KAM for quasi-linear KdV

Massimiliano Berti

Toronto, 10-1-2014, Conference on
"Hamiltonian PDEs: Analysis, Computations and Applications"
for the 60-th birthday of Walter Craig
KdV

\[ \partial_t u + u_{xxx} - 3\partial_x u^2 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0, \quad x \in \mathbb{T} \]

Quasi-linear Hamiltonian perturbation

\[ \mathcal{N}_4 := -\partial_x \{(\partial_u f)(x, u, u_x)\} + \partial_{xx} \{(\partial_{u_x} f)(x, u, u_x)\} \]

\[ \mathcal{N}_4 = a_0(x, u, u_x, u_{xx}) + a_1(x, u, u_x, u_{xx})u_{xxx} \]

\[ \mathcal{N}_4(x, \varepsilon u, \varepsilon u_x, \varepsilon u_{xx}, \varepsilon u_{xxx}) = O(\varepsilon^4), \quad \varepsilon \to 0 \]

\[ f(x, u, u_x) = O(|u|^5 + |u_x|^5), \quad f \in C^q(\mathbb{T} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \]

Physically important for perturbative derivation from water-waves (that I learned from Walter Craig)
Hamiltonian PDE

\[ u_t = X_H(u), \quad X_H(u) := \partial_x \nabla_{L^2} H(u) \]

Hamiltonian KdV

\[ H = \int_T \frac{u_x^2}{2} + u^3 + f(x, u, u_x) \, dx \]

where the density \( f(x, u, u_x) = O(||(u, u_x)||^5) \)

Phase space

\[ H_0^1(T) := \left\{ u(x) \in H^1(T, \mathbb{R}) : \int_T u(x) \, dx = 0 \right\} \]

Non-degenerate symplectic form:

\[ \Omega(u, v) := \int_T (\partial_x^{-1} u) \, v \, dx \]
**Goal:** look for small amplitude quasi-periodic solutions

**Definition:** quasi-periodic solution with $n$ frequencies

$$u(t, x) = U(\omega t, x)$$

where $U(\varphi, x) : \mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$,

$\omega \in \mathbb{R}^n (\text{= frequency vector})$ is irrational $\omega \cdot k \neq 0$, $\forall k \in \mathbb{Z}^n \setminus \{0\}$

$\implies$ the linear flow $\{\omega t\}_{t \in \mathbb{R}}$ is DENSE on $\mathbb{T}^n$

The torus-manifold

$$\mathbb{T}^n \ni \varphi \mapsto u(\varphi, x) \in \text{phase space}$$

is invariant under the flow evolution of the PDE
Linear Airy eq.

\[ u_t + u_{xxx} = 0, \quad x \in \mathbb{T} \]

Solutions: (superposition principle)

\[ u(t, x) = \sum_{j \in \mathbb{Z}\backslash\{0\}} a_j e^{ij^3 t} e^{ijx} \]

Eigenvalues \( j^3 = \text{"NORMAL FREQUENCIES"} \)

Eigenfunctions: \( e^{ijx} = \text{"NORMAL MODES"} \)

All solutions are 2\(\pi\)-periodic in time: COMPLETELY RESONANT

\(\Rightarrow\) Quasi-periodic solutions are a completely nonlinear phenomenon
The problem

Linear Airy eq.

\[ u_t + u_{xxx} = 0, \quad x \in \mathbb{T} \]

Solutions: (superposition principle)

\[ u(t, x) = \sum_{j \in \mathbb{Z}\setminus\{0\}} a_j e^{ij^3 t} e^{ijx} \]

Eigenvalues \( j^3 = "NORMAL\ \text{FREQUENCIES}" \)
Eigenfunctions: \( e^{ijx} = "NORMAL\ \text{MODES}" \)

All solutions are 2\(\pi\)-periodic in time: COMPLETELY RESONANT

\[ \Rightarrow \] Quasi-periodic solutions are a completely nonlinear phenomenon
KdV is completely integrable

\[ u_t + u_{xxx} - 3\partial_x u^2 = 0 \]

All solutions are periodic, quasi-periodic, almost periodic

What happens under a small perturbation?
KAM theory

Kuksin '98, Kappeler-Pöschel '03: KAM for KdV

\[ u_t + u_{xxx} + uu_x + \varepsilon \partial_x f(x, u) = 0 \]

1. **SEMIC LINEAR PERTURBATION**  \( \partial_x f(x, u) \)
2. Also true for Hamiltonian perturbations

\[ u_t + u_{xxx} + uu_x + \varepsilon \partial_x |\partial_x|^{1/2} f(x, |\partial_x|^{1/2} u) = 0 \]

of order 2

\[ |j^3 - i^3| \geq i^2 + j^2, \; i \neq j \implies \text{KdV gains up to 2 spatial derivatives} \]

Is for quasi-linear KdV? **OPEN PROBLEM**
KAM theory

Kuksin '98, Kappeler-Pöschel '03: KAM for KdV

\[ u_t + u_{xxx} + uu_x + \varepsilon \partial_x f(x, u) = 0 \]

1. **SEMICLINEAR PERTURBATION** \( \partial_x f(x, u) \)

2. Also true for Hamiltonian perturbations

\[ u_t + u_{xxx} + uu_x + \varepsilon \partial_x |\partial_x|^{1/2} f(x, |\partial_x|^{1/2} u) = 0 \]

of order 2

\[ |j^3 - i^3| \geq i^2 + j^2, \quad i \neq j \implies \text{KdV gains up to 2 spatial derivatives} \]

3. **for QUASI-LINEAR KdV? OPEN PROBLEM**
### Literature: KAM for "unbounded" perturbations

<table>
<thead>
<tr>
<th>Liu-Yuan '10 for Hamiltonian DNLS (and Benjamin-Ono)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( iu_t - u_{xx} + M_\sigma u + i\epsilon f(u, \bar{u})u_x = 0 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Zhang-Gao-Yuan '11 Reversible DNLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( iu_t + u_{xx} =</td>
</tr>
</tbody>
</table>

Craig-Wayne periodic solutions, Lyapunov-Schmidt + Nash-Moser

<table>
<thead>
<tr>
<th>Bourgain '96, Derivative NLW</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_{tt} - y_{xx} + my + y_t^2 = 0, \quad m \neq 0, )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Craig '00, Hamiltonian DNLW</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_{tt} - y_{xx} + g(x)y = f(x, D^\beta y), \quad D := \sqrt{-\partial_{xx} + g(x)} ),</td>
</tr>
</tbody>
</table>
quasi-periodic solutions

Berti-Biasco-Procesi '12, '13, reversible DNLW

\[ u_{tt} - u_{xx} + mu = g(x, u, u_x, u_t) \]

For quasi-linear PDEs: Periodic solutions:
- Iooss-Plotinikov-Toland, Iooss-Plotnikov, '01-'10,
- Water waves: quasi-linear equation,
  new ideas for conjugation of linearized operator
quasi-periodic solutions

Berti-Biasco-Procesi '12, '13, reversible DNLW

$$u_{tt} - u_{xx} + mu = g(x, u, u_x, u_t)$$

For quasi-linear PDEs: Periodic solutions:
- Iooss-Plotinikov-Toland, Iooss-Plotnikov, '01-'10,
  Water waves: quasi-linear equation,
  new ideas for conjugation of linearized operator
Main results

Hamiltonian density:
\[ f(x, u, u_x) = f_5(u, u_x) + f_{\geq 6}(x, u, u_x) \]
\( f_5 \) polynomial of order 5 in \((u, u_x)\); \( f_{\geq 6}(x, u, u_x) = O(|u| + |u_x|)^6 \)

Reversibility condition:
\[ f(x, u, u_x) = f(-x, u, -u_x) \]

KdV-vector field \( X_H(u) := \partial_x \nabla H(u) \) is reversible w.r.t the involution

\[ \varrho u := u(-x) \quad \varrho^2 = I \quad -\varrho X_H(u) = X_H(\varrho u) \]
Theorem ('13, P. Baldi, M. Berti, R. Montalto)

Let $f \in C^q$ (with $q := q(n)$ large enough). Then, for “generic” choice of the "TANGENTIAL SITES"

$$S := \{-\bar{j}_n, \ldots, -\bar{j}_1, \bar{j}_1, \ldots, \bar{j}_n\} \subset \mathbb{Z} \setminus \{0\},$$

the hamiltonian and reversible KdV equation

$$\partial_t u + u_{xxx} - 3\partial_x u^2 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0, \quad x \in \mathbb{T},$$

possesses small amplitude quasi-periodic solutions with Sobolev regularity $H^s$, $s \leq q$, of the form

$$u = \sum_{j \in S} \sqrt{\xi_j} e^{i\omega^\infty_j(\xi)} t e^{ijx} + o(\sqrt{\xi}), \quad \omega^\infty_j(\xi) = j^3 + O(|\xi|)$$

for a "Cantor-like" set of "initial conditions" $\xi \in \mathbb{R}^n$ with density 1 at $\xi = 0$. The linearized equations at these quasi-periodic solutions are reduced to constant coefficients and are stable.

If $f = f_{\geq 7} = O(\|(u, u_x)|^7)$ then any choice of tangential sites
Tangential sites

Explicit conditions:

- **Hypothesis** \((S_3)\) \(j_1 + j_2 + j_3 \neq 0\) for all \(j_1, j_2, j_3 \in S\)
- **Hypothesis** \((S_4)\) \(\not\exists j_1, \ldots, j_4 \in S\) such that
  \[
  j_1 + j_2 + j_3 + j_4 \neq 0, \quad j_1^3 + j_2^3 + j_3^3 + j_4^3 - (j_1 + j_2 + j_3 + j_4)^3 = 0
  \]

1. \((S_3)\) used in the linearized operator. If \(f_5 = 0\) then not needed
2. If also \(f_6 = 0\) then \((S_4)\) not needed (used in Birkhoff-normal-form)

“genericity”: After fixing \(\{\bar{j}_1, \ldots, \bar{j}_n\}\), in the choice of \(\bar{j}_{n+1} \in \mathbb{N}\) there are only Finitely many forbidden values
A similar result holds for

\[ \partial_t u + u_{xxx} \pm \partial_x u^3 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0, \quad x \in \mathbb{T} \]

for all the tangential sites \( S := \{-\bar{j}_n, \ldots, -\bar{j}_1, \bar{j}_1, \ldots, \bar{j}_n\} \) such that

\[ \frac{2}{2n - 1} \sum_{i=1}^{n} \bar{j}_i^2 \notin \mathbb{N} \]

1. If \( f = f(u, u_x) \) the result is \textbf{true} for all the tangential sites \( S \)
2. Also for generalized KdV (not integrable), with normal form techniques of Procesi-Procesi
Linear stability

(L): linearized equation \( \partial_t h = \partial_x \partial_u \nabla H(u(\omega t, x)) h \)

\[
h_t + a_3(\omega t, x) h_{xxx} + a_2(\omega t, x) h_{xx} + a_1(\omega t, x) h_x + a_0(\omega t, x) h = 0
\]

There exists a quasi-periodic (Floquet) change of variable

\[
h = \Phi(\omega t)(\psi, \eta, v), \quad \psi \in \mathbb{T}^\nu, \eta \in \mathbb{R}^\nu, v \in H_x^s \cap L_{S_{\perp}}^2
\]

which transforms (L) into the constant coefficients system

\[
\begin{aligned}
\dot{\psi} &= b \eta \\
\dot{\eta} &= 0 \\
\dot{v}_j &= i \mu_j v_j, \quad j \notin S, \; \mu_j \in \mathbb{R}
\end{aligned}
\]

\[\Rightarrow \eta(t) = \eta_0, v_j(t) = v_j(0)e^{i\mu_j t} \Rightarrow \|v(t)\|_s = \|v(0)\|_s: \text{stability}\]
Forced quasi-linear perturbations of Airy

Use $\omega = \lambda \vec{\omega} \in \mathbb{R}^n$ as 1-dim. parameter

**Theorem (Baldi, Berti, Montalto , to appear Math. Annalen)**

There exist $s := s(n) > 0$, $q := q(n) \in \mathbb{N}$, such that:

Let $\vec{\omega} \in \mathbb{R}^n$ diophantine. For every quasi-linear Hamiltonian nonlinearity $f \in C^q$ for all $\varepsilon \in (0, \varepsilon_0)$ small enough, there is a Cantor set $\mathcal{C}_\varepsilon \subset [1/2, 3/2]$ of asymptotically full measure, i.e.

$$|\mathcal{C}_\varepsilon| \to 1 \quad \text{as} \quad \varepsilon \to 0,$$

such that for all $\lambda \in \mathcal{C}_\varepsilon$ the perturbed Airy equation

$$\partial_t u + \partial_{xxx} u + \varepsilon f(\lambda \vec{\omega} t, x, u, u_x, u_{xx}, u_{xxx}) = 0$$

has a quasi-periodic solution $u(\varepsilon, \lambda) \in H^s$ (for some $s \leq q$) with frequency $\omega = \lambda \vec{\omega}$ and satisfying $\|u(\varepsilon, \lambda)\|_s \to 0$ as $\varepsilon \to 0$. 
Key: spectral analysis of quasi-periodic operator

\[ \mathcal{L} = \omega \cdot \partial \varphi + \partial_{xxx} + a_3(\varphi, x) \partial_{xxx} + a_2(\varphi, x) \partial_{xx} + a_1(\varphi, x) \partial_x + a_0(\varphi, x) \]

\[ a_i = O(\varepsilon), \; i = 0, 1, 2, 3 \]

Main problem: the non constant coefficients term \( a_3(\varphi, x) \partial_{xxx} \)!

Main difficulties:

1. The usual KAM iterative scheme is unbounded
2. We expect an estimate of perturbed eigenvalues

\[ \mu_j(\varepsilon) = j^3 + O(\varepsilon j^3) \]

which is NOT sufficient for verifying second order Melnikov

\[ |\omega \cdot \ell + \mu_j(\varepsilon) - \mu_i(\varepsilon)| \geq \frac{\gamma |j^3 - i^3|}{\langle \ell \rangle^\tau}, \quad \forall \ell, j, i \]
Idea to conjugate $\mathcal{L}$ to a diagonal operator

1. "REDUCTION IN DECREASING SYMBOLS"

\[ \mathcal{L}_1 := \Phi^{-1} \mathcal{L} \Phi = \omega \cdot \partial \varphi + m_3 \partial_{xxx} + m_1 \partial_x + R_0 \]

- $R_0(\varphi, x)$ pseudo-differential operator of order 0,
  $R_0(\varphi, x) : H^s_x \rightarrow H^s_x$, variable coefficients, $R_0 = O(\varepsilon)$,
- $m_3 = 1 + O(\varepsilon)$, $m_1 = O(\varepsilon)$, $m_1, m_3 \in \mathbb{R}$, CONSTANTS

Use suitable transformations "far" from the identity

2. "REDUCTION OF THE SIZE of $R_0$"

\[ \mathcal{L}_\nu := \Phi_{\nu}^{-1} \mathcal{L}_1 \Phi_{\nu} = \omega \cdot \partial \varphi + m_3 \partial_{xxx} + m_1 \partial_x + r^{(\nu)} + R_\nu \]

- $R_\nu = R_\nu(\varphi, x) = O(R_0^{2\nu})$
- $r^{(\nu)} = \text{diag}_{j\in\mathbb{Z}}(r_j^{(\nu)})$, $\sup_j |r_j^{(\nu)}| = O(\varepsilon)$,

KAM-type scheme, now transformations of $H^s_x \rightarrow H^s_x$
Idea to conjugate $\mathcal{L}$ to a diagonal operator

1. "REDUCTION IN DECREASING SYMBOLS"

$$\mathcal{L}_1 := \Phi^{-1} \mathcal{L} \Phi = \omega \cdot \partial \varphi + m_3 \partial_{xxx} + m_1 \partial_x + R_0$$

- $R_0(\varphi, x)$ pseudo-differential operator of order 0,
  $R_0(\varphi, x) : H_x^s \to H_x^s$, variable coefficients, $R_0 = O(\varepsilon)$,
- $m_3 = 1 + O(\varepsilon), m_1 = O(\varepsilon), m_1, m_3 \in \mathbb{R}$, CONSTANTS

Use suitable transformations "far" from the identity

2. "REDUCTION OF THE SIZE of $R_0"$

$$\mathcal{L}_\nu := \Phi^{-1}_\nu \mathcal{L}_1 \Phi_\nu = \omega \cdot \partial \varphi + m_3 \partial_{xxx} + m_1 \partial_x + r^{(\nu)} + R_\nu$$

- $R_\nu = R_\nu(\varphi, x) = O(\varepsilon^{1/2})$
- $r^{(\nu)} = \text{diag}_{j \in \mathbb{Z}}(r^{(\nu)}_j), \sup_j |r_j^{(\nu)}| = O(\varepsilon),$
  \[KAM\text{-type scheme}, \text{ now transformations of } H_x^s \to H_x^s\]
Higher order term

\[ L := \omega \cdot \partial \varphi + \partial_{xxx} + \varepsilon a_3(\varphi, x)\partial_{xxx} \]

**STEP 1:** Under the **symplectic** change of variables

\[ (Au) := (1 + \beta_x(\varphi, x))u(\varphi, x + \beta(\varphi, x)) \]

we get

\[ L_1 := A^{-1}LA = \omega \cdot \partial \varphi + (A^{-1}(1 + \varepsilon a_3)(1 + \beta_x)^3)\partial_{xxx} + O(\partial_{xx}) \]

\[ = \omega \cdot \partial \varphi + c(\varphi)\partial_{xxx} + O(\partial_{xx}) \]

imposing

\[ (1 + \varepsilon a_3)(1 + \beta_x)^3 = c(\varphi), \]

There exist solution \( c(\varphi) \approx 1, \beta = O(\varepsilon) \)
**STEP 2: Rescaling time**

\[(Bu)(\varphi, x) = u(\varphi + \omega q(\varphi), x)\]

we have

\[B^{-1}L_1B = B^{-1}(1 + \omega \cdot \partial_\varphi q)(\omega \cdot \partial_\varphi) + B^{-1}c(\varphi)\partial_{xxx} + O(\partial_{xx})\]

\[= \mu(\varepsilon) B^{-1}c(\varphi)(\omega \cdot \partial_\varphi) + B^{-1}c(\varphi)\partial_{xxx} + O(\partial_{xx})\]

solving

\[1 + \omega \cdot \partial_\varphi q = \mu(\varepsilon)c(\varphi), \quad q(\varphi) = O(\varepsilon)\]

Dividing for \(\mu(\varepsilon)B^{-1}c(\varphi)\) we get

\[L_2 := \omega \cdot \partial_\varphi + m_3(\varepsilon)\partial_{xxx} + O(\partial_x), \quad m_3(\varepsilon) := \mu^{-1}(\varepsilon) = 1 + O(\varepsilon)\]

which has the leading order with **constant coefficients**
Autonomous KdV

New further difficulties:

- **No external parameters.** The frequency of the solutions is **NOT** fixed a-priori. Frequency-amplitude modulation.
- **KdV is completely resonant**
- **Construction of an approximate inverse**

Ideas:

- **Weak Birkhoff-normal form**
- General method to decouple the "tangential dynamics" from the "normal dynamics", developed with P. Bolle
  Procedure which reduces autonomous case to the forced one
Fix the “tangential sites” \( S := \{-\bar{j}_n, \ldots, -\bar{j}_1, \bar{j}_1, \ldots, \bar{j}_n\} \subset \mathbb{Z} \setminus \{0\} \)

Split the dynamics:

\[
\begin{align*}
    u(x) &= v(x) + z(x) \\
    v(x) &= \sum_{j \in S} u_j e^{ijx} = "\text{tangential component}" \\
    z(x) &= \sum_{j \notin S} u_j e^{ijx} = "\text{normal component}" 
\end{align*}
\]

Hamiltonian

\[
H = \frac{1}{2} \int_T v_x^2 + \frac{1}{2} \int_T z_x^2 dx + \int_T v^3 dx + 3 \int_T v^2 z dx + \int_T v^3 dx + 3 \int_T v^2 z dx + 3 \int_T vz^2 dx + \int_T z^3 dx + \int_T f(u, u_x)
\]

Goal: eliminate terms linear in \( z \implies \{z = 0\} \) is invariant manifold
Theorem (Weak Birkhoff normal form)

There is a symplectic transformation $\Phi_B : H^1_0(\mathbb{T}_x) \to H^1_0(\mathbb{T}_x)$

$$\Phi_B(u) = u + \Psi(u), \quad \Psi(u) = \Pi_E \psi(\Pi_E u),$$

where $E := \text{span}\{ e^{ijx}, 0 < |j| \leq 6|S| \}$ is finite-dimensional, s.t.

$$H := H \circ \Phi_B = H_2 + H_3 + H_4 + H_5 + H_{\geq 6},$$

$$H_3 := \int_{\mathbb{T}} z^3 dx + 3 \int_{\mathbb{T}} vz^2 dx, \quad H_4 := -\frac{3}{2} \sum_{j \in S} \frac{|u_j|^4}{j^2} + H_{4,2} + H_{4,3},$$

$$H_{4,2} := 6 \int_{\mathbb{T}} vz \Pi_S ((\partial_x^{-1} v)(\partial_x^{-1} z)) dx + 3 \int_{\mathbb{T}} z^2 \pi_0 (\partial_x^{-1} v)^2 dx,$$

$$H_{4,3} := R(vz^3), \quad H_5 := \sum_{q=2}^5 R(v^{5-q}z^q),$$

and $H_{\geq 6}$ collects all the terms of order at least six in $(v, z)$. 
**Fourier representation**

\[ u(x) = \sum_{j \in \mathbb{Z} \setminus \{0\}} u_j e^{ijx}, \quad u(x) \longleftrightarrow (u_j)_{j \in \mathbb{Z} \setminus \{0\}} \]

**First Step.** Eliminate the \( u_{j_1} u_{j_2} u_{j_3} \) of \( H_3 \) with at most one index outside \( S \). Since \( j_1 + j_2 + j_3 = 0 \) they are finitely many.

**Φ := the time 1-flow map generated by**

\[ F(u) := \sum_{j_1+j_2+j_3=0} F_{j_1,j_2,j_3} u_{j_1} u_{j_2} u_{j_3} \]

The vector field \( X_F \) is supported on finitely many sites

\[ X_F(u) = \Pi_{H_2S} X_F(\Pi_{H_2S} u) \]

\[ \implies \text{the flow is a finite dimensional perturbation of the identity} \]

\[ \Phi = Id + \Psi, \quad \Psi = \Pi_{H_2S} \psi \Pi_{H_2S} \]
For the other steps:

- Normalize the quartic monomials $u_{j_1} u_{j_2} u_{j_3} u_{j_4}$, $j_1, j_2, j_3, j_4 \in S$. The fourth order system $\mathcal{H}_4$ restricted to $S$ turns out to be integrable, i.e.

$$-\frac{3}{2} \sum_{j \in S} \frac{|u_j|^4}{j^2} \quad (\text{non–isochronous rotators})$$

Now $\{z = 0\}$ is an invariant manifold for $\mathcal{H}_4$ filled by quasi-periodic solutions with a frequency which varies with the amplitude.
Difference w.r.t. other Birkhoff normal forms

1. **Kappeler-Pöschel (KdV), Kuksin-Pöschel (NLS), complete Birkhoff-normal form:**
   they remove/normalize also the terms $O(z^2), O(z^3), O(z^4)$

2. **Pöschel (NLW), semi normal Birkhoff normal form:**
   normalized only the term $O(z^2)$

3. **Kappeler** Global Birkhoff normal form for KdV, 1-d-cubic-NLS

The above transformations are

\begin{align*}
(1) \quad & I + \text{bounded}, \\
(2) \quad & I + O(\partial_x^{-1}), \\
(3) \quad & \Phi = F + O(\partial_x^{-1}),
\end{align*}

It is **NOT** enough for quasi-linear perturbations!

Our $\Phi = Id + \text{finite dimensional} \implies$ it changes very little the third order differential perturbations in KdV
Rescaled action-angle variables:

\[ u := \varepsilon v_\varepsilon(\theta, y) + \varepsilon z := \varepsilon \sum_{j \in S} \sqrt{\xi_j + |j|y_j} e^{i\theta_j} e^{ijx} + \varepsilon z \]

Hamiltonian:

\[ H_\varepsilon = N + P, \quad N(\theta, y, z, \xi) = \alpha(\xi) \cdot y + \frac{1}{2} (N(\theta, \xi)z, z)_{L^2(\mathbb{T})} \]

where

Frequency-amplitude map:

\[ \alpha(\xi) = \bar{\omega} + \varepsilon^2 A\xi \]

Variable coefficients normal form:

\[ \frac{1}{2} (N(\theta, \xi)z, z)_{L^2(\mathbb{T})} = \frac{1}{2} ((\partial_z \nabla H_\varepsilon)(\theta, 0, 0)[z], z)_{L^2(\mathbb{T})} \]
We look for quasi-periodic solutions of \( X_{H\varepsilon} \) with

**Diophantine frequencies:**

\[
\omega = \bar{\omega} + \varepsilon^2 A \xi
\]

**Embedded torus equation:**

\[
\partial_\omega i(\varphi) - X_{H\varepsilon}(i(\varphi)) = 0
\]

**Functional setting**

\[
\mathcal{F}(\varepsilon, X) := \begin{pmatrix}
\partial_\omega \theta(\varphi) - \partial_y H_{\varepsilon}(i(\varphi)) \\
\partial_\omega y(\varphi) + \partial_\theta H_{\varepsilon}(i(\varphi)) \\
\partial_\omega z(\varphi) - \partial_x \nabla_z H_{\varepsilon}(i(\varphi))
\end{pmatrix} = 0
\]

unknown: \( X := i(\varphi) := (\theta(\varphi), y(\varphi), z(\varphi)) \)
Main Difficulty

Invert linearized operator at approximate solution $i_0(\varphi)$:

$$D_i \mathcal{F}(i_0(\varphi))[\hat{i}] =$$

$$\partial_\omega \hat{\theta} - \partial_{\theta y} H_\varepsilon(i_0)[\hat{\theta}] - \partial_{yy} H_\varepsilon(i_0)[\hat{y}] - \partial_{z y} H_\varepsilon(i_0)[\hat{z}]$$

$$\partial_\omega \hat{y} + \partial_{\theta \theta} H_\varepsilon(i_0)[\hat{\theta}] + \partial_{\theta y} H_\varepsilon(i_0)[\hat{y}] + \partial_{\theta z} H_\varepsilon(i_0)[\hat{z}]$$

$$\partial_\omega \hat{z} - \partial_x \{ \partial_\theta \nabla_z H_\varepsilon(i_0)[\hat{\theta}] + \partial_y \nabla_z H_\varepsilon[\hat{y}] + \partial_z \nabla_z H_\varepsilon[\hat{z}] \}$$
Approximate inverse. Zehnder

A linear operator $T(X)$, $X := i(\varphi)$ is an approximate inverse of $dF(X)$ if

$$\|dF(X)T(X) - Id\| \leq \|F(X)\|$$

1. $T(X)$ is an exact inverse of $dF(X)$ at a solution
2. It is sufficient to invert $dF(X)$ at a solution

Use the general method to construct an approximate inverse, reducing to the inversion of quasi-periodically forced systems, Berti-Bolle for autonomous NLS-NLW with multiplicative potential.
How to take advantage that $i_0$ is a solution?

The invariant torus $i_0(\varphi) := (\theta_0(\varphi), y_0(\varphi), z_0(\varphi))$ is ISOTROPIC

$\Rightarrow$

the transformation $G$ of the phase space $\mathbb{T}^n \times \mathbb{R}^n \times H_{S\perp}$

\[
\begin{pmatrix}
\theta \\
y \\
z
\end{pmatrix} := G \begin{pmatrix}
\psi \\
\eta \\
w
\end{pmatrix} := \begin{pmatrix}
\theta_0(\psi) \\
y_0(\psi) + D\theta_0(\psi)^{-T} \eta + D\tilde{z}_0(\theta_0(\psi))^T \partial_{x}^{-1} w \\
z_0(\psi) + w
\end{pmatrix}
\]

where $\tilde{z}_0(\theta) := z_0(\theta^{-1}(\theta))$, is SYMPLECTIC

In the new symplectic coordinates, $i_0$ is the trivial embedded torus

$(\psi, \eta, w) = (\varphi, 0, 0)$
The problem

Main results

Proof: forced case

Proof: Autonomous case

Transformed Hamiltonian

\[ K := H_\varepsilon \circ G = K_{00}(\psi) + K_{10}(\psi)\eta + (K_{01}(\psi), w)_{L^2_x} + \frac{1}{2} K_{20}(\psi)\eta \cdot \eta \]

\[ + (K_{11}(\psi)\eta, w)_{L^2_x} + \frac{1}{2} (K_{02}(\psi)w, w)_{L^2_x} + O(|\eta| + |w|)^3 \]

Hamiltonian system in new coordinates:

\[
\begin{align*}
\dot{\psi} &= K_{10}(\psi) + K_{20}(\psi)\eta + K_{11}^T(\psi)w + O(\eta^2 + w^2) \\
\dot{\eta} &= -\partial_\psi K_{00}(\psi) - \partial_\psi K_{10}(\psi)\eta - \partial_\psi K_{01}(\psi)w + O(\eta^2 + w^2) \\
\dot{w} &= \partial_x (K_{01}(\psi) + K_{11}(\psi)\eta + K_{02}(\psi)w) + O(\eta^2 + w^2)
\end{align*}
\]

Since \((\psi, \eta, w) = (\omega t, 0, 0)\) is a solution

\[ \partial_\psi K_{00}(\psi) = 0, \quad K_{10}(\psi) = \omega, \quad K_{01}(\psi) = 0 \]
\[ K := H_\epsilon \circ G = \text{const} + \omega \cdot \eta + \frac{1}{2} K_{20}(\psi) \eta \cdot \eta + (K_{11}(\psi) \eta, w)_{L^2} \]
\[ + \frac{1}{2} (K_{02}(\psi) w, w)_{L^2} + O(|\eta| + |w|)^3 \]

Hamiltonian system in new coordinates:
\[
\begin{aligned}
\dot{\psi} &= \omega + K_{20}(\psi) \eta + K_{11}^T(\psi) w + O(\eta^2 + w^2) \\
\dot{\eta} &= O(\eta^2 + w^2) \\
\dot{w} &= \partial_x (K_{11}(\psi) \eta + K_{02}(\psi) w) + O(\eta^2 + w^2)
\end{aligned}
\]

\[ \Rightarrow \text{in the NEW variables the linearized equations at } (\varphi, 0, 0) \text{ simplify!} \]
Linearized equations at the invariant torus \((\varphi, 0, 0)\)

\[
\begin{pmatrix}
\partial_\omega \hat{\psi} - K_{20}(\varphi)\hat{\eta} - K_{11}^T(\varphi)\hat{w} \\
\partial_\omega \hat{\eta} \\
\partial_\omega \hat{w} - \partial_x K_{11}(\varphi)\hat{\eta} - \partial_x K_{02}(\varphi)\hat{w}
\end{pmatrix} = \begin{pmatrix}
\Delta a \\
\Delta b \\
\Delta c
\end{pmatrix}
\]

may be solved in a TRIANGULAR way.

**Step 1: solve second equation**

\[
\hat{\eta} = \partial_\omega^{-1} \Delta b + \eta_0 , \quad \eta_0 \in \mathbb{R}^\nu
\]

Remark: \(\Delta b\) has zero average by reversibility, \(\eta_0\) fixed later.

**Step 2: solve third equation**

\[
\mathcal{L}_\omega \hat{w} = \Delta c + \partial_x K_{11}(\varphi)\hat{\eta}, \quad \mathcal{L}_\omega := \omega \cdot \partial_\varphi - \partial_x K_{02}(\varphi)
\]

This is a quasi-periodically forced linear KdV operator!
Reduction of the linearized op. on the normal directions

\[ \mathcal{L}_\omega h = \prod_{S_{\perp}} \left( \omega \cdot \partial \varphi h + \partial_{xx} (a_1 \partial_x h) + \partial_x (a_0 h) - \varepsilon^2 \partial_x R_2[h] - \partial_x R_*[h] \right) \]

\[ a_1 - 1 := O(\varepsilon^3), \quad a_0 := \varepsilon p_1 + \varepsilon^2 p_2 + \ldots \]

The remainders \( R_2, R_* \) are finite range (very regularizing!)

Reduce \( \mathcal{L}_\omega \) to constant coefficients as in forced case, hence invert it

1. Terms \( O(\varepsilon), O(\varepsilon^2) \) are NOT perturbative: \( \varepsilon \gamma^{-1}, \varepsilon^2 \gamma^{-1} \) is large! \( \gamma = o(\varepsilon^2) \)

2. These terms eliminated by algebraic arguments (integrability property of Birkhoff normal form)
Step 3: solve first equation

\[ \partial_\omega \hat{\psi} = K_{20}(\varphi)\hat{\eta} + K_{11}^T(\varphi)\hat{\omega} - \Delta a \]

Since

\[ K_{20}(\varphi) = 3\varepsilon^2 Id + o(\varepsilon^2) \]

the matrix \( K_{20} \) is invertible and we choose \( \eta_0 \) (the average of \( \hat{\eta} \)) so that the right hand side has zero average. Hence

\[ \hat{\psi} = \partial_\omega^{-1} \left( K_{20}(\varphi)\hat{\eta} + K_{11}^T(\varphi)\hat{\omega} - \Delta a \right) \]

This completes the construction of an approximate inverse.
HAPPY BIRTHDAY !!