Connes-amenability of $B(G)$

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Amenable, locally compact groups

**Definition**

Let $G$ be a locally compact group. A mean on $L^\infty(G)$ is a functional $M \in L^\infty(G)^*$ such that $\langle 1, M \rangle = \|M\| = 1$.

**Definition (J. von Neumann 1929; M. M. Day, 1949)**

$G$ is amenable if there is a mean on $L^\infty(G)$ that is left invariant, i.e.,

$$\langle L_x \phi, M \rangle = \langle \phi, M \rangle \quad (x \in G, \phi \in L^\infty(G)),$$

where

$$(L_x \phi)(y) := \phi(xy) \quad (y \in G).$$
Some amenable and non-amenable groups

Examples

1. Compact groups are amenable: $M = \text{Haar measure}$.
2. Abelian groups are amenable: use Markov–Kakutani to get $M$.
3. If $G$ is amenable and $H < G$, then $H$ is amenable.
4. If $G$ is is amenable and $N \triangleleft G$, then $G/N$ is amenable.
5. If $G$ and $N \triangleleft G$ are such that $N$ and $G/N$ are amenable, then $G$ is amenable.
6. $F_2$, the free group in two generators, is not amenable.
7. If $G$ contains $F_2$ as a closed subgroup, then $G$ is not amenable.
Banach $\mathcal{A}$-bimodules and derivations

**Definition**

Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \to E$ is called a **derivation** if

$$D(ab) := a \cdot Db + (Da) \cdot b \quad (a, b \in \mathcal{A}).$$

If there is $x \in E$ such that

$$Da = a \cdot x - x \cdot a \quad (a \in \mathcal{A}),$$

we call $D$ an **inner derivation**.
Remark

If $E$ is a Banach $\mathcal{A}$-bimodule, then so is $E^*$:

$$\langle x, a \cdot \phi \rangle := \langle x \cdot a, \phi \rangle \quad (a \in \mathcal{A}, \phi \in E^*, x \in E)$$

and

$$\langle x, \phi \cdot a \rangle := \langle a \cdot x, \phi \rangle \quad (a \in \mathcal{A}, \phi \in E^*, x \in E).$$

We call $E^*$ a dual Banach $\mathcal{A}$-bimodule.

Definition (B. E. Johnson, 1972)

$\mathcal{A}$ is called amenable if, for every dual Banach $\mathcal{A}$-bimodule $E$, every derivation $D : \mathcal{A} \to E$, is inner.
Approximate and virtual diagonals, I

Definition (B. E. Johnson, 1972)

1. An approximate diagonal for $\mathcal{A}$ is a bounded net $(d_\alpha)_\alpha$ in the projective tensor product $\mathcal{A} \hat{\otimes} \mathcal{A}$ such that

$$a \cdot d_\alpha - d_\alpha \cdot a \to 0 \quad (a \in \mathcal{A})$$

and

$$a \Delta d_\alpha \to a \quad (a \in \mathcal{A})$$

with $\Delta : \mathcal{A} \hat{\otimes} \mathcal{A} \to \mathcal{A}$ denoting multiplication.

2. A virtual diagonal for $\mathcal{A}$ is an element $D \in (\mathcal{A} \hat{\otimes} \mathcal{A})^{**}$ such that

$$a \cdot D = D \cdot a \quad \text{and} \quad a \cdot \Delta^{**} D = a \quad (a \in \mathcal{A}).$$
Theorem (B. E. Johnson, 1972)

The following are equivalent for a Banach algebra $\mathcal{A}$:

1. $\mathcal{A}$ has an approximate diagonal;
2. $\mathcal{A}$ has a virtual diagonal;
3. $\mathcal{A}$ is amenable.
The meaning of amenability, I

Theorem (B. E. Johnson, 1972)

The following are equivalent for a locally compact group $G$:

1. $L^1(G)$, the group algebra of $G$, is amenable;
2. $G$ is amenable.

Grand theme

Let $C$ be a class of Banach algebras. Characterize the amenable members of $C$!
The meaning of amenability, II

Theorem (A. Connes, U. Haagerup, et al.)

The following are equivalent for a C*-algebra $\mathcal{A}$:

1. $\mathcal{A}$ is amenable;
2. $\mathcal{A}$ is nuclear.


The following are equivalent for a locally compact group $G$:

1. $M(G)$, the measure algebra of $G$, is amenable;
2. $G$ is amenable and discrete.
The meaning of amenability, III

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### Theorem (B. E. Forrest & VR, 2005)

The following are equivalent for a locally compact group $G$:

1. $A(G)$, the Fourier algebra of $G$, is amenable;
2. $G$ is almost abelian, i.e., has an abelian subgroup of finite index.

### Corollary

The following are equivalent for a locally compact group $G$:

1. $B(G)$, the Fourier–Stieltjes algebra of $G$, is amenable;
2. $G$ is almost abelian and compact.
Dual Banach algebras

Definition

A dual Banach algebra is a pair \((A, A^*)\) of Banach spaces such that:

1. \(A = (A^*)^*\);
2. \(A\) is a Banach algebra, and multiplication in \(A\) is separately \(\sigma(A, A^*)\) continuous.

Examples

1. Every von Neumann algebra;
2. \((M(G), C_0(G))\) for every locally compact group \(G\);
3. \((M(S), C(S))\) for every compact, semitopological semigroup \(S\);
4. \((B(G), C^*(G))\) for every locally compact group \(G\).
Definition (R. Kadison, BEJ, & J. Ringrose, 1972)

Let $\mathcal{M}$ be a von Neumann algebra, and let $E$ be a dual Banach $\mathcal{M}$-bimodule. Then $E$ is called normal if the module actions

$$\mathcal{M} \times E \to E, \quad (a, x) \mapsto \begin{cases} a \cdot x \\ x \cdot a \end{cases}$$

are separately weak*-weak* continuous. If $E$ is normal, we call a derivation $D : \mathcal{M} \to E$ normal if it is weak*-weak* continuous.


A von Neumann algebra $\mathcal{M}$ is Connes-amenable if, for every normal Banach $\mathcal{M}$-bimodule $E$, every normal derivation $D : \mathcal{M} \to E$ is inner.
Injectivity, semidiscreteness, and hyperfiniteness

Definition

A von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called

1. **injective** if there is a norm one projection $E : \mathcal{B}(\mathcal{H}) \to \mathcal{M}'$ (this property is independent of the representation of $\mathcal{M}$ on $\mathcal{H}$);

2. **semidiscrete** if there is a net $(S_\lambda)_\lambda$ of unital, weak*-weak* continuous, completely positive finite rank maps such that

$$S_\lambda a \xrightarrow{\text{weak}^*} a \quad (a \in \mathcal{M});$$

3. **hyperfinite** if there is a directed family $(\mathcal{M}_\lambda)_\lambda$ of finite-dimensional $*$-subalgebras of $\mathcal{M}$ such that $\bigcup_\lambda \mathcal{M}_\lambda$ is weak* dense in $\mathcal{M}$. 
Theorem (A. Connes, et al.)

The following are equivalent:

1. $\mathcal{M}$ is Connes-amenable;
2. $\mathcal{M}$ is injective;
3. $\mathcal{M}$ is semidiscrete;
4. $\mathcal{M}$ is hyperfinite.

The notions of normality and Connes-amenability make sense for every dual Banach algebra...
Normal, virtual diagonals, I

Notation

For a dual Banach algebra $\mathcal{A}$, let $\mathcal{L}^2_\sigma(\mathcal{A}, \mathbb{C})$ denote the separately weak$^*$ continuous bilinear functionals on $\mathcal{A}$.

Observations

1. $\mathcal{L}^2_\sigma(\mathcal{A}, \mathbb{C})$ is a closed submodule of $(\mathcal{A} \hat{\otimes} \mathcal{A})^*$.

2. $\Delta^* \mathcal{A}^* \subset \mathcal{L}^2_\sigma(\mathcal{A}, \mathbb{C})$, so that $\Delta^{**} : (\mathcal{A} \hat{\otimes} \mathcal{A})^{**} \rightarrow \mathcal{A}^{**}$ drops to a bimodule homomorphism $\Delta_\sigma : \mathcal{L}^2_\sigma(\mathcal{A}, \mathbb{C})^* \rightarrow \mathcal{A}$. 
Connes-amenability of $B(G)$

Volker Runde

Amenability...
...for locally compact groups
...and for Banach algebras

Dual Banach algebras

Connes-amenability

Diagonal-type elements

Normal, virtual diagonals $C^*_\sigma$-diagonals

The Fourier–Stieltjes algebra

Normal, virtual diagonals, II

**Definition (E. G. Effros, 1988; for von Neumann algebras)**

Let $\mathcal{A}$ be a dual Banach algebra. Then $D \in B^2_\sigma(\mathcal{A}, \mathbb{C})^*$ is called a **normal, virtual diagonal** for $\mathcal{A}$ if

$$a \cdot D = D \cdot a \quad (a \in \mathcal{A})$$

and

$$a\Delta_\sigma D = a \quad (a \in \mathcal{A}).$$

**Proposition**

Suppose that $\mathcal{A}$ has a normal, virtual diagonal. Then $\mathcal{A}$ is **Connes-amenable**.
Normal, virtual diagonals and Connes-amenability

Question

Is the converse true?

Theorem (E. G. Effros, 1988)

A von Neumann algebra $\mathcal{M}$ is Connes-amenable if and only if $\mathcal{M}$ has a normal virtual diagonal.

Theorem (VR, 2003)

The following are equivalent for a locally compact group $G$:

1. $G$ is amenable;
2. $M(G)$ is Connes-amenable;
3. $M(G)$ has a normal virtual diagonal.
Definition

A bounded continuous function $f : G \to \mathbb{C}$ is called \textbf{weakly almost periodic} if $\{L_x f : x \in G\}$ is relatively weakly compact in $C_b(G)$. We set

$$\mathcal{WAP}(G) := \{f \in C_b(G) : f \text{ is weakly almost periodic}\}.$$ 

Remark

$\mathcal{WAP}(G)$ is a commutative $C^*$-algebra. Its character space $\hat{\mathcal{WAP}}$ is a compact, semitopological semigroup containing $G$ as a dense subsemigroup. This turns $\mathcal{WAP}(G)^* \cong M(\hat{\mathcal{WAP}})$ into a dual Banach algebra.
Connes-amenability without a normal, virtual diagonal

**Proposition**

*The following are equivalent:*

1. $G$ is amenable;
2. $\mathcal{WAP}(G)^*$ is Connes-amenable.

**Theorem (VR, 2006 & 2013)**

*Suppose that $G$ is a [SIN]-group. Then the following are equivalent:*

1. $\mathcal{WAP}(G)^*$ has a normal virtual diagonal;
2. $G$ is compact.*
\( C^w_\sigma \)-elements, I

**Definition**

Let \( \mathcal{A} \) be a dual Banach algebra, and let \( E \) be a Banach \( \mathcal{A} \)-bimodule. We call \( x \in E \) a \( C^w_\sigma \)-element if the maps

\[
\mathcal{A} \to E, \quad a \mapsto \begin{cases} a \cdot x \\ x \cdot a \end{cases}
\]

are weak\(^*\)-weakly continuous.

**Notation**

\[
C^w_\sigma(E) := \{ x \in E : x \text{ is a } C^w_\sigma \text{-element} \}.
\]
$C^w_\sigma$-elements, II

Observations

1. $C^w_\sigma(E)$ is a closed submodule of $E$.
2. $C^w_\sigma(E)^*$ is normal.
3. $E^*$ is normal if and only if $E = C^w_\sigma(E)$.
4. If $\theta : E \to F$ is a bounded, $\mathcal{A}$-bimodule homomorphism, then $\theta(C^w_\sigma(E)) \subset C^w_\sigma(F)$.
5. As $\mathcal{A}^*_\sigma \subset C^w_\sigma(\mathcal{A}^*)$, we have $\Delta^*\mathcal{A}^*_\sigma \subset C^w_\sigma((\mathcal{A} \hat{\otimes} \mathcal{A})^*)$, and so $\Delta^{**} : (\mathcal{A} \hat{\otimes} \mathcal{A})^{**} \to \mathcal{A}^{**}$ drops to a bimodule homomorphism $\Delta^w_\sigma : C^w_\sigma((\mathcal{A} \hat{\otimes} \mathcal{A})^*) \to \mathcal{A}$. 
\( C_\sigma^w \)-diagonals and Connes-amenability

**Definition (VR, 2004)**

Let \( \mathcal{A} \) be a dual Banach algebra. Then \( D \in C_\sigma^w((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^* \) is called a \( C_\sigma^w \)-diagonal for \( \mathcal{A} \) if

\[
a \cdot D = D \cdot a \quad (a \in \mathcal{A})
\]

and

\[
a \Delta_{\sigma}^w D = a \quad (a \in \mathcal{A}).
\]

**Theorem (VR, 2004)**

For a dual Banach algebra \( \mathcal{A} \), the following are equivalent:

1. \( \mathcal{A} \) is Connes-amenable;
2. \( \mathcal{A} \) has a \( C_\sigma^w \)-diagonal.
From $C^*(G \times G)$ into $C^*_\sigma(B(G)\hat{\otimes}B(G))^*$.

**Lemma**

Let $\mathcal{A}$ be a dual Banach algebra. Then the canonical map from $\mathcal{A}_*\hat{\otimes}\mathcal{A}_*$ into $(\mathcal{A}\hat{\otimes}\mathcal{A})^*$ is an isometric $\mathcal{A}$-bimodule homomorphism with range in $C^*_\sigma((\mathcal{A}\hat{\otimes}\mathcal{A})^*)$.

**Corollary**

Let $G$ be a locally compact group. Then there is a canonical contractive $B(G)$-bimodule homomorphism from $C^*(G \times G)$ into $C^*_\sigma(B(G)\hat{\otimes}B(G))^*)$. 
... and from $C^w_\sigma(B(G)\hat{\otimes} B(G))^*)^*$ into $B(G_d \times G_d)$
Proposition

Let $G$ be a locally compact group such that $B(G)$ is Connes-amenable, and let $D \in C^{w}_{\sigma}((B(G)\hat{\otimes}B(G))^{*})^{*}$ be a $C^{w}_{\sigma}$-diagonal for $B(G)$. Then $\Theta(D) \in B(G_{d} \times G_{d})$ is the indicator function of the diagonal of $G \times G$, i.e., of

$$\{(x, x) : x \in G\}.$$
Theorem (VR & F. Uygul, 2013)

The following are equivalent for a locally compact group $G$:

1. $B(G)$ is Connes-amenable;
2. $B(G)$ has a $C^w_\sigma$-diagonal;
3. $B(G)$ has a normal, virtual diagonal;
4. $G$ is almost abelian.
Proof.

We shall prove (ii) $\implies$ (iv).
For $f \in B(G)$, define $\tilde{f} \in B(G)$ by

$$\tilde{f}(x) := f(x^{-1}).$$

Let

$$\vee: B(G) \to B(G), \quad f \mapsto \tilde{f}.$$  

Easy:

$$(\text{id} \otimes \vee)^*: (B(G)\hat{\otimes}B(G))^* \to (B(G)\hat{\otimes}B(G))^*$$ 

maps $C^w_\sigma((B(G)\hat{\otimes}B(G))^*)$ into itself.
Proof (continued).

Let \( D \in \mathcal{C}_\sigma^w ((B(G) \hat{\otimes} B(G))^*)^* \) be a \( \mathcal{C}_\sigma^w \)-diagonal for \( B(G) \), and set
\[
\chi := \theta((\text{id} \otimes \vee)^{**}(D)) \in B(G_d \times G_d).
\]
Then \( \chi \) is the indicator function of the anti-diagonal of \( G \times G \), i.e.,
\[
\{(x, x^{-1}) : x \in G\}.
\]
This means that \( \vee : B(G) \rightarrow B(G) \) is completely bounded, which is possible only if \( C^*(G) \) is subhomogeneous, i.e., \( G \) is almost abelian.