STABILITY OF NEAR-RESONANT GRAVITY-CAPILLARY WAVES

Olga Trichtchenko

Department of Applied Mathematics
University of Washington
ota6@uw.edu
Acknowledgements

This is joint work with my advisor Bernard Deconinck (University of Washington).
Funding provided by

NSF-DMS-1008001
Outline

1 Background

2 Solutions

3 Stability
Why consider surface tension and resonance

Henderson and Hammack (1987) looked at instabilities in the presence of surface tension (resonant triads):

- Consider a tank in deep water
- Generate waves at the back of the tank

Examine the frequency of the waves at different points
Why consider surface tension and resonance

Henderson and Hammack (1987) looked at instabilities in the presence of surface tension (resonant triads):

- Consider a tank in deep water
- Generate waves at the back of the tank
- Examine the frequency of the waves at different points

![Temporal wave profiles and corresponding periodograms](image)

**Figure 15.** Temporal wave profiles and corresponding periodograms for Wilton’s ripples (9.8 Hz): $sk_1 = 0.32$, $y = 0.$
Why consider surface tension and resonance

Waves generated at 19.6 Hz excited a harmonic at 9.8 Hz as they propagated

These phenomena are known as Wilton ripples. They are due to the presence of surface tension.
Why consider surface tension and resonance

Waves generated at 19.6 Hz excited a harmonic at 9.8 Hz as they propagated

These phenomena are known as Wilton ripples. They are due to the presence of surface tension.
Why consider surface tension and resonance

Waves generated at 19.6 Hz excited a harmonic at 9.8 Hz as they propagated.

These phenomena are known as Wilton ripples. They are due to the presence of surface tension.
Some Background

The field of water waves has a long history. A few notable and relevant works in this particular area include

- **Wilton (1915)** incorporated *capillary effects* in a *series solution* and showed it diverges for surface tension parameter equal to $1/n$ (for water of infinite depth).

- **Vanden-Broeck et al. (since 1978)** - studied the numerical solutions for solitary and periodic capillary-gravity waves with variable surface tension, including Wilton ripples (1D).

- **Henderson and Hammack (1987)** experimentally observed Wilton ripples in a deep water wave tank.

- **Akers and Gao (2012)** looked at *Wilton ripples in nonlinear model equations* and computed the perturbation series expansions.
The field of water waves has a long history. A few notable and relevant works in this particular area include

- Wilton (1915) incorporated capillary effects in a series solution and showed it diverges for surface tension parameter equal to \( \frac{1}{n} \) (for water of infinite depth).
- Vanden-Broeck et al. (since 1978) - studied the numerical solutions for solitary and periodic capillary-gravity waves with variable surface tension, including Wilton ripples (1D).
- Henderson and Hammack (1987) experimentally observed Wilton ripples in a deep water wave tank.
Some Background

The field of water waves has a long history. A few notable and relevant works in this particular area include

- Wilton (1915) incorporated capillary effects in a series solution and showed it diverges for surface tension parameter equal to $1/n$ (for water of infinite depth).
- Vanden-Broeck et al. (since 1978) - studied the numerical solutions for solitary and periodic capillary-gravity waves with variable surface tension, including Wilton ripples (1D).
- Henderson and Hammack (1987) experimentally observed Wilton ripples in a deep water wave tank.
Some Background

The field of water waves has a long history. A few notable and relevant works in this particular area include:

- Wilton (1915) incorporated capillary effects in a series solution and showed it diverges for surface tension parameter equal to $1/n$ (for water of infinite depth).
- Vanden-Broeck et al. (since 1978) - studied the numerical solutions for solitary and periodic capillary-gravity waves with variable surface tension, including Wilton ripples (1D).
- Henderson and Hammack (1987) experimentally observed Wilton ripples in a deep water wave tank.
Outline

1. Background

2. Solutions

3. Stability
For an inviscid, incompressible fluid with velocity potential $\phi(x, z, t)$

\[
\begin{align*}
\phi_{xx} + \phi_{zz} &= 0, & \quad (x, z) \in D, \\
\phi_z &= 0, & \quad z = -h, \\
\eta_t + \eta_x \phi_x &= \phi_z, & \quad z = \eta(x, t), \\
\phi_t + \frac{1}{2} \left( \phi_x^2 + \phi_z^2 \right) + g \eta &= \sigma \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}, & \quad z = \eta(x, t),
\end{align*}
\]

where $g$: gravity, $\sigma$: coefficient of surface tension, $D$: a periodic domain and $\eta(x, t)$: variable surface (in 1D) with period $L = 2\pi$ and depth $h$. 
Our approach to investigating stability of stationary solutions is a two-step process:

1. Reformulate the problem using the approach by Ablowitz, Fokas and Musslimani and construct solutions for periodic water waves in the travelling frame of reference.

2. Check to see if constructed solutions are spectrally stable by using the Floquet-Fourier-Hill (Bloch) method.
Our approach to investigating stability of stationary solutions is a two-step process:

1. Reformulate the problem using the approach by Ablowitz, Fokas and Musslimani and construct solutions for periodic water waves in the travelling frame of reference.

2. Check to see if constructed solutions are spectrally stable by using the Floquet-Fourier-Hill (Bloch) method.
So far

Gravity waves with and without surface tension are unstable.

Figure: Eigenvalues of the stability problem for gravity waves with no surface tension (in black) and waves with a small coefficient of surface tension (in red).


So far

Gravity waves with and without surface tension are unstable

Figure: Eigenvalues of the stability problem for gravity waves with no surface tension (in black) and waves with a small coefficient of surface tension (in red).

So far

Gravity waves with and without surface tension are *unstable*

![Graph showing eigenvalues for stability problem](image)

**Figure:** Eigenvalues of the stability problem for gravity waves with no surface tension (in black) and waves with a small coefficient of surface tension (in red).


Examine stability of periodic travelling gravity-capillary water waves near resonance.
Starting with Euler’s equations

- Setting \( q(x, t) = \phi(x, \eta(x, t), t) \) (Zakharov, 1968), the kinematic condition and the Bernoulli equation give

\[
q_t + \frac{1}{2} q_x^2 + g \eta - \frac{1}{2} \left( \eta_t + \eta_x q_x \right)^2 = \sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}}.
\]

- Using Laplace’s equation and the boundary conditions,

\[
\int_0^{2\pi} e^{ikx} \left( i \eta_t \cosh(k(\eta + h)) + q_x \sinh(k(\eta + h)) \right) dx = 0,
\]

\( \forall k \in \mathbb{Z}, \ k \neq 0 \).
Starting with Euler’s equations

- Setting \( q(x, t) = \phi(x, \eta(x, t), t) \) (Zakharov, 1968), the kinematic condition and the Bernoulli equation give

\[
q_t + \frac{1}{2} q_x^2 + g \eta - \frac{1}{2} \left( \eta_t + \eta_x q_x \right)^2 = \sigma \frac{\eta_{xx}}{\left( 1 + \eta_x^2 \right)^{3/2}}.
\]

- Using Laplace’s equation and the boundary conditions,

\[
\int_0^{2\pi} e^{ikx} \left( i\eta_t \cosh(k(\eta + h)) + q_x \sinh(k(\eta + h)) \right) dx = 0,
\]

\[\forall k \in \mathbb{Z}, \ k \neq 0.\]
Reformulation

- Switching to the travelling frame by setting \((x, t) \rightarrow (x - ct, t)\).
- Looking at the steady-state problem, set \(\eta_t = q_t = 0\).
- Use the local equation to obtain \(q_x\).
- The non-local equation becomes

\[
\int_0^{2\pi} e^{ikx} \sqrt{(1 + \eta_x^2) \left( c^2 - 2g\eta + 2\sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \right)} \sinh(k(\eta + h)) \, dx = 0
\]

\(\forall k \in \mathbb{Z}, \ k \neq 0\).

How do we solve this?

1. Stokes’ expansion (to see where the resonances are)
2. Numerical continuation employing Newton’s method at each step
Reformulation

- Switching to the travelling frame by setting \((x, t) \rightarrow (x-ct, t)\).
- Looking at the steady-state problem, set \(\eta_t = q_t = 0\).
- Use the local equation to obtain \(q_x\).
- The non-local equation becomes

\[
\int_0^{2\pi} e^{ikx} \sqrt{(1 + \eta_x^2)} \left( c^2 - 2g\eta + 2\sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \right) \sinh(k(\eta + h)) \, dx = 0
\]

\(\forall k \in \mathbb{Z}, \; k \neq 0\).

How do we solve this?

1. Stokes’ expansion (to see where the resonances are)
2. Numerical continuation employing Newton’s method at each step
Reformulation

- Switching to the travelling frame by setting \((x, t) \rightarrow (x-ct, t)\).
- Looking at the steady-state problem, set \(\eta_t = q_t = 0\).
- Use the local equation to obtain \(q_x\).
- The non-local equation becomes

\[
\int_0^{2\pi} e^{ikx} \sqrt{(1 + \eta_x^2)} \left( c^2 - 2g\eta + 2\sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \right) \sinh(k(\eta + h)) \, dx = 0
\]

\[\forall k \in \mathbb{Z}, \ k \neq 0.\]

How do we solve this?

1. Stokes’ expansion (to see where the resonances are)
2. Numerical continuation employing Newton’s method at each step
Reformulation

- Switching to the travelling frame by setting \((x, t) \rightarrow (x-ct, t)\).
- Looking at the steady-state problem, set \(\eta_t = q_t = 0\).
- Use the local equation to obtain \(q_x\).
- The non-local equation becomes

\[
\int_{0}^{2\pi} e^{ikx} \sqrt{(1 + \eta^2_x) \left( c^2 - 2g\eta + 2\sigma \frac{\eta_{xx}}{(1 + \eta^2_x)^{3/2}} \right)} \sinh(k(\eta + h)) \, dx = 0
\]

\(\forall k \in \mathbb{Z}, \; k \neq 0\).

How do we solve this?

1. Stokes’ expansion (to see where the resonances are)
2. Numerical continuation employing Newton’s method at each step
Reformulation

- Switching to the travelling frame by setting \((x, t) \rightarrow (x - ct, t)\).
- Looking at the steady-state problem, set \(\eta_t = q_t = 0\).
- Use the **local equation** to obtain \(q_x\).
- The **non-local equation** becomes

\[
\int_0^{2\pi} e^{ikx} \sqrt{(1 + \eta_x^2) \left(c^2 - 2g\eta + 2\sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}}\right)} \sinh(k(\eta + h)) \, dx = 0
\]

\[\forall k \in \mathbb{Z}, \, k \neq 0.\]

How do we solve this?

1. **Stokes’ expansion** (to see where the resonances are)
2. **Numerical continuation** employing Newton’s method at each step
Reformulation

• Switching to the travelling frame by setting $(x, t) \rightarrow (x-ct, t)$.
• Looking at the steady-state problem, set $\eta_t = q_t = 0$.
• Use the local equation to obtain $q_x$.
• The non-local equation becomes

$$\int_0^{2\pi} e^{ikx} \sqrt{(1 + \eta_x^2) \left( c^2 - 2g\eta + 2\sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \right)} \sinh(k(\eta + h)) dx = 0$$

$\forall k \in \mathbb{Z}, k \neq 0$.

How do we solve this?

1. Stokes’ expansion (to see where the resonances are)
2. Numerical continuation employing Newton’s method at each step
Reformulation

- Switching to the travelling frame by setting \((x, t) \rightarrow (x - ct, t)\).
- Looking at the steady-state problem, set \(\eta_t = q_t = 0\).
- Use the local equation to obtain \(q_x\).
- The non-local equation becomes

\[
\int_0^{2\pi} e^{ikx} \sqrt{(1 + \eta_x^2)} \left( c^2 - 2g\eta + 2\sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \right) \sinh(k(\eta + h)) \, dx = 0
\]

\[\forall k \in \mathbb{Z}, \ k \neq 0.\]

How do we solve this?

1. Stokes’ expansion (to see where the resonances are)
2. Numerical continuation employing Newton’s method at each step
Reformulation

- Switching to the travelling frame by setting \((x, t) \rightarrow (x - ct, t)\).
- Looking at the steady-state problem, set \(\eta_t = q_t = 0\).
- Use the local equation to obtain \(q_x\).
- The non-local equation becomes

\[
\int_0^{2\pi} e^{ikx} \sqrt{(1 + \eta_x^2) \left(c^2 - 2g\eta + 2\sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}}\right)} \sinh(k(\eta + h)) \, dx = 0
\]

\(\forall k \in \mathbb{Z}, \ k \neq 0\).

How do we solve this?

1. Stokes’ expansion (to see where the resonances are)
2. Numerical continuation employing Newton’s method at each step
Stokes’ Expansion

The algorithm is

1. Set

\[ c = \sum_{j=0}^{\infty} \epsilon^j c_j \] and \[ \eta = \sum_{j=0}^{\infty} \epsilon^j \eta_j \]

2. Substitute into

\[ \int_{0}^{2\pi} e^{ikx} \sqrt{(1 + \eta_x^2)} \left( c^2 - 2g\eta + \frac{2\sigma\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \right) \sinh(k(\eta + h)) \, dx = 0 \]

3. Group terms by order of \( \epsilon^n \)

4. Solve the recursion relation such that

\[ c_n = f_1(c_{n-1}, c_{n-2}, \ldots, c_0) \] and \[ \eta_n = f_2(\eta_{n-1}, \eta_{n-2}, \ldots, \eta_0) \]

very messy, but explicit!
Stokes’ Expansion

The algorithm is

1. Set

\[
c = \sum_{j=0}^{\infty} \epsilon^j c_j \quad \text{and} \quad \eta = \sum_{j=0}^{\infty} \epsilon^j \eta_j
\]

2. Substitute into

\[
\int_0^{2\pi} e^{ikx} \sqrt{(1 + \eta_x^2) \left( c^2 - 2g\eta + \frac{2\sigma\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \right) \sinh(k(\eta + h))} \, dx = 0
\]

3. Group terms by order of \( \epsilon^n \)

4. Solve the recursion relation such that

\[
c_n = f_1(c_{n-1}, c_{n-2}, \ldots, c_0) \quad \text{and} \quad \eta_n = f_2(\eta_{n-1}, \eta_{n-2}, \ldots, \eta_0)
\]

very messy, but explicit!
Stokes’ Expansion

The algorithm is

1. Set

\[
c = \sum_{j=0}^{\infty} \epsilon^j c_j \quad \text{and} \quad \eta = \sum_{j=0}^{\infty} \epsilon^j \eta_j
\]

2. Substitute into

\[
\int_{0}^{2\pi} e^{ikx} \sqrt{(1 + \eta_x^2) \left( c^2 - 2g\eta + \frac{2 \sigma \eta_{xx}}{(1 + \eta_x^2)^{3/2}} \right) \sinh(k(\eta + h))} \, dx = 0
\]

3. Group terms by order of \( \epsilon^n \)

4. Solve the recursion relation such that

\[
c_n = f_1(c_{n-1}, c_{n-2}, \ldots, c_0) \quad \text{and} \quad \eta_n = f_2(\eta_{n-1}, \eta_{n-2}, \ldots, \eta_0)
\]

very messy, but explicit!
Stokes’ Expansion

The algorithm is

1. Set

\[ c = \sum_{j=0}^{\infty} \epsilon^j c_j \text{ and } \eta = \sum_{j=0}^{\infty} \epsilon^j \eta_j \]

2. Substitute into

\[
\int_{0}^{2\pi} e^{ikx} \sqrt{1 + \eta_x^2} \left( c^2 - 2g\eta + \frac{2\sigma\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \right) \sinh(k(\eta + h)) \, dx = 0
\]

3. Group terms by order of \( \epsilon^n \)

4. Solve the recursion relation such that

\[ c_n = f_1(c_{n-1}, c_{n-2}, \ldots, c_0) \text{ and } \eta_n = f_2(\eta_{n-1}, \eta_{n-2}, \ldots, \eta_0) \]

very messy, but explicit!
Stokes’ Expansion

The algorithm is

1. Set

\[ c = \sum_{j=0}^{\infty} \epsilon^j c_j \quad \text{and} \quad \eta = \sum_{j=0}^{\infty} \epsilon^j \eta_j \]

2. Substitute into

\[ \int_0^{2\pi} e^{ikx} \sqrt{\left(1 + \eta_x^2\right) \left(c^2 - 2g\eta + \frac{2\sigma\eta_{xx}}{(1 + \eta_x^2)^{3/2}}\right)} \sinh(k(\eta + h)) \, dx = 0 \]

3. Group terms by order of \( \epsilon^n \)

4. Solve the recursion relation such that

\[ c_n = f_1(c_{n-1}, c_{n-2}, \ldots, c_0) \quad \text{and} \quad \eta_n = f_2(\eta_{n-1}, \eta_{n-2}, \ldots, \eta_0) \]

very messy, but explicit!
Stokes’ Expansion

The algorithm is

1. Set

\[ c = \sum_{j=0}^{\infty} \epsilon^j c_j \text{ and } \eta = \sum_{j=0}^{\infty} \epsilon^j \eta_j \]

2. Substitute into

\[
\int_{0}^{2\pi} e^{ikx} \sqrt{(1 + \eta_x^2) \left( c^2 - 2g\eta + \frac{2\sigma\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \right)} \sinh(k(\eta + h)) \, dx = 0
\]

3. Group terms by order of \( \epsilon^n \)

4. Solve the recursion relation such that

\[ c_n = f_1(c_{n-1}, c_{n-2}, \ldots, c_0) \text{ and } \eta_n = f_2(\eta_{n-1}, \eta_{n-2}, \ldots, \eta_0) \]

very messy, but explicit!
In infinite depth \((h = \infty)\), obtain

\[c_0 = \sqrt{1 + \sigma}\]
\[c_1 = 0\]
\[c_2 = -\frac{2\sigma^2 + \sigma + 8}{4(1 + \sigma)^{1/2}(2\sigma - 1)}\]
\[c_3 = 0\]
\[\vdots\]

\[\eta_0 = 0\]
\[\eta_1 = 2\cos(x)\]
\[\eta_2 = -\frac{2(1 + \sigma)}{2\sigma - 1}\cos(2x)\]
\[\eta_3 = \frac{3}{2}\frac{2\sigma^2 + 7\sigma + 2}{(3\sigma - 1)(2\sigma - 1)}\cos(3x)\]
\[\vdots\]

Note: blow up if \(\sigma = \frac{1}{n}\)
Resonance Condition

Isolating for the coefficient of surface elevation in finite depth, we get the following:

\[
(c_0^2 - (\sigma k^2 + g) \tanh(kh)) \hat{\eta}_k = "a mess"
\]

Resonance if

\[
\sigma = \frac{g}{k} \left( \frac{\tanh(hk) - k \tanh(h)}{\tanh(h) - k \tanh(hk)} \right) \quad \text{with} \quad k \in \mathbb{Z}
\]

Near resonance (small divisor problem) if

\[
c_0^2 - (\sigma k^2 + g) \tanh(kh) \approx 0 \quad \text{with} \quad c_0 = \sqrt{(\sigma + g) \tanh(h)}
\]

Fix \(g\) and \(h\), solve for \(\sigma\) with a variety of \(k\) values near 20 or near 10.
Isolating for the coefficient of surface elevation in finite depth, we get the following:

\[
\left( c_0^2 - (\sigma k^2 + g) \tanh(kh) \right) \hat{\eta}_k = "a mess"
\]

Resonance if

\[
\sigma = \frac{g}{k} \left( \frac{\tanh(hk) - k \tanh(h)}{\tanh(h) - k \tanh(hk)} \right) \quad \text{with} \ k \in \mathbb{Z}
\]

Near resonance (small divisor problem) if

\[
\frac{c_0^2}{k} - (\sigma k^2 + g) \tanh(kh) \approx 0 \quad \text{with} \ c_0 = \sqrt{(\sigma + g) \tanh(h)}
\]

Fix \( g \) and \( h \), solve for \( \sigma \) with a variety of \( k \) values near 20 or near 10.
Resonance Condition

Isolating for the coefficient of surface elevation in finite depth, we get the following:

\[
(c_0^2 - (\sigma k^2 + g) \tanh(kh)) \hat{\eta}_k = "a mess"
\]

Resonance if

\[
\sigma = \frac{g}{k} \left( \frac{\tanh(hk) - k \tanh(h)}{\tanh(h) - k \tanh(hk)} \right) \quad \text{with } k \in \mathbb{Z}
\]

Near resonance (small divisor problem) if

\[
c_0^2 - (\sigma k^2 + g) \tanh(kh) \approx 0 \quad \text{with } c_0 = \sqrt{(\sigma + g) \tanh(h)}
\]

Fix \( g \) and \( h \), solve for \( \sigma \) with a variety of \( k \) values near 20 or near 10.
Numerical Continuation

Recall

\[
\int_0^{2\pi} e^{ikx} \sqrt{\left(1 + \eta_x^2 \right)} \left( c^2 - 2g\eta + 2\sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \right) \sinh(k(\eta + h)) \, dx = 0.
\]

We want to generate a bifurcation diagram:

1. Assume in general \( \eta_N(x) = \sum_{j=1}^{N} a_j \cos(jx). \)

2. Linearizing we can find the bifurcation will start when \( c = \sqrt{(g + \sigma)} \tanh(h) \) and \( \eta(x) = a \cos(x). \)

3. Use this guess in Newton’s method to compute the true solution.

4. Scale the previous solution to get a guess for the new bifurcation parameter.

5. Apply Newton’s method to find the solution.
Recall

\[
\int_0^{2\pi} e^{ikx} \sqrt{(1 + \eta_x^2)} \left( c^2 - 2g\eta + 2\sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \right) \sinh(k(\eta + h)) \, dx = 0.
\]

We want to generate a bifurcation diagram:

1. Assume in general \( \eta_N(x) = \sum_{j=1}^{N} a_j \cos(jx) \).

2. Linearizing we can find the bifurcation will start when \( c = \sqrt{(g + \sigma) \tanh(h)} \) and \( \eta(x) = a \cos(x) \).

3. Use this guess in Newton’s method to compute the true solution.

4. Scale the previous solution to get a guess for the new bifurcation parameter.

5. Apply Newton’s method to find the solution.
Numerical Continuation

Recall

\[ \int_{0}^{2\pi} e^{ikx} \sqrt{(1 + \eta_x^2) \left( c^2 - 2g\eta + 2\sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \right)} \sinh(k(\eta + h)) \, dx = 0. \]

We want to generate a bifurcation diagram:

1. **Assume** in general \( \eta_N(x) = \sum_{j=1}^{N} a_j \cos(jx) \).

2. **Linearizing** we can find the bifurcation will start when \( c = \sqrt{(g + \sigma) \tanh(h)} \) and \( \eta(x) = a \cos(x) \).

3. Use this guess in Newton’s method to **compute** the true solution.

4. **Scale** the previous solution to get a guess for the new bifurcation parameter.

5. Apply Newton’s method to **find** the solution.
Near Resonant Solutions - near $k = 20$

Let $h = 0.05$ and compute $\sigma$ for $k = 20.5$

Figure: Bifurcation branch

Figure: Physical profile of the wave

Figure: Fourier coefficients of the profile
Near Resonant Solutions - near $k = 20$

Let $h = 0.05$ and compute $\sigma$ for $k = 20.05$

**Figure:** Bifurcation branch

**Figure:** Physical profile of the wave

**Figure:** Fourier coefficients of the profile
Near Resonant Solutions - near $k = 10$

Let $h = 0.05$ and compute $\sigma$ for $k = 10.5$

**Figure:** Bifurcation branch

**Figure:** Physical profile of the wave

**Figure:** Fourier coefficients of the profile
Near Resonant Solutions - near $k = 10$

Let $h = 0.05$ and compute $\sigma$ for $k = 10.05$.
Comparisons of Profiles - near $k = 20$

$$\sigma \approx 7.80 \times 10^{-4} \ (k = 20.5)$$

$$\sigma \approx 7.82 \times 10^{-4} \ (k = 20.05)$$
Comparisons of Profiles - near $k = 10$

$\sigma \approx 8.18 \times 10^{-4}$ ($k = 10.5$)  \hspace{1cm} $\sigma \approx 8.19 \times 10^{-4}$ ($k = 10.05$)
Outline

1 Background

2 Solutions

3 Stability
Recall the local equation

\[ q_t - cq_x + \frac{1}{2} q_x^2 + g\eta - \frac{1}{2} \left( \eta_t - c\eta_x + q_x\eta_x \right)^2 = \sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \]

and the nonlocal equation

\[ \int_0^{2\pi} e^{ikx} \left[ i(\eta_t - c\eta_x) \cosh(k(\eta + h)) + q_x \sinh(k(\eta + h)) \right] dx = 0. \]

1. Let \( q(x, t) = q_0(x) + cq_1(x)e^{\lambda t} + \ldots \text{ and } \eta(x) = \eta_0(x) + c\eta_1(x)e^{\lambda t} + \ldots. \)
2. Using Floquet decompositions, we set \( \eta_1 = e^{i\mu x} \tilde{\eta}_1 \text{ and } q_1 = e^{i\mu x} \tilde{q}_1. \)
3. Apply Fourier decomposition with \( \tilde{\eta}_1 = \sum_{m=-\infty}^{\infty} \tilde{N}_m e^{imx} \text{ and } \tilde{q}_1 = \sum_{m=-\infty}^{\infty} \tilde{Q}_m e^{imx}. \)
4. To allow perturbations of a different period, introduce spatial averaging in the nonlocal equation.
Stability Eigenvalue Problem

Recall the local equation

\[ q_t - cq_x + \frac{1}{2} q_x^2 + g\eta - \frac{1}{2} \frac{\left( \eta_t - c\eta_x + q_x\eta_x \right)^2}{1 + \eta_x^2} = \sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \]

and the nonlocal equation

\[
\int_{0}^{2\pi} e^{ikx} \left[ i(\eta_t - c\eta_x) \cosh(k(\eta + h)) + q_x \sinh(k(\eta + h)) \right] dx = 0.
\]

1. Let \( q(x, t) = q_0(x) + \epsilon q_1(x)e^{\lambda t} + \ldots \) and \( \eta(x) = \eta_0(x) + \epsilon \eta_1(x)e^{\lambda t} + \ldots \).

2. Using Floquet decompositions, we set \( \eta_1 = e^{i\mu x} \tilde{\eta}_1 \) and \( q_1 = e^{i\mu x} \tilde{q}_1 \).

3. Apply Fourier decomposition with \( \tilde{\eta}_1 = \sum_{m=-\infty}^{\infty} \hat{N}_m e^{imx} \) and \( \tilde{q}_1 = \sum_{m=-\infty}^{\infty} \hat{Q}_m e^{imx} \).

4. To allow perturbations of a different period, introduce spatial averaging in the nonlocal equation.
Recall the local equation

\[ q_t - cq_x + \frac{1}{2} q_x^2 + g\eta - \frac{1}{2} \left( \frac{\eta_t - c\eta_x + q_x\eta_x}{1 + \eta_x^2} \right)^2 = \sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \]

and the nonlocal equation

\[
\int_0^{2\pi} e^{ikx} \left[ i(\eta_t - c\eta_x) \cosh(k(\eta + h)) + q_x \sinh(k(\eta + h)) \right] dx = 0.
\]

1. Let \( q(x, t) = q_0(x) + \epsilon q_1(x) e^{\lambda t} + \ldots \) and \( \eta(x) = \eta_0(x) + \epsilon \eta_1(x) e^{\lambda t} + \ldots \).

2. Using Floquet decompositions, we set \( \eta_1 = e^{i\mu x} \tilde{\eta}_1 \) and \( q_1 = e^{i\mu x} \tilde{q}_1 \).

3. Apply Fourier decomposition with \( \tilde{\eta}_1 = \sum_{m=-\infty}^{\infty} \hat{N}_m e^{imx} \) and \( \tilde{q}_1 = \sum_{m=-\infty}^{\infty} \hat{Q}_m e^{imx} \).

4. To allow perturbations of a different period, introduce spatial averaging in the nonlocal equation.
Recall the local equation
\[ q_t - c q_x + \frac{1}{2} q_x^2 + g \eta - \frac{1}{2} \left( \eta_t - c \eta_x + q_x \eta_x \right)^2 = \sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \]
and the nonlocal equation
\[ \int_0^{2\pi} e^{ikx} \left[ i(\eta_t - c \eta_x) \cosh(k(\eta + h)) + q_x \sinh(k(\eta + h)) \right] dx = 0. \]

1. Let \( q(x, t) = q_0(x) + \epsilon q_1(x)e^{\lambda t} + \ldots \) and \( \eta(x) = \eta_0(x) + \epsilon \eta_1(x)e^{\lambda t} + \ldots \).
2. Using Floquet decompositions, we set \( \eta_1 = e^{i\mu x} \tilde{\eta}_1 \) and \( q_1 = e^{i\mu x} \tilde{q}_1 \).
3. Apply Fourier decomposition with \( \tilde{\eta}_1 = \sum_{m=-\infty}^{\infty} \hat{N}_m e^{imx} \) and \( \tilde{q}_1 = \sum_{m=-\infty}^{\infty} \hat{Q}_m e^{imx} \).
4. To allow perturbations of a different period, introduce spatial averaging in the nonlocal equation.
Recall the local equation
\[ q_t - cq_x + \frac{1}{2} q_x^2 + g \eta - \frac{1}{2} \left( \eta_t - c \eta_x + q_x \eta_x \right)^2 = \sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \]
and the nonlocal equation
\[
\int_0^{2\pi} e^{ikx} \left[ i(\eta_t - c\eta_x) \cosh(k(\eta + h)) + q_x \sinh(k(\eta + h)) \right] dx = 0.
\]

1. Let \( q(x, t) = q_0(x) + \epsilon q_1(x) e^{\lambda t} + \ldots \) and \( \eta(x) = \eta_0(x) + \epsilon \eta_1(x) e^{\lambda t} + \ldots \).
2. Using Floquet decompositions, we set \( \eta_1 = e^{i\mu x} \tilde{\eta}_1 \) and \( q_1 = e^{i\mu x} \tilde{q}_1 \).
3. Apply Fourier decomposition with \( \tilde{\eta}_1 = \sum_{m=-\infty}^\infty \hat{N}_m e^{imx} \) and \( \tilde{q}_1 = \sum_{m=-\infty}^\infty \hat{Q}_m e^{imx} \).
4. To allow perturbations of a different period, introduce spatial averaging in the nonlocal equation.
After all the substitutions, obtain

$$\Rightarrow \begin{bmatrix} S & T \\ U & V \end{bmatrix} \begin{pmatrix} \hat{N} \\ \hat{Q} \end{pmatrix} = \lambda \begin{bmatrix} A & I \\ C & 0 \end{bmatrix} \begin{pmatrix} \hat{N} \\ \hat{Q} \end{pmatrix}$$

The local equation gives the row in blue and the nonlocal equation gives the row in green.

**Generalized eigenvalue problem**

$$\lambda = \lambda(\mu, m, \sigma)$$

The problem is Hamiltonian and due to symmetries,

$$\mathbb{R}\{\lambda\} \neq 0 \Rightarrow \text{instability.}$$
After all the substitutions, obtain

$$\Rightarrow \begin{bmatrix} S & T \\ U & V \end{bmatrix} \begin{pmatrix} \hat{N} \\ \hat{Q} \end{pmatrix} = \lambda \begin{bmatrix} A & I \\ C & 0 \end{bmatrix} \begin{pmatrix} \hat{N} \\ \hat{Q} \end{pmatrix}$$

The local equation gives the row in blue and the nonlocal equation gives the row in green.

Generalized eigenvalue problem

$$\lambda = \lambda(\mu, m, \sigma)$$

The problem is Hamiltonian and due to symmetries,

$$\mathbb{R}\{\lambda\} \neq 0 \Rightarrow \text{instability.}$$
Instability

For flat water, can compute the eigenvalues explicitly

$$\lambda_{\mu+m}^{\pm} = ic(\mu + m) \pm i \sqrt{[g(\mu + m) + \sigma(\mu + m)^3] \tanh((\mu + m)h)}$$

⇒ flat water is spectrally stable

How does an instability arise?

- Eigenvalues are continuous with respect to the wave amplitude
- As amplitude increases they may develop a non-zero real part

A necessary condition for loss of stability is

$$\lambda_{\mu}^{\pm} = \lambda_{\mu+m}^{\pm}$$
Instability

For flat water, can compute the eigenvalues explicitly

\[ \lambda_{\mu+m}^{\pm} = ic(\mu + m) \pm i \sqrt{[g(\mu + m) + \sigma(\mu + m)^3] \tanh((\mu + m)h)} \]

⇒ flat water is spectrally stable

How does an instability arise?

- Eigenvalues are continuous with respect to the wave amplitude
- As amplitude increases they may develop a non-zero real part

A necessary condition for loss of stability is

\[ \lambda_{\mu}^{\pm} = \lambda_{\mu+m}^{\pm} \]
For flat water, can compute the eigenvalues explicitly

$$\lambda_{\mu+m}^{\pm} = ic(\mu + m) \pm i \sqrt{g(\mu + m) + \sigma(\mu + m)^3} \tanh((\mu + m)h)$$

$$\Rightarrow \text{flat water is spectrally stable}$$

How does an instability arise?

- Eigenvalues are continuous with respect to the wave amplitude
- As amplitude increases they may develop a non-zero real part

A necessary condition for loss of stability is

$$\lambda_{\mu}^{\pm} = \lambda_{\mu+m}^{\pm}$$
Instability

For flat water, can compute the eigenvalues explicitly

$$\lambda_{\mu+m}^{\pm} = ic(\mu + m) \pm i \sqrt{g(\mu + m) + \sigma(\mu + m)^3} \tanh((\mu + m)h)$$

$$\Rightarrow$$ flat water is spectrally stable

How does an instability arise?

- Eigenvalues are continuous with respect to the wave amplitude
- As amplitude increases they may develop a non-zero real part

A necessary condition for loss of stability is

$$\lambda_{\mu}^{\pm} = \lambda_{\mu+m}^{\pm}$$
Instabilities near $k = 20$

**Figure:** Wave profile

**Figure:** Eigenvalues in the complex plane
Instabilities near $k = 20$

**Figure:** Wave profile

**Figure:** Eigenvalues in the complex plane
Instabilities near $k = 10$

Figure: Wave profile

Figure: Eigenvalues in the complex plane
Instabilities near $k = 10$

**Figure:** Wave profile

**Figure:** Eigenvalues in the complex plane
Conclusions

- Solutions can be computed near resonance.
- A larger coefficient of surface tension does not stabilize the solutions.
- As the parameter of surface tension gets larger, the waves become more unstable.
Future Work

- Compute solutions to a higher precision (quadruple precision, with Jon Wilkening at Berkeley).
- Compute the stability spectra for more values of the Floquet parameter.
- Track the new instabilities along the bifurcation branch.
- Track the instabilities as the surface tension parameter is varied.
- Examine the form of the perturbations that lead to the new instabilities.
THANK YOU FOR YOUR ATTENTION