Singularities of the L^2 curvature flow

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April 2014

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Yang-Mills connections are natural candidates to consider as "optimal" connections on P, and their study has played a central role in various physical, geometric, and topological theories, especially in dimension $n = 4$.

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From a Lagrangian-theoretic standpoint, critical metrics are natural choices of “best metrics” on smooth manifolds, especially in dimension 4, which is the scale-invariant dimension for $\mathcal{F}$.

Gradient Flows

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Conjecturally, the long time behavior of both flows is governed by the principle of dimensional criticality.

Long time existence properties

Both functionals $\mathcal{YM}$ and $\mathcal{F}$ obey scaling laws which render dimensions $n \leq 3$ "subcritical", dimension $n = 4$ "critical", and dimension $n \geq 5$ "supercritical".

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(______ 2012) Let (M^n, g_t) be a solution to the L^2 flow which exists on a maximal time interval $[0, T)$, $T < \infty$. Then

$$\limsup_{t \to T} |\mathrm{Rm}| = \infty.$$

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Proposition

Let $(M^n, g(t))$ be a solution to the L^2 flow, $n = 2, 3$. Suppose $g(t)$ exists on a maximal time interval $[0, T)$, $T < \infty$. Let $\{(x_i, t_i)\}$ be a sequence of points such that $t_i \to T$ and $\left|\mathrm{Rm}\right|(x_i, t_i) = \sup_{[0,t_i]} \left|\mathrm{Rm}\right|$. Then

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$$\lim_{i \to \infty} \mathrm{inj}_g^2(x_i) \, |\mathrm{Rm}| (x_i) = 0.$$

Moreover, for $n \geq 6$, local collapse actually occurs, as evidenced by homogeneous metrics on $S^5 \times S^1$.

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In the rest of the talk I will describe a new, dimension-dependent technique for controlling the local collapsing along the L^2 flow, which yields new structure theorems for finite time singularities, as well as new compactness/diffeomorphism-finiteness results in Riemannian geometry.

Singularity Decompositions

We show that in dimension $n = 3$, any point of curvature blowup is eventually arbitrarily close to arbitrarily collapsed points. The set $\mathcal{T}_\mu(g_t)$ is roughly points which are μ-collapsed on the scale of curvature.

Theorem (______ 2013)

Let (M^3, g_0) be a compact manifold, and suppose the solution to the L^2 flow with initial condition g_0 exists on a maximal time interval $[0, T)$, $T < \infty$. Then for any $x \in M$ such that $\limsup_{t \to T} |\mathrm{Rm}|(x, t) = \infty$ and any $\mu > 0$,

$$\liminf_{t \to T} d(x, \mathcal{T}_\mu(g_t), t) = 0.$$

Before stating our result for $n = 4$, let's recall the Struwe "bubbling" statement.

Theorem (Struwe)

"Finite time singularities of the Yang-Mills flow in dimension 4 occur via energy concentration." I.e., if T denotes a finite maximal existence time of the Yang-Mills flow, there exist finitely many points $\{x_i\} \in M$ and an $\epsilon > 0$ depending on the given bundle such that for all $R > 0$,

$$\limsup_{t \to T} \int_{B_R(x_i)} |F|^2 \geq \epsilon.$$

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Note that it is still an open question if these finite time singularities actually occur.

Singularity Decompositions

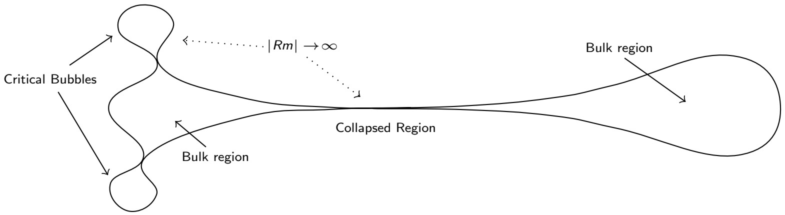
Theorem (______ 2013)

(Concentration-Collapse Decomposition) For any $E, \mu > 0$ there exists $\epsilon(E,\mu) > 0$ so that if (M^4, g_t) is a solution to the L^2 flow satisfying

$$\mathcal{F}(g_0) \leq E,$$

then for any $T \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, and any $x \in M$ such that $\limsup_{t \to T} |\mathrm{Rm}|(x,t) = \infty$, we have either:

1. *For all $r > 0$,* $\limsup_{t \to T} \int_{B_r(x,t)} |\mathrm{Rm}|^2 \geq \epsilon$,
2. $\liminf_{t \to T} d(x, T_\mu(g_t), t) = 0$.



Definition

Let (M^n, g) be a Riemannian manifold. Fix $0 < \delta < 1$, and let ω_n denote the volume of the unit n-ball in $\mathbb{R}^n$. Given $x \in M$, define the δ-*volume radius at* x to be

$$r_\delta(x) := \sup \left\{ r \geq 0 \;\middle|\; \forall s \leq r, \frac{\operatorname{Vol} B_s(x)}{s^n} \geq \delta \omega_n \right\}.$$

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Theorem (______ 2013)

Given $0 < \delta < 1$, there exists $\epsilon, \iota, A > 0$ depending only on δ so that if (M^4, g) is a compact Riemannian manifold satisfying

1. $r_\delta \geq \rho$,
2. $\mathcal{F}(g) \leq \epsilon$,

the L^2 flow with initial condition g exists on $\left[0, \rho^4\right]$ and moreover satisfies the estimates

1. $\left|\mathrm{Rm}\right|_{g_t} \leq A\mathcal{F}^{\frac{1}{6}}(g_t)t^{-\frac{1}{2}}$,
2. $\mathrm{inj}_{g_t} \geq \iota t^{\frac{1}{4}}$,

Corollaries on compactness and diffeofiniteness

Let us recall a typical statement of "pinching" for Riemannian manifolds:

Theorem

(Gao, Anderson, Anderson-Cheeger) Given $V, D, H > 0$ *there exists* $\epsilon = \epsilon(V, D, H)$ *so that if* (M^4, g) *is a compact Riemannian manifold satisfying:*

1. $|\mathrm{Rc}| \leq H$
2. $\mathrm{diam} \leq D$
3. $\mathrm{Vol} \geq V$,
4. $\int_M |\mathrm{Rm}|^2 \leq \epsilon$,

then M *admits a flat metric.*

Corollaries on compactness and diffeofiniteness

Let us recall a typical statement of "pinching" for Riemannian manifolds:

Theorem

(Gao, Anderson, Anderson-Cheeger) Given $V, D, H > 0$ there exists $\epsilon = \epsilon(V, D, H)$ so that if (M^4, g) is a compact Riemannian manifold satisfying:

1. $|\mathrm{Rc}| \leq H$
2. $\mathrm{diam} \leq D$
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These results all rely, implicitly or explicitly, on smoothing properties for second order elliptic or parabolic PDE (Ricci flow), usually through the technique of Moser iteration, which requires a "supercritical" estimate to work.

Corollary (______ 2013)

Given $0 < \delta < 1$ *and* $\rho, V > 0$ *there exists* $\epsilon = \epsilon(\delta, \rho, V) > 0$ *such that given* (M^4, g) *a compact Riemannian manifold satisfying*

1. $\text{Vol} \leq V$,
2. $r_\delta \geq \rho$,
3. $\mathcal{F}(g) \leq \epsilon$,

then the solution to the L^2 *flow with initial condition* g *exists for all time and converges exponentially to a flat metric.*

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Given $A, V > 0$ there exists $\epsilon = \epsilon(A, V) > 0$ so that if (M^4, g) is a compact Riemannian four-manifold satisfying

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2. $C_S \leq A$,
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Corollary (______ 2013)

Given $0 < \delta < 1$, *there exists* $\epsilon(\delta) > 0$ *so that for any* $\rho, V > 0$, *there are only finitely many diffeomorphism types of compact Riemannian manifolds* (M^4, g) *satisfying*

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Proof of Smoothing Result

By an overall parabolic rescaling, it suffices to show that if $r_\delta \geq 1$, for ϵ chosen sufficiently small with respect to δ the solution to L^2 flow exists on $[0, 1]$, and satisfies estimates

$$\left|\mathrm{Rm}\right|_{g_t} < A\mathcal{F}^{\frac{1}{6}}(g_t)t^{-\frac{1}{2}}, \qquad \mathrm{inj}_{g_t} > \iota t^{\frac{1}{4}}.$$

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By continuity, these estimates hold on $[0, \epsilon)$. If they failed to hold on $[0, 1]$, choose the first time τ at which one of the above is an equality, and rescale again so that the equality occurs at $[0, 1]$.

Our task is to derive contradictions from either possible equality.

Case I: Curvature inequality fails

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$$\mathcal{F}(g_1) \geq \int_{B_{\mu K}(x)} \left|\overline{\mathrm{Rm}}\right|^2 \geq \frac{K^2}{4} \mathrm{Vol}(B_{\mu K_i}(x_i)) \geq c\mu^4 K^6 = c\mu^4 A^6 \mathcal{F}(g_1) > \mathcal{F}(g_1),$$

a contradiction.

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Let $U = B_{\frac{1}{2}}(x, g_0)$. By construction we have $\mathrm{Vol}_{g_0}(U) \geq \frac{\delta\omega_4}{16}$. We will show two estimates:

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Together these imply that the volume ratio of unit balls at time $t = 1$ is at least some controlled fraction of δ, which will yield the final contradiction provided ι is chosen small with respect to δ.

Estimate 1: We can directly compute the evolution of volume of an open set U via

$$\frac{d}{dt}\int_U dV_g = \int_U \mathrm{tr}_g \,\mathrm{grad}\,\mathcal{F} dV_g \geq -C \left|\left|\mathrm{grad}\,\mathcal{F}\right|\right|_{L^2} \mathrm{Vol}_{g_t}(U)^{\frac{1}{2}}.$$

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But $\operatorname{grad}\mathcal{F} = \nabla^2 \operatorname{Rm} + \operatorname{Rm}^{*2} \approx t^{-1}$, so this is not integrable!.

Proof of Smoothing Result

To overcome this crucial obstacle we need to find a way to use our one "supercritical" estimate: the fundamental energy estimate

$$\int_0^t \int_M |\mathrm{grad}\,\mathcal{F}|^2 \, dV_g \, dt = \mathcal{F}(g_0) - \mathcal{F}(g_t) \leq \epsilon.$$

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The crucial idea to obtain the required distance estimate is to replace the integral of grad $\mathcal{F}$ along a curve by averaging over a tubular neighborhood.

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Thus, by averaging over the disc or radius r_t orthogonal to γ, one obtains

$$\left|\operatorname{grad} \mathcal{F}\right|(p, t) \leq \operatorname{Area}(D_{r_t}(p))^{-\frac{1}{2}} \left[\int_{D_{r_t}(p)} \left|\operatorname{grad} \mathcal{F}\right|^2 (q) dA(q)\right]^{\frac{1}{2}} + CRt^{\alpha - \frac{5}{4}}.$$

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Ideally, provided $r_t \leq \mathrm{inj}_{g_t} \approx t^{\frac{1}{4}}$, one has $\mathrm{Area}(D_{r_t}(p)) \geq r_t^{n-1} \geq C(R)t^{\alpha(n-1)}$.

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$$\begin{aligned}\frac{d}{dt}L(\gamma) &\leq CR^{1-n}t^{-\frac{\alpha(n-1)}{2}} \int_\gamma \left[\int_{D_{r_t}} |\text{grad}\,\mathcal{F}|^2\, dA\right]^{\frac{1}{2}} + CRL(\gamma)t^{\alpha - \frac{5}{4}} \\ &\leq CR^{1-n}L(\gamma)^{\frac{1}{2}}t^{-\frac{\alpha(n-1)}{2}} \left[\int_\gamma \int_{D_{r_t}} |\text{grad}\,\mathcal{F}|^2\, dA\right]^{\frac{1}{2}} + CRL(\gamma)t^{\alpha - \frac{5}{4}}.\end{aligned}$$

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Integrating over time yields

$$L(\gamma_T) \lesssim L(\gamma_0) + R^{1-n} \int_0^T t^{-\frac{\alpha(n-1)}{2}} \left[\int_\gamma \int_{D_{r_t}} |\text{grad}\, \mathcal{F}|^2 \, dA \right]^{\frac{1}{2}} + CR \int_0^T t^{\alpha - \frac{5}{4}}$$

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$$L(\gamma_T) \lesssim L(\gamma_0) + R^{1-n} \int_0^T t^{-\frac{\alpha(n-1)}{2}} \left[\int_\gamma \int_{D_{r_t}} |\text{grad}\, \mathcal{F}|^2 \, dA \right]^{\frac{1}{2}} + CR \int_0^T t^{\alpha - \frac{5}{4}}$$

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The third term is

- Finite $\leftrightarrow \alpha > \frac{1}{4}$.
- Small if R is chosen small.

With this choice made, the second term is

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Thus there is an appropriate choice of α if and only if $n \leq 4$, as required!

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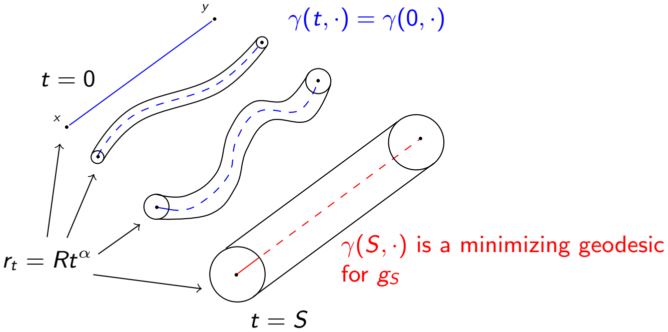
- Allow a given curve to acquire some geodesic curvature, but ensure the disc neighborhood can be constructed
- Once too much geodesic curvature is acquired, pick a new geodesic and continue!



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- One then shows new local smoothing estimates for parabolic balls with curvature bounds to obtain higher order control.
- With these preliminaries, one can construct a local blowup limit, which by energy estimates, is automatically critical.
- Finally, via a new ϵ-regularity result ensures that at least some energy must have concentrated near the blowup point.

Calabi Flow

Let (M^{2n}, g, J) be a compact Kähler manifold.

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This flow bears a close family resemblance to the L^2 flow, and many analytic techniques can be shared between these two flows. In particular, the singularity decomposition for $n = 4$ applies to Calabi flow on complex surfaces.

Thank You!