Overlapping Patches for Dynamic Surface Problems

C. Carlo Fazioli

Drexel University

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A collection of mappings \( \mathbf{X}_i(\alpha, \beta, t) : \mathbb{R}^2 \rightarrow S \) from the plane onto the free surface.

Associated partition of unity \( \{ \psi_i \}, \sum \psi_i = 1 \) on \( S \).

Advantages:
- adaptivity
- complex surfaces
- isothermal coordinates
Time Evolution of Patches

Preserve partition of unity by preserving on normal lines:

$$\psi_t = X_t \cdot \nabla_s \psi$$

Preserve physical quantities on material lines:

$$\mu_t = -U \cdot \nabla_s \mu$$

Upwind considerations require:

$$(\psi \mu)_t = \mu \psi_t + \psi \mu_t = \mu X_t \cdot \nabla_s \psi - \psi U \cdot \nabla_s \mu$$

With reconstruction as:

$$\mu = (\sum \psi_i) \mu = \sum (\psi_i \mu)$$
Interpolation

At a point $X^*$ on one patch, need the value of $\Psi_\mu$ from other patches.

$X^*$ has unknown preimage

$\alpha = \alpha_{ij} + \Delta \alpha$.

Once $\alpha$ is known, can easily interpolate $\Psi_\mu$ there (say, bicubic).
Physical Problem

Vortex sheet \((S)\) motion in ideal flow:

\[
U_t^\pm + U^\pm \cdot \nabla U^\pm + \nabla p = 0 \quad \text{in } D^\pm
\]

\[
\nabla \cdot U^\pm = 0 \quad \text{in } D^\pm
\]

\[
\nabla \times U^\pm = 0 \quad \text{on } S
\]

\[
U^+ \cdot n = U^- \cdot n \quad \text{on } S
\]

Vortex sheet with strength \(\mu\) induces vector potential

\[
A(x) = \frac{1}{4\pi} \int_S \mu(x') n(x') \times \nabla_{x'} \left( \frac{1}{|x - x'|} \right) dS(x')
\]

Physical velocity:

\[
U \cdot n = (n \cdot \nabla \times A) n = [(A \cdot T_1)_2 - (A \cdot T_2)_1] n
\]
Kernel Smoothing

Regularize the fundamental solution to $G_\delta$

$$G_\delta = -\frac{1}{4\pi} \frac{\text{erf}(r/\delta)}{r} = G(x)\text{erf}(r/\delta)$$
Correction Terms

\[ \int - \sum \delta = \left( \int - \int_{\delta} \right) + \left( \int_{\delta} - \sum \delta \right) \]

Regularization correction:

\[ \epsilon = \int_{S} n(x') \times \left[ \nabla_{x'} G_{\delta}(x - x') - \nabla_{x'} G(x - x') \right] \mu(x') dS(x') \]

\[ = \frac{\delta}{2\sqrt{\pi}} \left\{ T_{2}\mu_{1} - T_{1}\mu_{2} \right\} + O(\delta^{3}) \]

Discretization correction based on estimates of the Fourier coefficients of the regularized kernel, but won’t fit onto a slide.
Acknowledgements and References

Joint work with M. Siegel and M. Booty (NJIT), and D. Ambrose (Drexel).

References:

- Caflisch and Li: “Lagrangian Theory for 3D Vortex Sheets”
- Baker and Nie: “Application of Adaptive Quadrature to Vortex Sheet Motion”