Kinematic Vortices in a Thin Film Driven by an Applied Current

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Joint work with Lydia Peres Hari and Jacob Rubinstein
Technion
Consider a thin film superconductor subjected to an applied current of magnitude $I$ (fed through the sides) and a perpendicular applied magnetic field of magnitude $h$. 

Magnetic Field = $h \hat{z}$
Goal: Understanding anomalous vortex behavior


However, experiments and numerics based on a Ginzburg-Landau type model reveal unexpected behavior in the present setting.

- Oscillatory (periodic) behavior characterized by oppositely 'charged' vortex pairs either
  - nucleating inside the sample and then exiting on opposite sides
  - entering the sample on opposite sides and ultimately colliding and annihilating each other in the middle.

- Vortex emergence even with zero magnetic field: "Kinematic vortices" Andronov, Gordion, Kurin, Nefedov, Shereshevsky '93, Berdiyorov, Elmurodov, Peeters, Vodolazov, Milosevic '09, Du '03
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Ginzburg-Landau formulation of problem

\[ \psi_t + i\phi \psi = (\nabla - ihA_0)^2 \psi + (\Gamma - |\psi|^2)\psi \text{ for } (x, y) \in \mathcal{R}, \ t > 0, \]
\[ \Delta \phi = \nabla \cdot \left( \frac{i}{2} \{ \psi \nabla \psi^* - \psi^* \nabla \psi \} - |\psi|^2 hA_0 \right) \text{ for } (x, y) \in \mathcal{R}, \ t > 0, \]

where \( \mathcal{R} = [-L, L] \times [-K, K], \ A_0 = (-y, 0) \) and \( \Gamma > 0 \) prop. to \( T_c - T \).
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**Boundary conditions for \( \Psi \):**

\[
\Psi(\pm L, y, t) = 0 \text{ for } |y| < \delta,
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(\nabla - ihA_0) \Psi \cdot \mathbf{n} = 0 \text{ elsewhere on } \partial \mathcal{R}.
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\[ \phi_x(\pm L, y, t) = \begin{cases} -1 & \text{for } |y| < \delta, \\ 0 & \text{for } \delta < |y| < K, \end{cases} \]

\[ \phi_y(x, \pm K, t) = 0 \text{ for } |x| \leq L. \]
Rigorous bifurcation from normal state

**Normal State**: At high temp. (Γ small) and/or large magnetic field or electric current, expect to see no superconductivity:

\[ \Psi \equiv 0, \quad \phi = I \phi^0 \]

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Note: One easily checks that \( \phi^0 \) is odd in \( x \) and even in \( y \):

\[ \phi^0(-x, y) = -\phi^0(x, y) \quad \text{and} \quad \phi^0(x, -y) = \phi^0(x, y). \]
Linearization about Normal State:

\[ \psi_t = \mathcal{L}[\psi] + \Gamma \psi \quad \text{in } \mathcal{R}, \]

where

\[ \mathcal{L}[\psi] := (\nabla - i\hbar A_0)^2 \psi - i\Im \phi^0 \psi. \]

subject to boundary conditions

\[ \psi(\pm L, y, t) = 0 \text{ for } |y| < \delta, \]

\[ (\nabla - i\hbar A_0) \psi \cdot \mathbf{n} = 0 \quad \text{elsewhere on } \partial \mathcal{R}, \]

\[ \mathcal{L} = \text{Imaginary perturbation of (self-adjoint) magnetic Schrödinger operator}. \]
Spectral Properties of $\mathcal{L}$

Note that $\mathcal{L}$, and hence its spectrum, depend on $L, K, \delta, h$ and $I$. 
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- Spectrum of $\mathcal{L}$ consists only of point spectrum:

$$\mathcal{L}[u_j] = -\lambda_j u_j \quad \text{in } \mathcal{R} \quad \text{+ boundary cond.'s, } j = 1, 2, \ldots$$

with $0 < \Re \lambda_1 \leq \Re \lambda_2 \leq \ldots$, and $|\Im \lambda_j| < \|\phi^0\|_{L^\infty} I$
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Hence, if $(\lambda_j, u_j)$ is an eigenpair then so is $(\lambda_j^\ast, u_j^\dagger)$ where

$$u_j^\dagger(x, y) := u_j^\ast(-x, y).$$

If $\lambda_j$ is real, then $u_j = u_j^\dagger$, and indeed each $\lambda_j$ is real for $I$ small.
Collisions of first 4 eigenvalues for $L = 1$, $K = 2/3$, $\delta = 1/6$, $h = 0$. 
Tuning the temperature to capture bifurcation

From now on, fix $I > I_c$ so that $\text{Im} \lambda_1 \neq 0$. 
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Going back to linearized problem

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Set $\mathcal{L}_1 := \mathcal{L} + \text{Re} \lambda_1$, so that bottom of spectrum of $\mathcal{L}_1$ consists of purely imaginary eigenvalues:

$$\pm \text{Im} \lambda_1 i,$$

followed by eigenvalues having negative real part.
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To capture this (Hopf) bifurcation we take

$$\Gamma = \text{Re} \lambda_1 + \varepsilon \quad \text{for } 0 < \varepsilon \ll 1.$$
Formulation as a single nonlocal PDE:

With the choice $\Gamma = \text{Re} \lambda_1 + \varepsilon$ for $0 < \varepsilon \ll 1$, full problem then takes the form of a single nonlinear, nonlocal PDE:

$$\Psi_t = \mathcal{L}_1[\Psi] + \varepsilon \Psi + \mathcal{N}(\Psi),$$

where

$$\mathcal{N}(\Psi) := -|\Psi|^2 \Psi - i\tilde{\phi}[\Psi] \Psi,$$

with $\tilde{\phi} = \tilde{\phi}[\Psi]$ solving

$$\Delta \tilde{\phi} = \nabla \cdot \left( \frac{i}{2} \{\Psi \nabla \Psi^* - \Psi^* \nabla \Psi\} - |\Psi|^2 hA_0 \right) \quad \text{in } \mathcal{R}$$

along with homogeneous boundary conditions on $\Psi$ and $\tilde{\phi}$. 
There exists a value $\varepsilon_0 > 0$ such that for all positive $\varepsilon < \varepsilon_0$, the system undergoes a supercritical Hopf bifurcation to a periodic state $(\psi_\varepsilon, \phi_\varepsilon)$.

One has the estimate

$$\|\psi_\varepsilon - \left( a^\varepsilon(t)u_1 + a^\varepsilon(t)^* u_1^\dagger \right) \|_{H^2(\mathcal{R})} \leq C\varepsilon^{3/2}$$

with

$$a^\varepsilon(t) := C_0\varepsilon^{1/2} e^{-i\chi t} \text{ where } \chi = \text{Im } \lambda_1 + \gamma \varepsilon$$

and $C_0$ and $\gamma$ are constants depending on certain integrals of $u_1$.

Generalization of techniques from 1d problem by J.R., S. and K. Zumbrun.
A key element of the proof: Exploiting PT symmetry on center manifold.

• For each $\varepsilon$ small, there exists a graph $\Phi^\varepsilon : S \to H^2(\mathcal{R}; \mathbb{C})$ over center subspace $S := \text{Span}\{u_1, u_2\}$ and complex-valued functions $\alpha_1(t), \alpha_2(t)$ such that (for small initial data) solution to TDGL $\psi_\varepsilon$ describable as

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• Projection onto \( S \) leads to dynamical system for \( \alpha_1 \) and \( \alpha_2 \). Four real equations in four unknowns.
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- Projection onto $S$ leads to dynamical system for $\alpha_1$ and $\alpha_2$. Four real equations in four unknowns.
- One proves exponential attraction to PT-symmetric subset of center manifold.

$$\alpha_1(t)u_1 + \alpha_2(t)u_2 = (\alpha_1(t)u_1 + \alpha_2(t)u_2)^\dagger \iff \alpha_2 = \alpha_1^*.$$ 

Easy system for $\alpha_1$—explicitly solvable.
A kinematic vortex motion law

According to theorem, the leading order term \((O(\varepsilon^{1/2}))\) is:

\[
\psi = a^\varepsilon(t)u_1 + a^\varepsilon(t)^*u_1^\dagger \quad \text{with} \quad a^\varepsilon(t) = C_0 \varepsilon^{1/2} e^{-i \chi t}.
\]

Focusing our attention along the center line \(x = 0\) and writing

\[
u_1(0, y) = |u_1(0, y)| e^{i\beta(y)} \quad \text{for some phase} \ \beta(y)
\]

we find that

\[
\psi(0, y, t) = 2C_0 \varepsilon^{1/2} |u_1(0, y)| \cos (\beta(y) - \chi t).
\]

Hence, the order parameter vanishes on the center line \(x = 0\) whenever the equation

\[
\chi t = \beta(y) + \pi/2 + n\pi, \quad n = 0, \pm 1, \pm 2, \ldots
\]

is satisfied. Recall that \(\chi = \text{Im} \lambda_1 + o(1)\).
Using shape of \( \beta \) to explain anomalous vortex behavior

Case 1: No magnetic field, \( h = 0 \). Recall that \( \beta = \text{phase of} \ u_1(0, y) \). Numerical computations reveal sensitive dependence on \( I \).

Here \( L = 1, K = 2/3, \delta = 4/15 \). Note symmetry of \( \beta \).
Case 2: Graphs of $\beta$ when magnetic field present: $h > 0$.

Symmetry broken so vortices enter/exit boundaries $y = K$ and $y = -K$ at different times. Here we have taken $h = 0.05$. 
Remarks on numerical experiments

• When magnetic field strength $h$ is small, one only sees vortices on the center line (kinematic).

\[ \nabla - i h A_0 \frac{d^2}{d x^2} - i I \phi_0 u_1 = -\lambda_1 u_1 \]
Remarks on numerical experiments

- When magnetic field strength $h$ is small, one only sees vortices on the center line (kinematic).

- As $h$ increases, many new effects:
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  (i) vortices enter/exit the top and bottom at different times.
  (ii) some vortices move along and then slightly off center line (in a periodic manner)
  (iii) ‘magnetic vortices’ appear far from center line, presumably associated with vortices of ground-state $u_1$ of perturbed magnetic Schrödinger operator

\[
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Conclusions

• Through a rigorous center manifold approach we have identified a Hopf bifurcation from the normal state to stable periodic solutions.
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• The creation and motion of ‘kinematic vortices’ moving along the center line $x = 0$ traced to PT symmetry and nature of first eigenfunction $u_1$ of linear operator along this line.

• Anomalous vortex behavior explained through sensitive dependence of shape of phase of $u_1(0, y)$ on the value of applied current $I$.

• When magnetic field $h$ is large enough, one sees motion of both ‘magnetic vortices’ off the center line and ‘kinematic vortices’ on or near the center line.

• What happens deep in the nonlinear regime? (No longer small amplitude)
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