Operators on Hilbert Spaces

Let $H$ be a Hilbert space with inner product

$$\langle \xi \mid \eta \rangle$$

for $\xi, \eta \in H$. We obtain a norm

$$\|\xi\| = \langle \xi \mid \xi \rangle^{\frac{1}{2}}.$$

A linear operator $x : H \to H$ is bounded if

$$\|x\| = \sup\{\|x\xi\| : \|\xi\| \leq 1, \xi \in H\}.$$  

Then $B(H)$, the space of bounded linear operators on $H$, is a Banach space.
Involution on $B(H)$

$B(H)$ with this operator norm is unital Banach algebra since

$$\|xy\| \leq \|x\|\|y\|.$$

There exist an involution $*$ on $B(H)$ given by

$$\langle x^* \xi \mid \eta \rangle = \langle \xi \mid x\eta \rangle.$$

$B(H)$ with this involution is an involutive Banach algebra since it satisfies

1. $(\alpha x + \beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^*$,  
2. $(xy)^* = y^* x^*$,  
3. $(x^*)^* = x$.

Moreover it also satisfies

4. $\|x^* x\| = \|x\|^2$.

Therefore, $B(H)$ is a unital C*-algebra.
**C*-algebras**

In general, a **C*-algebra** is an involutiva Banach algebra satisfying the condition (4), i.e. it satisfies

\[ \|x^*x\| = \|x\|^2. \]

It is clear that every norm closed *-subalgebra

\[ A \subseteq B(H) \]

is a C*-algebra. Here we say that \( A \) is *-subalgebra if \( x^* \in A \) whenever \( x \in A \).

**Theorem [Gelfand-Naimark 1943]:** Let \( A \) be a C*-algebra, i.e. let \( A \) be an involutive Banach algebra satisfying the condition (4). Then there exists a Hilbert space \( H \) and an isometric *-homomorphism

\[ \pi : A \to \pi(A) \subseteq B(H). \]

This shows that every C*-algebra can be represented on some Hilbert space.
Examples of C*-algebras

- $B(H)$ for some Hilbert space $H$.

In particular the matrix algebra $M_n(\mathbb{C}) = B(\mathbb{C}^n)$, for which the multiplication is given by matrix product

$$[x_{ij}][y_{jk}] = \left[ \sum_j x_{ij}y_{jk} \right]$$

and the involution is given by $[x_{ij}]^* = [\overline{x}_{ji}]$

- Finite dimensional C*-algebras

$$M_{n_1}(\mathbb{C}) \oplus_\infty M_{n_2}(\mathbb{C}) \oplus_\infty \cdots \oplus_\infty M_{n_k}(\mathbb{C})$$

- The space $K(H) \subseteq B(H)$ of all compact linear operators on $H$

- Any norm closed ideal $J$ of a C*-algebra $A$, and its quotient $A/J$

- The Calkin algebra $Q(H) = B(H)/K(H)$
**Commutative C*-algebras**

Let $\Omega$ be a compact topological space. Then $A = C(\Omega)$ with norm

$$\|f\|_\infty = \sup \{|f(t)| : t \in \Omega\}$$

and involution $f^*(t) = \overline{f(t)}$ is a unital commutative C*-algebra.

Indeed, for any $f, g \in C(\Omega)$, we have

$$\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$$

and we have

$$\|f^*f\|_\infty = \sup \{|f(t)f(t)| : t \in \Omega\} = \|f\|_\infty^2.$$ 

Therefore, $(C(\Omega), \| \cdot \|_\infty)$ is a unital commutative C*-algebra.
**Theorem:** For every unital commutative C*-algebra $A$, there exists a compact topological space $\Omega$ such that we have the isometric *-isomorphism

$$A = C(\Omega).$$

**Proof:** Let $A$ be a unital commutative C*-algebra and let

$$\Omega = \Delta(A)$$

be the set of all unital *-homomorphism (i.e. unital contractive homomorphism) from $A$ to $\mathbb{C}$. Then $\Omega$ is a weak* closed and thus weak* compact subset of $A^*_1$. Let

$$a \in A \rightarrow \hat{a} \in A^{**}$$

be the canonical isometric inclusion given by

$$\hat{a}(\varphi) = \varphi(a)$$

for $\varphi \in A^*$. Then the Gelfand Transformation

$$a \in A \rightarrow \hat{a}|_{\Omega} \in C(\Omega)$$

is an isometric *-isomorphism from $A$ onto $C(\Omega)$,
Remark:

Let $\Omega$ be a compact topological space. For each $t \in \Omega$, the point-evaluation

$$\varphi_t : f \in C(\Omega) \to f(t) \in \mathbb{C}$$

is a unital $*$-homomorphism from $C(\Omega)$ into $\mathbb{C}$. This defines a homeomorphism

$$\tau : t \in \Omega \leftrightarrow \varphi_t \in \Delta(C'(\Omega)).$$

Therefore, the above Theorem establishes a duality correspondence between

Compact Topological Spaces $\Omega$

and

Unital Comm $C^*$-algebras $A = C(\Omega)$. 
We also have a natural duality correspondence between

Locally Compact Topological Spaces $\Omega$

and

Commutative C*-algebras $C_0(\Omega)$
Therefore, we may regard general C*-algebras as Noncommutative Topological Spaces.
More Examples of C*-algebras
Group C*-algebras $C^*_\lambda(G)$

Let $G$ be a discrete group and $H = \ell_2(G)$. For each $s \in G$, we obtain a unitary operator $\lambda_s$ on $\ell_2(G)$ given by

$$(\lambda_s \xi)(t) = \xi(s^{-1}t).$$

We have

$$\lambda_s \lambda_t = \lambda_{st} \text{ and } \lambda^*_s = \lambda_{s^{-1}}.$$

Then $C^*_\lambda(G) = \{ \sum_{s \in G} \alpha_s \lambda_s \}^{-\|\cdot\|}$ is a unital C*-subalgebra of $B(\ell_2(G))$. We call $C^*_\lambda(G)$ the reduced group C*-algebra.

If $G$ is an abelian group, then $C^*_\lambda(G)$ is a unital comm C*-algebra. In this case, each unital *-homomorphism $\varphi : C^*_\lambda(G) \to \mathbb{C}$ uniquely corresponds to a group homomorphism

$$\chi_\varphi : s \in G \to \varphi(\lambda_s) \in \mathbb{T} \subseteq \mathbb{C}.$$

In this case, $\Delta(C^*_\lambda(G))$ is just the dual group $\hat{G} = \{ \chi : G \to \mathbb{T} \}$ all (continuous) characters of $G$. 

• If $G = \mathbb{Z}$, then $\tilde{G} = \mathbb{T}$ and thus

$$C^*_\lambda(\mathbb{Z}) = C(\mathbb{T}).$$

• If $G = \mathbb{Z} \times \mathbb{Z}$, then

$$C^*_\lambda(\mathbb{Z} \times \mathbb{Z}) = C(\mathbb{T} \times \mathbb{T}).$$

• If $G = \mathbb{F}_2$ is the free group of 2-generators, then $C^*_\lambda(\mathbb{F}_2)$ represents a noncommutative topological space.

Suppose that $\mathbb{F}_2$ is the free group with two generators $u$ and $v$. Then $\mathbb{F}_2$ consists of all reduced words: $e$ (empty word), $u, v, u^{-1}, v^{-1}$ (words of length 1), $uu, uv, uv^{-1}, vv, vu, vu^{-1}, u^{-1}u^{-1}, \ldots$ (words of length 2), $\ldots$.

**Question:** How many elements of length $|s| = n$?

Then $\mathbb{F}_2$ is a non-abelian group with multiplication and inverse given by

$$(uvu^{-1})(uvvu) = uvvvu$$
and

$$(uvu^{-1})^{-1} = uv^{-1}u^{-1}.$$ 

The empty word $e$ is the unital element of $\mathbb{F}_2$. 
**Reduced Free Group C*-algebras**

**Theorem [Powers 1975]:** $C^*_\lambda(F_2)$ is a simple C*-algebra, i.e. has no non-trivial closed two-sided ideals.

**Remark:** The simplicity of $C^*_\lambda(F_2)$ means that the corresponding “space” is highly noncommutative.

**Theorem [Pimsner and Voiculescu 1982 ] and [Connes 1986]:** $C^*_\lambda(F_2)$ has no non-trivial projection.

**Remark:** If we have a non-trivial projection $p = \chi_E$ in $C(\Omega)$, then the corresponding set $E$ must be closed and open in $\Omega$. Therefore, $\Omega$ must be disconnected.

Therefore, the above theorem shows that $C^*_\lambda(F_2)$ determines a

“highly noncommutative and connected space.”
**Rotation Algebras**

Let us first recall that we can identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$ via the function $z(t) = e^{2\pi i t}$. We let $H = L_2(\mathbb{T}) = L_2(\mathbb{R}/\mathbb{Z})$.

Let $\theta$ be a real number in $[0, 1)$. We can obtain two unitary operators $U$ and $V$ on $H$ given by

$$U\xi(t) = z(t)\xi(t) \text{ and } V\xi(t) = \xi(t - \theta).$$

A simple calculation shows that

$$UV = e^{2\pi i \theta} VU.$$

Let $A_\theta$ be the universal C*-algebra generated by the unitary operators $\tilde{U}$ and $\tilde{V}$ satisfying the above relation. We call $A_\theta$ the rotation algebra.
If \( \theta = 0 \), we get \( UV = VU \). In this case,

\[ A_0 \cong C(\mathbb{T} \times \mathbb{T}) \]

is a unital commutative C*-algebra.

We are particularly interested in the case when \( \theta \) is irrational.

**Theorem [Rieffel 1981]:** If \( \theta \) is an irrational number, then \( A_\theta \) is a unital simple C*-algebra.

Since \( VU = e^{-2\pi i \theta} UV \), we get \( VU = e^{2\pi i (1-\theta)} UV \), and thus

\[ A_\theta = A_{1-\theta}. \]

However, for distinct irrationals \( \theta \) in \( [0, \frac{1}{2}] \), \( A_\theta \) are all distinct (i.e. non-isomorphic).
CAR Algebra

Let us consider the canonical embeddings

\[ M_2 \hookrightarrow M_2 \otimes M_2 \hookrightarrow M_2 \otimes M_2 \otimes M_2 \hookrightarrow \ldots \]

Take the norm closure, we get a C*-algebra \( A_{2\infty} \), which is called the CAR algebra.

If we consider all projections in the diagonal of \( A_{2\infty} \). These projections generates a unital commutative C*-algebra \( B = C(\Omega) \), where \( \Omega \) is nothing, but the Cantor set.
von Neumann Algebras

Let $H$ be a Hilbert space. We say that a net of operators $\{x_\alpha\}$ converges to $x$ in the **strong operator topology** in $B(H)$ if

$$\|x_\alpha \xi - x \xi\| \to 0 \quad \text{for all } \xi \in H.$$ 

A **von Neumann algebra** on a Hilbert space $H$ is a **strong operator closed** $\ast$-subalgebra $M \subseteq B(H)$. So every von Neumann algebra is a C*-algebra and is a dual space with a unique predual. In general speaking, von Neumann algebras are exactly dual C*-algebras.

Let $(X, \mu)$ be a measure space. Then $L_\infty(X, \mu)$ is a commutative von Neumann algebra on $L_2(X, \mu)$. In fact, every commutative von Neumann algebra $M$ can be written as $M = L_\infty(X, \mu)$.

There is a correspondence between

Measure Spaces $(X, \mu)$

and

Commutative von Neumann Algebras $L_\infty(X, \mu)$
Therefore, we may regard general von Neumann Algebras as Noncommutative Measure Spaces
Examples

Let $G$ be a discrete group. Then the group von Neumann algebra

$$VN_{\lambda}(G) = span\{\lambda_s : s \in G\}^{-s.o.t.}.$$

is a von Neumann algebra.

If $G = \mathbb{Z}$, then $VN_{\lambda}(\mathbb{Z}) = L_{\infty}(\mathbb{T})$.

If $G = \mathbb{Z} \times \mathbb{Z}$, then $VN_{\lambda}(\mathbb{Z} \times \mathbb{Z}) = L_{\infty}(\mathbb{T} \times \mathbb{T})$.

In general, we may regard $VN_{\lambda}(G) \cong L_{\infty}(\hat{G})$ as the duality of $L_{\infty}(G)$.

Here $\hat{G}$ is just a notation to indicate the ‘duality’ of $G$.

There exists a unique normal tracial state $\tau$ on $VN_{\lambda}(G)$ given by

$$\tau(x) = \langle x\delta_e | \delta_e \rangle$$

which corresponding to the canonical Haar measure on $\hat{G}$. 
Hyperfinite $II_1$-Factor

• Consider the canonical embeddings

$$M_2 \hookrightarrow M_2 \otimes M_2 \hookrightarrow M_2 \otimes M_2 \otimes M_2 \hookrightarrow \ldots$$

We may take a “weak closure” and obtain a von Neumann algebra $R_{2\infty}$.

• We can, similarly, consider the von Neumann algebra $R_{3\infty}$ generated by $3 \times 3$ matrices.

It turns out that these von Neumann algebras are equal! They are all hyperfinite $II_1$-factor.

A von Neumann algebra $M$ on a Hilbert space $H$ is called a factor if

$$M \cap M' = \mathbb{C}1,$$

where $M' = \{x \in B(H) : xy = yx, y \in M\}$ is the commutant of $M$. A von Neumann algebra is called hyperfinite if it contains sufficiently many finite dim C*-subalgebras.
Appendix I

Let $G$ be a discrete group. Then $\ell_1(G)$ is a unital involutive Banach algebra with the multiplication given by the convolution

$$f \ast g(t) = \sum_{s \in G} f(s)g(s^{-1}t)$$

and the involution given by

$$f^*(t) = \overline{f(t^{-1})}.$$

Let $\delta_s$ denote the characteristic function at $s$. Then for $s,t \in G$, we have

$$\delta_s \ast \delta_t = \delta_{st}.$$

From this it is easy to see that $\delta_e$ is the unit element of $\ell_1(G)$.

**Theorem:** If $|G| \geq 2$, $\ell_1(G)$ is not a C*-algebra, i.e. it fails to have

$$\|f^* \ast f\|_1 = \|f\|_1^2.$$
Example 1: We can look at $\ell_1(\mathbb{Z})$, and consider $f = \delta_0 + i\delta_1 + \delta_2$. It is easy to see that $\|f\|_1 = 3$. But

$$f^* * f = (\delta_0 - i\delta_{-1} + \delta_{-2}) * (\delta_0 + i\delta_1 + \delta_2) = \delta_{-2} + 3\delta_0 + \delta_2.$$ 

So

$$\|f^* * f\|_1 = 5 < 9 = \|f\|_1^2.$$ 

Example 2: Find a function $f \in \ell_1(\mathbb{Z}_2)$ such that

$$\|f^* * f\|_1 \neq \|f\|_1^2.$$
Appendix II

Let $A$ be a C*-algebra. Then

\[ A_{s.a} = \{ a \in A : a^* = a \} , \]

the space of all selfadjoint operators in $A$, is a real subspace of $A$.

An operator $a \in A$ is **positive** if $a$ is selfadjoint and its spectrum $\sigma(a) \subseteq [0, \infty)$. An operator $a \in A$ is positive if and only if $a = b^*b$ for some $b \in A$. Then $A^+$, the set of all positive operators in $A$, is a proper positive cone in $A_{s.a}$. This defines an order on $A_{s.a}$, i.e. $a \leq b$ if $b - a \geq 0$.

**Theorem:** Every selfadjoint element $a \in A_{s.a}$ can be uniquely decomposed to

\[ a = a^+ - a^- \text{ with } a^+ a^- = 0. \]

**Example:** Let $A = C(\Omega)$. Then $A_{s.a} = C(\Omega, \mathbb{R})$ and $A^+ = C(\Omega, [0, \infty))$.  

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**GNS Representation**

A linear functional $\varphi : A \rightarrow \mathbb{C}$ is **positive** if

$$\varphi : A^+ \rightarrow [0, \infty).$$

Every positive linear functional is bounded with $\|\varphi\| = \varphi(1)$.

**Theorem [Gelfand-Naimark-Segal]:** Let $\varphi : A \rightarrow \mathbb{C}$ be a positive linear functional. There exist a Hilbert space $H_\varphi$, a unital $\ast$-homomorphism $\pi_\varphi : A \rightarrow B(H_\varphi)$, and a vector $\xi_\varphi \in H_\varphi$ such that

$$\varphi(x) = \langle \pi_\varphi(x) \xi_\varphi | \xi_\varphi \rangle.$$

We can choose $H_\varphi$ such that $\pi_\varphi(A) \xi_\varphi$ is norm dense in $H_\varphi$. In this case, we call $(\pi_\varphi, H_\varphi, \xi_\varphi)$ a (cyclic) **GNS representation** of $\varphi$. 
Outline of Proof: First, we can define a semi-inner product on $A$ given by

$$\langle a|b\rangle_\phi = \varphi(b^*a).$$

Let $N_\phi = \{a \in A : \varphi(a^*a) = 0\}$. Then $N_\phi$ is a left ideal of $A$, and the above semi-inner product induces an inner product

$$\langle [a]|[b]\rangle_\phi = \varphi(b^*a) \text{ for } [a], [b] \in A/N_\phi.$$

We let $H_\phi$ denote the norm completion of $A/N_\phi$.

For each $x \in A$, we can define a bounded operator

$$\pi_\phi(x) : [a] \in A/N_\phi \to [xa] \in A/N_\phi$$

with $\|\pi_\phi(x)\| \leq \|x\|$. We use $\pi_\phi(x)$ denote the extension to $H_\phi$. Then

$$\pi_\phi : x \in A \to \pi_\phi(x) \in B(H_\phi).$$

is a unital *-homomorphism. Finally, we let $\xi_\phi = [1] \in H_\phi$ and get

$$\varphi(x) = \varphi(1^*x) = \langle [x]|[1]\rangle_\phi = \langle \pi_\phi(x)\xi_\phi|\xi_\phi\rangle_\phi.$$

The representation is cyclic since $\pi_\phi(A)\xi_\phi = A/N_\phi$ is norm dense in $H_\phi$. 

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Appendix III

Using GNS representation theorem, we can prove Gelfand-Naimark theorem for C*-algebras. The idea is to consider

$$\pi = \bigoplus_\varphi \pi_\varphi : a \in A \rightarrow \bigoplus_\varphi \pi_\varphi(a) \in B(\bigoplus_\varphi H_\varphi),$$

where \( \varphi \) run through all states, i.e. positive linear functional of norm one, on \( A \).
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