequence of polynomials defined by

$$R_0(x) = 1,$$
$$R_1(x) = \frac{1}{a_0} (x - b_0),$$
$$R_1(x)R_n(x) = a_nR_{n+1}(x) + b_nR_n(x) + c_nR_{n-1}(x), \quad n \geq 1,$$

Then,

$$R_n(x)R_m(x) = \sum_{k=|n-m|}^{n+m} g(n, m; k)R_k(x)$$

where $g(n, m; k) \in \mathbb{R}^+$ for all $|n - m| \leq k \leq n + m$. 
Example 4. Polynomial hypergroups

Define $*$ on $\ell^1(\mathbb{N}_0)$ such that

$$
\delta_n \ast \delta_m = \sum_{k=|n-m|}^{n+m} g(n, m; k) \delta_k
$$

and $\check{n} = n$.

Then $(\mathbb{N}_0, *, \check{\cdot})$ is a discrete commutative hypergroup with the unit element 0 which is called the polynomial hypergroup on $\mathbb{N}_0$ induced by $(R_n)_{n \in \mathbb{N}_0}$. 
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LEFT INvariant MEAN

[Skantharajah ’92]:

A linear functional $m \in L^\infty(H)^*$ is called a mean if it has norm 1 and is non-negative, i.e. $f \geq 0$ a.e. implies $m(f) \geq 0$.

$m$ is called left invariant mean if $m(L_x f) = m(f)$. 
**LEFT INVARİANT MEAN**

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A linear functional $m \in L^\infty(H)^*$ is called a **mean** if it has norm 1 and is non-negative, i.e. $f \geq 0$ a.e. implies $m(f) \geq 0$.

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A hypergroup $H$ is called **amenable** if it has a **left invariant mean**.
A linear functional $m \in L^\infty(H)^*$ is called a **mean** if it has **norm 1** and is **non-negative**, i.e. $f \geq 0$ a.e. implies $m(f) \geq 0$.

$m$ is called **left invariant mean** if $m(L_x f) = m(f)$.

A hypergroup $H$ is called **amenable** if it has a **left invariant mean**.

**Theorem.**
Every **commutative** and/or **compact** hypergroup is amenable.
REITER’S CONDITIONS

[Skantharajah ’92]:

\[ H \text{ satisfies } (P_r), 1 \leq r < \infty, \text{ if whenever } \epsilon > 0 \text{ and a compact set } E \subseteq H \text{ are given, then there exists } f \in L'\{H\}, f \geq 0, \|f\|_r = 1 \text{ such that } \]

\[ \|L_x f - f\|_r < \epsilon \quad (x \in E). \]
**REITER’S CONDITIONS**

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H satisfies \((P_r)\), \(1 \leq r < \infty\), if whenever \(\epsilon > 0\) and a compact set \(E \subseteq H\) are given, then there exists \(f \in L^r(H), f \geq 0, \|f\|_r = 1\) such that

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[Skantharajah ’92]:

Amenablity \(\Leftrightarrow (P_1) \Leftrightarrow (P_2) \Leftrightarrow (P_r)\) \(1 < r < \infty\)
**Reiter’s Conditions**

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$H$ satisfies $(P_r)$, $1 \leq r < \infty$, if whenever $\epsilon > 0$ and a compact set $E \subseteq H$ are given, then there exists $f \in L^r(H), f \geq 0, \|f\|_r = 1$ such that

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[Skantharajah '92]:

Amenablity $\iff (P_1) \iff (P_2) \iff (P_r)_{1 < r < \infty} \iff 1 \in \text{supp}(\pi)$.

When $H$ is commutative.
REITER’S CONDITIONS

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Amenablity $\Leftrightarrow (P_1) \Leftarrow (P_2) \Leftrightarrow (P_r)_{1 < r < \infty} \Leftrightarrow 1 \in \text{supp}(\pi)$.

When $H$ is commutative.
Note that $(P_1) \not\Rightarrow (P_2)$. 
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**Leptin condition**

[Singh ’96]:

\(\text{(L) } H \text{ satisfies the Leptin condition if for every compact subset } K \text{ of } H \text{ and } \epsilon > 0, \exists V \text{ measurable in } H \text{ such that } 0 < h(V) < \infty \text{ and} \)

\[ \frac{h(K \ast V)}{h(V)} < 1 + \epsilon. \]
**Leptin Condition**

[Singh ’96]:

\((L)\) \(H\) satisfies the Leptin condition if for every compact subset \(K\) of \(H\) and \(\epsilon > 0\), \(\exists V\) measurable in \(H\) such that

\[0 < h(V) < \infty\]

and

\[
\frac{h(K \ast V)}{h(V)} < 1 + \epsilon.
\]

**Theorem.** [Singh 96]

Let \(H\) be a hypergroup satisfying \((L)\). Then it does \((P_r)\) for \(1 \leq r < \infty\).
Leptin Hypergroups

Hypergroups satisfying \((L)\) condition:

- Amenable locally compact groups. (Leptin ’68)
- Some simple polynomial hypergroups. (Singh ’96)
- Every compact hypergroup.
- \(\widehat{SU(2)}\). (A. ’13)
**Modified Leptin Condition**

[A. ’14]:

\((L_D)\) \(H\) satisfies the \(D\)-Leptin condition for some \(D \geq 1\) if for every compact subset \(K\) of \(H\) and \(\epsilon > 0\), \(\exists V\) measurable in \(H\) such that \(0 < h(V) < \infty\) and

\[
\frac{h(K \ast V)}{h(V)} < D + \epsilon.
\]
**Examples of \((L_D)\) hypergroups.**

[A. ]:

- Let \(G\) be an FD group. Then \(\text{Conj}(G)\) satisfies the \(D\)-Leptin condition for \(D = |G'|\).
EXAMPLES OF $(L_D)$ HYPERGROUPS.

[A.]:

- Let $G$ be an FD group. Then $\text{Conj}(G)$ satisfies the $D$-Leptin condition for $D = |G'|$.
- $\widehat{SU(3)}$ satisfies $3^8$-Leptin condition.
**Examples of \((L_D)\) Hypergroups.**

[A. ]:

- Let \(G\) be an FD group. Then \(\text{Conj}(G)\) satisfies the \(D\)-Leptin condition for \(D = |G'|.\)
- \(\widehat{SU}(3)\) satisfies \(3^8\)-Leptin condition.
- By [Banica- Vergnioux ’09]: dual of connected simply connected compact real Lie group satisfies some \(D\)-Leptin condition:
EXAMPLES OF \( (L_D) \) HYPERGROUPS.

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- \( \widehat{SU}(3) \) satisfies \( 3^8 \)-Leptin condition.
- By [Banica- Vergnioux ’09]: dual of connected simply connected compact real Lie group satisfies some \( D \)-Leptin condition:

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THEOREM. [Choi-Ghahramani ’12]
Every proper Segal algebra of $\mathbb{T}^d$ is not approximately amenable.
AN APPLICATION OF LEPTIN CONDITION

Theorem. [Choi-Ghahramani ’12] Every proper Segal algebra of $\mathbb{T}^d$ is not approximately amenable.

Theorem. [A. ’14] Let $G$ be a compact group such that $\hat{G}$ satisfies $D$-Leptin condition. Every proper Segal algebra of $G$ is not approximately amenable.
**D-Leptin and \((P_2)\)?**

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**Question.**

Let \(H\) be a hypergroup satisfying \((L_D)\). Does it satisfy \((P_2)\)?
**D-Leptin and \((P_2)\)?**

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If \(H\) is a **locally compact group**: Yes!
**D-Leptin and \((P_2)\)?**

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**Question.**
Let \(H\) be a hypergroup satisfying \((L_D)\). Does it satisfy \((P_2)\)?

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If \(H\) is a **locally compact group**: Yes!

1. \((L_D)\) implies that \(A(H)\) has a \(D\)-bounded approximate identity.
2. **Leptin’s Theorem**: \(A(H)\) has a bounded approximate identity if and only if \(H\) is amenable.
3. \(H\) is amenable if and only if \((P_2)\).
(\(L_D\)) \Rightarrow (P_2) \text{ FOR HYPERGROUPS}

**Proposition.** [A. ’14]

Let \(H\) be a regular Fourier hypergroup. If \(H\) satisfies \((L_D)\). Then \(A(H)\) has a \(D\)-bounded approximate identity.
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Leptin’s Theorem for Hypergroups. [A. ’14]
If \( H \) is a regular Fourier hypergroup. Then \( A(H) \) has a bounded approximate identity if and only if \( H \) satisfies \((P_2)\).
(\(L_D\)) \Rightarrow (\(P_2\)) FOR HYPERGROUPS

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If \(H\) is a **regular Fourier hypergroup**. Then \(A(H)\) has a bounded approximate identity if and only if \(H\) satisfies \((P_2)\). Then there is a 1-bounded approximate identity for \(A(H)\).
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Let $H$ be a regular Fourier hypergroup. If $H$ satisfies \((L_D)\). Then $A(H)$ has a $D$-bounded approximate identity.

Leptin’s Theorem for Hypergroups. [A. ’14]
If $H$ is a regular Fourier hypergroup. Then $A(H)$ has a bounded approximate identity if and only if $H$ satisfies \((P_2)\). Then there is a 1-bounded approximate identity for $A(H)$.

\[(L_D) \Rightarrow (D \text{ - b.a.i} \Leftrightarrow )(P_2).\]
Application of Leptin’s Theorem.

Corollary. [A. ’14]
Let $G$ be a locally compact group. Then $G//K$ satisfies $(P_2)$ for every compact subgroup $K$ if and only if $G$ is amenable.
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Let $G$ be a locally compact group. Then $G//K$ satisfies $(P_2)$ for every compact subgroup $K$ if and only if $G$ is amenable.

**Proof.**
- $G//K$ is a **regular Fourier hypergroup** and $A(G//K)$ is $f \in A(G)$ which are constant on double cosets of $K$. 
  (Murugunandam ’08)
APPLICATION OF LEPTIN’S THEOREM.

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- (i) If $G$ is amenable if and only if $A(G)$ has a **bounded approximate identity**. (Leptin ’68)

- (ii) $A(G)$ has a **bounded approximate identity** if and only if $A(G//K)$ has a **bounded approximate identity**.
APPLICATION OF LEPTIN’S THEOREM.

Corollary. [A. ’14]
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- $G//K$ is a regular Fourier hypergroup and $A(G//K)$ is $f \in A(G)$ which are constant on double cosets of $K$. (Murugunandam ’08)

(i) If $G$ is amenable if and only if $A(G)$ has a bounded approximate identity. (Leptin ’68)

(ii) $A(G)$ has a bounded approximate identity if and only if $A(G//K)$ has a bounded approximate identity.

(iii) By Leptin’s Theorem, $G//K$ satisfies $(P_2)$ if and only if $A(G//K)$ has a b.a.i.
Let $H$ be a regular Fourier hypergroup.

(\textit{m}) $H$ is amenable.

(\textit{L}_D) $H$ satisfies the $D$-Leptin condition for some $D \geq 1$.

(\textit{B}_D) $A(H)$ has a $D$-bounded approximate identity for some $D \geq 1$.

(\textit{P}_2) $H$ satisfies (\textit{P}_2).

(\textit{P}_1) $H$ satisfies Reiter condition.

(\textit{AM}) $L^1(H)$ is an amenable Banach algebra.

\[
\begin{array}{cccc}
(\textit{L}_1) & \rightarrow & (P_2) & \not\leftrightarrow (P_1) \leftrightarrow (AM) \\
\downarrow & & \uparrow & \uparrow \\
(\textit{L}_D) & \rightarrow & (B_D) & \leftrightarrow (B_1) \\
\end{array}
\]
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**Amenable Hypergroup Algebras**

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Chebyshev polynomial hypergroup on \( \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \):

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- $Z_{\pm 1}A(\mathbb{T})$ is a subalgebra of an amenable Banach algebra ($A(\mathbb{T})$) invariant under a finite subgroup of $Aut(A(\mathbb{T}))$.  
[Kepert ’94]: $Z_{\pm 1}A(\mathbb{T})$ is amenable.
AMENABILITY OF DISCRETE HYPERGROUP ALGEBRAS

**Theorem.** [Lasser ‘07]

Let $\mathbb{N}_0$ be a polynomial hypergroup and for each $N > 0$, \( \{ x \in \mathbb{N}_0 : h(x) \leq N \} \) is finite. Then $\ell^1(\mathbb{N}_0)$ is not amenable.

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**Conjecture.**
\( \ell^1(H) \) is **amenable** if and only if \( \sup_{x \in H} h(x) < \infty \).
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Let $G$ be an RDPF group. Then $\ell^1(\text{Conj}(G)) (= Z\ell^1(G))$ is amenable if and only if $G$ is FD.
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If \(G\) is a **non-abelian connected compact group**, then \(L^1(\text{Conj}(G))(= \mathbb{Z}L^1(G))\) is **not amenable**.
Thank You