Spectral Synthesis and Ideal Theory
Lecture 3

Eberhard Kaniuth

University of Paderborn, Germany

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The Restriction Map $A(G) \rightarrow A(H)$

**Theorem**

Let $H$ be a closed subgroup of $G$. For every $u \in A(H)$, there exists $v \in A(G)$ such that

$$v|_H = u \quad \text{and} \quad \|v\|_{A(G)} = \|u\|_{A(H)}.$$ 

This important result was independently shown by McMullen and Herz:


**Remark**

If $H$ is open in $G$, then $v$ can be defined to be zero on $G \setminus H$. In the general case, the proof is fairly technical. We give a brief indication for second countable groups.
Suppose that $G$ is second countable. Then there exists a Borel subset $S$ of $G$ with the following properties:

- $S \cap H = \{e\}$
- $S$ intersects each right coset of $H$ in exactly one point
- for each compact subset $C$ of $G$, $HC \cap S$ is relatively compact
- there exists a closed neighbourhood $V$ of $e$ in $G$ such that $HV = V$ and $V \cap S$ is relatively compact.

For $x \in G$, let $\beta(x)$ denote the unique element of $H$ such that $x = \beta(x)s$ for some $s \in S$. For any function $f$ on $G$, define $f_V$ on $G$ by

$$f_V(x) = f(\beta(x))1_V(x), \quad x \in G.$$
Lemma

Let \( G, H, S, V, \ldots \) be as above. There exists a constant \( c > 0 \) such that \( f \to c f_V \) is a linear isometry of \( L^2(H) \) into \( L^2(G) \). Moreover, for all \( f, g \in L^2(H) \) and \( h \in H \),

\[
c^2 (f_V *_G \tilde{g}_V)(h) = (f *_H \tilde{g})(h).
\]

Remark

What is \( c \)?

If \( f \in C_c(H) \), then \( f_V \) is bounded and measurable and has compact support. Thus we can define a linear functional \( I \) on \( C_c(H) \) by

\[
I(f) = \int_G f_V(x) \, dx.
\]

Check that \( I \) is left invariant and if \( f \geq 0 \) and \( f \neq 0 \), then \( I(f) > 0 \). Thus \( I \) is a left Haar integral on \( H \). By uniqueness, there exists \( c > 0 \) such that

\[
c \int_G f_V(x) \, dx = \int_H f(h) \, dh.
\]
Amenable Groups

**Definition**

A locally compact group $G$ is called *amenable* if there exists a left invariant mean, i.e. a linear functional $m$ on $L^\infty(G)$ such that $m(f) = m(f)$ for all $f \in L^\infty(G)$, $m(f) \geq 0$ if $f \geq 0$ and $m(1) = 1$.

Amenability of $G$ can also be characterized through the existence of left invariant means on various other function spaces on $G$.

**Examples**

(1) Compact groups and abelian locally compact groups

(2) If $N$ is a closed normal subgroup of $G$ and $N$ and $G/N$ are both amenable, then $G$ is amenable

(3) Closed subgroup of amenable groups are amenable
Further Examples

(4) If there exists an increasing sequence

$$\{e\} = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_r = G$$

of closed subgroups of $G$ such that $H_{j-1}$ is normal in $H_j$ and every quotient group $H_j/H_{j-1}$ is amenable, $1 \leq j \leq r$, then $G$ is amenable.

(5) Free groups and $SL(n, \mathbb{Z})$ are not amenable.

(6) Noncompact semisimple Lie groups is not amenable.

(7) If $G = \bigcup_{\alpha} H_\alpha$, where $(H_\alpha)_\alpha$ is an upwards directed system of closed amenable subgroups of $G$, then $G$ is amenable.
Characterizations of Amenability

For a locally compact group $G$ with left Haar measure, let $\lambda_G$ denote the left regular representation, i.e. the representation on $L^2(G)$ defined by

$$\lambda_G(x)f(y) = f(x^{-1}y), \quad f \in L^2(G), \ x \in G.$$ 

The coordinate functions of $\lambda_G$ are the functions of the form

$$u_{f,g}(x) = \langle \lambda_G(x)f, g \rangle, \quad f, g \in L^2(G).$$

**Theorem**

For a locally compact group $G$, the following are equivalent:

1. $G$ is amenable
2. $1_G$ is weakly contained in $\lambda_G$: the function $1$ can be approximated uniformly on compact subsets of $G$ by functions $u_{f,g}$
3. For every $f \in L^1(G), f \geq 0, \|\lambda_G(f)\| = \|f\|_1$. 
Existence of a Bounded Approximate Identity in $A(G)$

**Theorem**

For a locally compact $G$, the following three conditions are equivalent:

1. $G$ is amenable
2. $A(G)$ has an approximate identity $(u_{\alpha})_{\alpha}$ such that, for every $\alpha$, $\|u_{\alpha}\| \leq 1$ and $u_{\alpha}$ is a positive definite function with compact support
3. $A(G)$ has a bounded approximate identity.


The proof outlined below is taken from an unpublished thesis of Nielson and appears in

Outline of Proof

Have to show \((1) \implies (2)\) and \((3) \implies (1)\)

\((1) \implies (2):\) Amenability of \(G\) is equivalent to that \(1_G\) is weakly contained in \(\lambda_G\) \(\implies\) given \(K \subseteq G\) compact and \(\epsilon > 0\), there exists \(u_{K,\epsilon} \in P(G)\) such that

- \(|u_{K,\epsilon} - 1| \leq \epsilon\) for all \(x \in K\)
- \(u_{K,\epsilon}\) is a coordinate function of \(\lambda_G\).

Since \(C_c(G)\) is dense in \(L^2(G)\), we can assume that \(u_{K,\epsilon}\) has compact support. \((2)\) follows now from the following lemma, applied to \(u = 1_G\).

**Lemma**

Let \((u_{\alpha})_{\alpha}\) be a net in \(P(G)\) and \(u \in P(G)\) such that \(u_{\alpha} \to u\) uniformly on compact subsets of \(G\). Then

\[\|(u_{\alpha} - u)v\|_{A(G)} \to 0\]

for every \(v \in A(G)\).
For (3) \( \implies \) (1) one shows that \( \| \lambda_G(f) \| = \| f \|_1 \) for every \( f \in C_c(G) \), \( f \geq 0 \).

This implies amenability of \( G \).

Let \( (u_\alpha)_\alpha \) be an approximate identity for \( A(G) \) bounded by \( c > 0 \). Let \( K = \text{supp}(f) \) and choose a compact symmetric neighbourhood \( V \) of \( e \) in \( G \). Set

\[
u = |V|^{-1} (1_V * 1_{VK}) \in A(G).
\]

Then \( u = 1 \) on \( K \) and hence, since \( \| u_\alpha u - u \|_{A(G)} \to 0 \), \( u_\alpha \to 1 \) uniformly on \( K \). This implies, since \( f \geq 0 \),

\[
\| f \|_1 = \lim_\alpha |\langle u_\alpha, f \rangle| = \lim_\alpha |\langle u_\alpha, \lambda_G(f) \rangle| \\
\leq c \| \lambda_G(f) \|.
\]
Replacing $f$ with the $n$-fold convolution product $f^n$, it follows that

$$\|f\|_1^n = \|f^n\|_1 \leq c \|\lambda_G(f^n)\| \leq c \|\lambda_G(f)\|^n$$

and therefore

$$\|f\|_1 \leq \|\lambda_G(f)\| \cdot \lim_{n \to \infty} c^{1/n} = \|\lambda_G(f)\| \leq \|f\|_1.$$ 

This completes the proof of (3) $\implies$ (1).
When does Spectral Synthesis hold for $A(G)$?

**Necessary Condition:** $u \in \overline{uA(G)}$ for every $u \in A(G)$.

**Sufficient Condition:** $G = \Delta(A(G))$ is discrete and $u \in \overline{uA(G)}$ for every $u \in A(G)$.

**Remark**

The hypothesis that $u \in \overline{uA(G)}$ for every $u \in A(G)$ is satisfied in the following cases:

- $G$ is amenable: then $A(G)$ has a bounded approximate identity
- $G = \mathbb{F}_2$, $G = SL(2, \mathbb{R})$ or $G = SL(2, \mathbb{R})$: then $A(G)$ has an approximate identity, which is bounded in the multiplier norm (Haagerup).

**Question:** Do we always have $u \in \overline{uA(G)}$ for every $u \in A(G)$?
Theorem

Let $G$ be an arbitrary locally compact group. Then spectral synthesis holds for $A(G)$ (if and) only if $G$ is discrete and $u \in uA(G)$ for each $u \in A(G)$.


Independently, this result was also shown in

Lemma

Let $H$ be a closed subgroup of $G$, and let

$$I(H) = \{ u \in A(G) : u|_H = 0 \}.$$  

Then the restriction map $A(G) \to A(H)$ induces an isometric isomorphism

$$A(G)/I(H) \to A(H), \quad u + I(H) \to u|_H.$$  

Proof.

The map $u + I(H) \to u|_H$ is an algebra isomorphism from $A(G)/I(H)$ into $A(H)$. By the restriction theorem, it is surjective, and it is an isometry, since

$$\|u|_H\|_{A(H)} = \inf \{ \|v\|_{A(G)} : v \in A(G), v|_H = u|_H \} = \inf \{ \|v\|_{A(G)} : v - u \in I(H) \} = \|u + I(H)\|$$

for every $u \in A(G)$.
Lemma

Let $K$ be a compact normal subgroup of $G$, $q : G \rightarrow G/K$ the quotient homomorphism and $E$ a closed subset of $G/K$. If $q^{-1}(E)$ is a set of synthesis for $A(G)$, then $E$ is a set of synthesis for $A(G/K)$.

Proof.

Given $u \in k(E)$ and $\epsilon > 0$, consider $u_1 = u \circ q$. Then $u_1 \in k(q^{-1}(E))$ and hence there exists $v_1 \in j(q^{-1}(E))$ such that $\|u_1 - v_1\| \leq \epsilon$. Define $v$ on $G/K$ by

$$v(xK) = \int_K v_1(xk) \, dk = \int_K (R_k v_1)(x) \, dk.$$ 

Then $v \in A(G/K)$ and

$$\|u - v\|_{A(G/K)} \left\| \int_K R_k(u_1 - v_1) \, dk \right\|_{A(G/K)} \leq \|u - v\|_{A(G)} \leq \epsilon.$$
Proof continued

Moreover, \( \nu \in j(E) \) since:

- \( C = \text{supp}(\nu_1) \) is compact and \( C \cap q^{-1}(E) = \emptyset \)

- hence there exists a symmetric neighbourhood \( V \) of \( e \) in \( G \) such that \( C \cap Vq^{-1}(E) = \emptyset \)

- \( \nu \) vanishes on the neighbourhood \( q(Vq^{-1}(E)) \) of \( E \) since \( \nu_1 = 0 \) von \( Vq^{-1}(E) \)

- \( \text{supp} V \subseteq q(C) \)
Lemma

Let $G$ be a connected locally compact group. If spectral synthesis holds for $A(G)$, then $G$ is trivial.

Proof.

Assume that $G \neq \{e\}$.

• $G$ connected $\implies G$ contains a compact normal subgroup $K$ such that $G/K$ is a Lie group

• spectral synthesis holds for $A(G/K)$

• the nontrivial connected Lie group $G/K$ contains a closed nondiscrete abelian subgroup $H$ (a one-parameter subgroup)

• spectral synthesis holds for $A(H)$ since $A(H)$ is a quotient of $A(G)$

• this contradicts Malliavin’s theorem for abelian groups
Proof of the Theorem

Suppose that synthesis holds for \( A(G) \)

- then synthesis holds for \( G_0 \), the connected component of the identity
- \( G_0 = \{ e \} \) by the preceding lemma, i.e. \( G \) is totally disconnected

Fix a compact open subgroup \( K \) of \( G \), and assume that \( K \) is infinite.

- by a theorem of Zelmanov, every infinite compact group contains an infinite abelian subgroup, say \( H \)
- then spectral synthesis holds for \( A(H) \), which contradicts Malliavin’s theorem
Fourier Algebras of Coset Spaces

$G$ a locally compact group, $K$ a compact subgroup of $G$ with normalized Haar measure

$G/K$ the space of left cosets of $K$, equipped with the quotient topology, $q : G \to G/K$ the quotient map

**Definition**

$A(G/K) = \{ u : G/K \to \mathbb{C} : u \circ q \in A(G) \}$ is called the *Fourier algebra of* $G/K$.

Let $p_K : A(G) \to A(G/K)$ be defined by

$$p_K(u)(xK) = \int_K u(xk) dk, \quad u \in A(G), \, x \in G.$$
Then $p_K$ maps the subalgebra

$$\{ u \in A(G) : u(xk) = u(x) \text{ for all } k \in K \text{ and all } x \in G \}$$

of $A(G)$ isometrically onto $A(G/K)$.

The spaces $A(G/K)$ are precisely the norm closed left translation invariant subspaces of $A(G)$ (Takesaki/Tatsuuma).

**Theorem**

1. $A(G/K)$ is regular and semisimple
2. $\Delta(A(G/K)) = G/K$: the map $xK \mapsto \varphi_{xK}$, where $\varphi_{xK}(u) = u(xK)$, is a homeomorphism
3. $A(G/K)$ has a bounded approximate identity if and only if $G$ is amenable

When does Spectral Synthesis hold for $A(G/K)$?

Yes, if $K$ is open in $G$ and $u \in \overline{uA(G/K)}$ for every $u \in A(G/K)$.

**Conjecture:** The converse is true.

**Theorem**

Let $G$ contain a nilpotent open subgroup. If $K$ is a compact subgroup of $G$ and spectral synthesis holds for $A(G/K)$, then $K$ is open in $G$.

**Corollary**

Suppose that $G_0$, the connected component of the identity, is nilpotent. If $K$ is a compact subgroup of $G$ and spectral synthesis holds for $A(G/K)$, then $G_0 \subseteq K$.

Lemma

Let $H$ be a closed subgroup and $K$ a compact subgroup of $G$. Then the restriction map

$$A(G/K) \to A(H/H \cap K), \quad u \to u|_H$$

is surjective in any of the two cases:
- $H$ is contained in the normalizer of $K$
- $H$ is open in $G$.

Lemma

Let $i : H/H \cap K \to G/K$, $x(H \cap K) \to xK$, $x \in H$, and suppose that

$$u \to u|_H, \quad A(G/K) \to A(H/H \cap K)$$

is surjective. Let $E$ be a closed subset of $H/H \cap K = \Delta(A(H/H \cap K))$. If $i(E)$ is a set of synthesis (Ditkin set) for $A(G/K)$, then $E$ is a set of synthesis (a Ditkin set) for $A(H/H \cap K)$. 
Corollary

1. Singletons \( \{xK\} \) are sets of synthesis for \( A(G/K) \)

2. If \( G \) is amenable, then finite subsets of \( G/K \) are Ditkin sets for \( A(G/K) \).

Proof.

Take \( H = K \) and recall that \( xK \) is a set of synthesis for \( A(G) \) and that \( xK \) is a Ditkin set if \( G \) is amenable.

(1) and (2) for sets of synthesis were already proved by Forrest l.c..