Spectral Synthesis and Ideal Theory
Lecture 2

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Synthesis Notions

A a regular and semisimple commutative Banach algebra. For a closed subset $E$ of $\Delta(A)$, let

$$j(E) = \{a \in A : \hat{a} \text{ has compact support disjoint from } E\}.$$ 

Then, if $I$ is any ideal in $A$ with $h(I) = E$,

$$j(E) \subseteq I \subseteq k(E).$$

Definition

$E$ is called a set of synthesis or spectral set if $\overline{j(E)} = k(E)$ (equivalently, $I = k(E)$ for any closed ideal $I$ with $h(I) = E$).

We say that spectral synthesis holds for $A$ if every closed subset of $\Delta(A)$ is a set of synthesis.
Definition

\( E \subseteq \Delta(A) \) closed is called \textit{Ditkin set} if \( a \in \overline{aj(E)} \) for every \( a \in k(E) \). Thus

- Every Ditkin set is a set of synthesis

- \( \emptyset \) is a Ditkin set if and only if given \( a \in A \) and \( \epsilon > 0 \), there exists \( b \in A \) such that \( \hat{b} \) has compact support and \( \|a - ab\| \leq \epsilon \) (in this case we also say that \( A \) satisfies \textit{Ditkin's condition at infinity})

\( A \) is called \textit{Tauberian} if the set of all \( a \in A \) such that \( \hat{a} \) has compact support, is dense in \( A \). Thus

- \( A \) is Tauberian if and only if \( \emptyset \) is a set of synthesis.

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When does Spectral Synthesis hold for $A$?

Spectral synthesis holds for $C_0(X)$, $X$ a locally compact Hausdorff space.

Spectral synthesis does not hold for $C^n[a, b]$, $n \geq 1$: singletons $\{t\}$, $t \in [a, b]$, are not sets of synthesis.

**Remark**

Suppose that spectral synthesis holds for $A$. Then $a \in a\overline{A}$ for each $a \in A$.

**Proof:**

Let $E = \{\varphi \in \Delta(A) : \varphi(a) = 0\}$. Then $E$ is closed in $\Delta(A)$ and $E = h(a\overline{A})$. Thus $a \in k(E) = a\overline{A}$ since $E$ is of synthesis.

The condition that $a \in a\overline{A}$ for every $a \in A$ is satisfied, if $A$ has an approximate identity.
Lemma

Let $A$ be a regular and semisimple commutative Banach algebra and $E$ an open and closed subset of $\Delta(A)$.

1. If $A$ is Tauberian and $a \in \overline{aA}$ for every $a \in k(E)$, then $E$ is a set of synthesis.

2. If $A$ satisfies Ditkin’s condition at infinity, then $E$ is a Ditkin set.

Proof of (2) Have to show that $a \in \overline{aj(E)}$ for each $a \in k(E)$:

- $E$ open and closed $\implies$

$$h(j(E) + j(\Delta(A) \setminus E)) = E \cap (\Delta(A) \setminus E) = \emptyset$$

and hence $j(\emptyset) \subseteq j(E) + j(\Delta(A) \setminus E)$

- $\emptyset$ Ditkin $\implies$ for every $a \in A$, there exist sequences $(u_n)_n \subseteq j(E)$ and $(v_n)_n \subseteq j(\Delta(A) \setminus E)$ such that $a(u_n + v_n) \to a$

- let $a \in k(E)$: then $\hat{a}v_n = \hat{a}\hat{v}_n$ vanishes on $E$ and on $\Delta(A) \setminus E$, hence $av_n = 0$. So $a = \lim_{n \to \infty} au_n \in \overline{aj(E)}$, as required.
From the first assertion of the lemma and the above remark it follows

**Corollary**

Suppose that $\Delta(A)$ is discrete and $A$ is Tauberian. Then spectral synthesis holds for $A$ if and only if $a \in \overline{aA}$ for each $a \in A$.

**Corollary**

Let $G$ be a compact abelian group. Then spectral synthesis holds for $L^1(G)$.

**Proof.**

- $L^1(G)$ has an approximate identity
- $L^1(G)$ is Tauberian
- $\hat{G} = \Delta(L^1(G))$ is discrete since $G$ is compact.
The Example of L. Schwartz

**Theorem**

For $n \geq 3$, the sphere $S^{n-1} = \{y \in \mathbb{R}^n : \|y\| = 1\} \subseteq \Delta(L^1(\mathbb{R}^n))$ fails to be a set of synthesis for $L^1(\mathbb{R}^n)$.

**Remark**

(1) L. Schwartz [Sur une propriété de synthèse spectrale dans les groupes noncompacts, C.R. Acad. Sci. Paris 227 (1948), 424-426] proved this result for $n = 3$, but the proof works for all $n \geq 3$.

(2) $S^1 \subseteq \mathbb{R}^2$ is a set of synthesis for $L^1(\mathbb{R}^2)$ [C. Herz, *Spectral synthesis for the circle*, Ann. Math. 68 (1958), 709-712]
Proof of Schwartz’ Theorem

Identify \( \hat{R}^n \) with \( R^n \) through \( y \rightarrow \gamma_y \), where \( \gamma_y(x) = \langle x, y \rangle \) for \( x \in R^n \).

- \( \hat{f}(y) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} f(x) e^{-i\langle x, y \rangle} \, dx, \quad f \in L^1(\hat{R}^n) \)
- \( \check{g}(x) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} g(y) e^{i\langle x, y \rangle} \, dy, \quad g \in L^1(\hat{R}^n) \)
- \( f \in L^1(\hat{R}^n) \cap L^2(\hat{R}^n) \) and \( \check{f} \in L^1(\hat{R}^n) \), then \( (\check{f})^\wedge = f \) in \( L^2(\hat{R}^n) \), hence \( (\check{f})^\wedge(x) = f(x) \) for all \( x \in R^n \) if \( f \) is continuous

Lemma

Let \( D(R^3) \) denote the set of all functions in \( L^1(R^3) \cap C_0(R^3) \) with the property that all partial derivatives exist and are in \( L^1(R^3) \cap C_0(R^3) \). Then \( \hat{f} \in L^1(R^3) \) and \( (\check{f})^\wedge = f \) for every \( f \in D(R^3) \).
Lemma

Let $S = S^2$ and $I = k(S) \subseteq L^1(\mathbb{R}^n)$, and

$$J = \left\{ f \in I : \hat{f} \in D(\mathbb{R}^n) \text{ and } \frac{\partial \hat{f}}{\partial y_1} = 0 \text{ on } S \right\}.$$  

Then $\overline{J}$ is an ideal in $L^1(\mathbb{R}^3)$ and $h(J) = S$.

To show that $\overline{J} \neq I$, it suffices to construct a bounded linear functional $F$ on $L^1(\mathbb{R}^3)$ such that $F(J) = \{0\}$, but $F(I) \neq \{0\}$. Such an $F$ can be constructed as follows:

There exists a unique probability measure $\mu$ on $S$, which is invariant under orthogonal transformations.

Define a function $\phi$ on $\mathbb{R}^3$ by

$$\phi(x) = \int_S e^{-i\langle x, y \rangle} d\mu(y).$$
Then the function $x \to x_1\phi(x)$ on $\mathbb{R}^3$ is continuous and bounded. More precisely, it can be shown that

$$|x_1\phi(x)| \leq \|x\| \cdot |\phi(x)| \leq \frac{4\pi}{3}, \quad x \in \mathbb{R}^3.$$ 

The required functional $F$ can now be defined by

$$F(f) = \int_{\mathbb{R}^3} f(x)x_1\phi(x) \, dx, \quad f \in L^1(\mathbb{R}^3).$$

Since

$$\frac{\partial \hat{f}}{\partial y_1}(y) = (-ix_1f(x))^\wedge(y) = \int_{\mathbb{R}^3} (-ix_1)f(x)e^{-i\langle x, y \rangle} \, dx,$$

we have

$$i \int_S \frac{\partial \hat{f}}{\partial y_1}(y) \, d\mu(y) = \int_S \left( \int_{\mathbb{R}^3} x_1f(x)e^{-i\langle x, y \rangle} \, dx \right) \, d\mu(y)$$

$$= \int_{\mathbb{R}^3} x_1f(x) \left( \int_S e^{-i\langle x, y \rangle} \, d\mu(y) \right) \, dx = \int_{\mathbb{R}^3} f(x)x_1\phi(x) \, dx = F(f).$$

Thus $F(f) = 0$ for every $f \in J$. 

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To show that $F(I) \neq \{0\}$, consider the function

$$f(x) = (\sqrt{2})^3 e^{-\|x\|^2} - e^{1/4} e^{-\|x\|^2/2}, \quad x \in \mathbb{R}^3.$$ 

Then $f \in L^1(\mathbb{R}^3)$, and

$$\hat{f}(y) = e^{-\|y\|^2/4} - e^{1/4} e^{-\|y\|^2/2}.$$ 

Hence $\hat{f}(y) = 0$ if $\|y\| = 1$, i.e. $f \in I$.

We claim that $F(L_\alpha f) \neq 0$ for some $\alpha \in \mathbb{R}^3$ (note that $L_\alpha f \in I$ since $I$ is a closed ideal). For arbitrary $f$, we have

$$\bar{L_\alpha f}(y) = e^{i \langle \alpha, y \rangle} \hat{f}(y) \implies \frac{\partial \bar{L_\alpha f}}{\partial y_1}(y) = e^{i \langle \alpha, y \rangle} \left[ i \alpha \hat{f}(y) + \frac{\partial \hat{f}}{\partial y_1}(y) \right].$$

If $f \in I$, then $\hat{f}(y) = 0$ for $y \in S$, and hence

$$F(L_\alpha f) = i \int_S \frac{\partial \bar{L_\alpha f}}{\partial y_1}(y) d\mu(y) = i \int_S e^{i \langle \alpha, y \rangle} \frac{\partial \hat{f}}{\partial y_1}(y) d\mu(y).$$
Now, for our special function $f$,
\[
\frac{\partial \hat{f}}{\partial y_1}(y) = -\frac{1}{2} y_1 e^{-\|y\|^2/4} + y_1 e^{1/4} e^{-\|y\|^2/2}
\]
and hence, for $y \in S$,
\[
\frac{\partial \hat{f}}{\partial y_1}(y) = \frac{1}{2} y_1 e^{-1/4} y_1.
\]

Finally, take $a = (\pi, 0, 0)$; then with $c = \frac{1}{2} e^{-1/4}$,
\[
F(L_a f) = i c \int_S e^{i\pi y_1} y_1 d\mu(y)
= i c \int_S y_1 \cos(\pi y_1) \mu(y) - c \int_S y_1 \sin(\pi y_1) \mu(y).
\]
The first integral is zero since $(y_1, y_2, y_3) \rightarrow (-y_1, y_2, y_3)$ is an orthogonal transformation. So
\[
F(L_a f) = c \int_S y_1 \sin(\pi y_1) \mu(y).
\]
Since $y_1 \sin(\pi y_1) > 0$ whenever $y_1 \neq 0, 1, -1$, it follows that $F(L_a f) \neq 0$. 

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Theorem

Let \( I = k(S^{n-1}) \subseteq L^1(\mathbb{R}^n) \), and for \( 1 \leq k \leq \lfloor \frac{n+1}{2} \rfloor \), let \( I^k \) denote the closed ideal of \( L^1(\mathbb{R}^n) \) generated by all convolution products \( f_1 \ast f_2 \ast \ldots \ast f_k, f_j \in I \). Then

\[
I = I^1 \supseteq I^2 \supseteq \ldots \supseteq I^{\lfloor \frac{n+1}{2} \rfloor} = \overline{j(S^{n-1})}.
\]

- All the inclusions are proper
- The ideals \( I^k \) are the only rotation invariant closed ideals of \( L^1(\mathbb{R}^n) \) with hull equal to \( S^{n-1} \).

Injection Theorem for Spectral Sets

A a regular and semisimple commutative Banach algebra, \( I \) a closed ideal of \( A \) and \( i : \Delta(A/I) \rightarrow \Delta(A) \) the usual embedding.

**Theorem**

Let \( E \) be a closed subset of \( \Delta(A/I) \).
- If \( i(E) \) is a set of synthesis (Ditkin set) for \( A \), then \( E \) is a set of synthesis for \( A/I \).
- Suppose that \( E \) is a set of synthesis for \( A/I \) and \( h(I) \) is a set of synthesis for \( A \). Then \( i(E) \) is a set of synthesis for \( A \).

**Remark**

In the second statement of the theorem, the hypothesis on \( h(I) \) cannot be dropped, and the analogue for Ditkin sets requires some additional strong hypothesis on \( A \).
Unions of sets of synthesis and Ditkin sets

Let $A$ be a regular and semisimple commutative Banach algebra.

**Theorem**

Let $E$ and $F$ be closed subsets of $\Delta(A)$ such that $E \cap F$ is a Ditkin set. Then $E \cup F$ is a set of synthesis if and only if both $E$ and $F$ are sets of synthesis.

**Theorem**

Let $E_1, E_2, \ldots \subseteq \Delta(A)$ be Ditkin sets. If $\bigcup_{i=1}^{\infty} E_i$ is closed in $\Delta(A)$, then $\bigcup_{i=1}^{\infty} E_i$ is a Ditkin set.
Problems

**Union Problem:** Let $E, F \subseteq \Delta(A)$ be sets of synthesis. Is then $E \cup F$ also a set of synthesis?

**The C-set/S-set Problem:** Is every set of synthesis a Ditkin set? (Ditkin sets are sometimes called C-sets, $C$ referring to Calderon)

Since finite unions of Ditkin sets are Ditkin sets, an affirmative answer to the C-set/S-set problem implies an affirmative answer to the union problem.

In general, the answer to both questions is negative!

Both problems are open for $L^1(G)$, $G$ a noncompact locally compact abelian group, even for $G = \mathbb{Z}$. 
The Mirkil Algebra

Definition

Identify $[-\pi, \pi]$ with the circle $\mathbb{T}$, and let $M$ be the space of all $f \in L^2(\mathbb{T})$ such that $f$ is continuous on the interval $[-\pi/2, \pi/2]$. Endow $M$ with the norm

$$\|f\| = \|f\|_2 + \|f|_{[-\pi/2,\pi/2]}\|_\infty$$

and convolution.

$M$ is a regular and semisimple commutative Banach algebra, and the spectrum $\Delta(M)$ can be identified with $\mathbb{Z}$ via $n \to \varphi_n$, where

$$\varphi_n(f) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-int} dt, \quad f \in M.$$
The algebra $M$ shows that in the general Banach algebra context the answer to both problems is negative:

- $4\mathbb{Z}$ and $4\mathbb{Z} + 2$ are both sets of synthesis, but their union $2\mathbb{Z}$ is not of synthesis
- $4\mathbb{Z}$ and $4\mathbb{Z} + 2$ fail to be Ditkin sets
- Every finite subset of $\Delta(M)$ is a set of synthesis, but not a Ditkin set (in particular, $\emptyset$ is not Ditkin).


Examples

(1) Every closed convex set in $\mathbb{R}^n$ is set of synthesis for $L^1(\mathbb{R}^n)$

(2) Let $D = \{y \in \mathbb{R}^n : \|y\| < 1\}$: then $\mathbb{R}^n \setminus D$ is a set of synthesis for $L^1(\mathbb{R}^n)$.

(3) $\overline{D} = \{y \in \mathbb{R}^n : \|y\| \leq 1\}$ is of synthesis by (1), but the intersection $S^{n-1} = \overline{D} \cap \mathbb{R}^n \setminus D$ is not of synthesis.

(4) $E \subseteq \hat{G}$ such that $\partial(E)$ is a Ditkin set, then $E$ is a Ditkin set for $L^1(G)$. In particular, if $\partial(E)$ is countable, then $E$ is a Ditkin set.

(5) Translates of sets of synthesis (Ditkin sets) are sets of synthesis (Ditkin sets).

(6) Let $\Gamma, \Gamma_1, \ldots, \Gamma_n$ be closed subgroups of $\hat{G}$ such that $\Gamma_j \subseteq \Gamma$ and $\Gamma_j$ is relatively open in $\Gamma$. Then, for any $\gamma_1, \ldots, \gamma_n \in \hat{G}$, the set $\Gamma \setminus \bigcup_{j=1}^n \gamma_j \Gamma_j$ is a Ditkin set.
Malliavin's Theorem

Let $G$ be a locally compact abelian group. If $G$ is compact (equivalently, if $\hat{G} = \Delta(L^1(G))$ is discrete), then spectral synthesis holds for $L^1(G)$, since $\emptyset$ is a Ditkin set.

**Theorem (Malliavin’s Theorem)**

*Spectral synthesis holds for $L^1(G)$ (if and) only if $G$ is compact.*


A more constructive proof than Malliavin’s was given by Varopoulos, using tensor product methods:

Steps of the Proof

(1) Let $\Gamma$ be a closed subgroup of $\hat{G}$ and

$$H = \{ x \in G : \gamma(x) = 1 \text{ for all } \gamma \in \Gamma \}.$$ 

Let $E$ be a closed subset of $\Gamma$ and suppose that $E$ is a set of synthesis for $L^1(G/H)$. Then $E$ is a set of synthesis for $L^1(G)$.

(2) If $\mathbb{T} = \Delta(\ell^1(\mathbb{Z}))$ contains a set which is not of synthesis for $\ell^1(\mathbb{Z})$, then $\mathbb{R}$ contains a nonspectral set for $L^1(\mathbb{R})$.

Every locally compact abelian group contains an open subgroup $H$ of the form $H = \mathbb{R}^n \times K$, where $K$ is compact and $n \in \mathbb{N}_0$. Therefore (1) and (2) imply

(3) If spectral synthesis does not hold for every infinite discrete abelian group, then it does not hold for every noncompact locally compact abelian group.