Smooth solutions to portfolio liquidation problems under price-sensitive market impact

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August 29, 2013

\textsuperscript{1}Based on Joint work with P. Graewe and E. Séré
Outline

- Portfolio liquidation/acquisition under market impact
  - liquidation with active orders
  - liquidation with active and passive orders
- Markovian Control Problem (with P. Graewe and E. Séré)
  - An HJB equation with singular terminal value
  - Existence of short-time solutions
  - Verification argument
- Non-Markovian Control Problem (with P. Graewe and J. Qiu)
  - A BSPDE with singular terminal value
  - Existence of solutions
  - Verification argument
- Conclusion
Portfolio Liquidation
Portfolio Liquidation

- Traditional financial market models assume that investors can buy sell arbitrary amounts at given prices.
- This neglects *market impact*: large transactions (1%-3% of ADV, or more) move prices in an unfavorable direction.
Portfolio Liquidation

- Economists have long studied models of optimal block trading
  - Their focus is often on informational asymmetries
  - *Stealth trading*: split large blocks into a series of smaller ones

- Mathematicians identified this topic only more recently
  - Their focus is often on ‘structural models’ (algorithmic trading)
  - Models of optimal portfolio liquidation give rise to novel stochastic control problems:
    - (‘Liquidation’) constraint on the terminal state
    - Value functions with *singular terminal value*
    - PDEs, BSDEs, BSPDEs, .... with singular terminal values
Almost all trading nowadays takes place in limit order markets.

- Limit order book: list of prices and available liquidity
- Limited liquidity available at each price level

There are (essentially) two types of orders one can submit:

- *active* orders submitted for immediate execution
- *passive* orders submitted for future execution

We allow active and passive orders; *price sensitive impact*

- Markovian model: PDE with singular terminal condition
- non-Markovian model: BSPDE with singular terminal condition
Liquidation with active orders

Consider an order to sell $X > 0$ shares by time $T > 0$:

- $\xi_t$ rate of trading (control)
- $X_t = X - \int_0^t \xi_s \, ds$ remaining position (controlled state)
- $S_t$ market/benchmark price (uncontrolled state)

The optimal liquidation problem is of the form

$$\min_{(\xi_t)} \mathbb{E} \left[ \int_0^T f(\xi_t, S_t, X_t) \, dt \right] \quad \text{s.t. } X_T- = 0$$

The liquidation constraint results in a singularity of the value function:

$$\lim_{t \to T-} V(t, S, X) = \begin{cases} +\infty & \text{for } X \neq 0 \\ 0 & \text{for } X = 0 \end{cases}$$
Benchmark: linear temporary impact

For some martingale \((S_t)\), the 
transaction price

is given by

\[
\tilde{S}_t = S_t - \eta \xi_t \quad (\eta = \text{market impact factor}).
\]

The *liquidity costs* are then defined as

\[
\mathcal{C} = \text{book value} - \text{revenue}
= S_0 X - \int_0^T \tilde{S}_t \xi_t \, dt = -\int_0^T X_t \, dS_t + \int_0^T \eta \xi_t^2 \, dt
\]

and the *expected liquidity costs* are

\[
\mathbb{E}[\mathcal{C}] = \int_0^T \eta \xi_t^2 \, dt.
\]

Usually, one minimizes expected liquidation + risk costs.
Literature review

• Almgren & Chriss (2000): mean-variance, $S_t$ BM

$$\int_0^T \eta \xi_t^2 + \lambda \sigma^2 X_t^2 \, dt \rightarrow \min$$

• Gatheral & Schied (2011): time-averaged VaR, $S_t$ GBM

$$\mathbb{E} \left[ \int_0^T \eta \xi_t^2 + \lambda S_t X_t \, dt \right] \rightarrow \min$$

• Ankirchner & Kruse (2012): similar but $dS_t = \sigma(S_t) dW_t$

$$\mathbb{E} \left[ \int_0^T \eta \xi_t^2 + \lambda (S_t) X_t^2 \, dt \right] \rightarrow \min$$

• and many others ....
Markovian Models
Liquidation with active and passive orders

Modeling the impact of active orders is comparably simple; the impact of passive orders is harder to model:

- how does the market react to passive order placement?
- using active and passive orders simultaneously may lead to market manipulation
- ....

To overcome this problem, we assume that passive orders are placed in a dark pool:

- passive orders are not openly displayed
- executed only when matching liquidity becomes available
- if executed, then at prices coming from some primary venue

Dark trading: *reduced trading costs vs. execution uncertainty.*
Liquidation with active and passive orders

We allow for active and passive orders:

- active order placements: \((\xi_t)_{t \in [0, T)}\)
- passive order placements: \((\nu_t)_{t \in [0, T)}\)

For \(X_0 = X\) the portfolio dynamics is given by

\[
dX_t = -\xi_t \, dt - \nu_t \, d\pi_t \quad \text{with} \quad X_{T-} = 0 \quad \text{a.s.}
\]

Our value function is given by

\[
V(T, S, X) = \inf_{(\xi, \nu) \in \mathcal{A}(T, X)} \mathbb{E} \left[ \int_0^T \eta(S_t)|\xi_t|^p + \gamma(S_t)|\nu_t|^p + \lambda(S_t)|X_t|^p \, dt \right]
\]

where the coefficients \(\eta, \sigma, \gamma, \lambda\) are nice enough and \(p > 1\).
Remark (Power-structure of cost function)

Kratz (2012) and H & Naujokat (2013) consider the cost function

\[
\mathbb{E} \left[ \int_0^T \eta |\xi_t|^2 + \gamma |\nu_t|^1 + \lambda |X_t|^2 \, dt \right].
\]

In this case, no passive orders are used after first execution. This property does not carry over to price-sensitive impact factors. We thus consider

\[
\mathbb{E} \left[ \int_0^T \eta(S_t)|\xi_t|^p + \gamma(S_t)|\nu_t|^p + \lambda(S_t)|X_t|^p \, dt \right].
\]
Theorem (Structure of the Value Function)

The value function is of the form (‘power-utility’)

\[ V(T, S, X) = v(T, S)|X|^p \]

and the optimal controls are:

\[ \xi^*_t = \frac{v(T - t, S_t)^\beta}{\eta(S_t)^\beta} X_t, \quad \nu^*_t = \frac{v(T - t, S_t)^\beta}{\gamma(S_t)^\beta + v(T - t, S_t)^\beta} X_t, \]

where \( \beta := \frac{1}{p-1} > 0 \) and the “inflator” \( v \) solves the PDE

\[ v_T = \frac{1}{2} \sigma^2(S)v_{SS} + \lambda(S) - \frac{1}{\beta \eta(S)^\beta} v^{\beta+1} - \theta \left( v - \frac{\gamma(S)v}{(\gamma(S)^\beta + v^\beta)^{1/\beta}} \right) \]

\[ F(S, v) \]
Boundary condition for $v$

The final position when following $\xi^*$ and $\nu^*$ is

$$X \exp \left( - \int_0^T \frac{v(T - t, S_t)^\beta}{\eta(S_t)^\beta} dt \right) \prod_{0 \leq t < T} \left( 1 - \frac{v(T - t, S_t)^\beta}{\gamma(S_t)^\beta + v(T - t, S_t)^\beta} \right).$$

- To ensure $X_{T^-}^* = 0$ one needs

$$\frac{v(T - t, S)^\beta}{\eta(S)^\beta} \rightarrow \infty \quad \text{as} \quad t \rightarrow T \quad (\text{uniformly in } S).$$

- Through a-priori estimates one shows that

$$v(T, S) \sim \frac{\eta(S)}{T^{\frac{1}{\beta}}} \quad \text{as} \quad T \rightarrow 0 \quad \text{uniformly in } S.$$

If $\eta \equiv \text{const}$, no passive orders, then this holds automatically.
Theorem (PDE for $v$)

After a change of variables, the inflator $v$ is the unique classical solution of

$$v_t = \frac{1}{2} \Delta v - \frac{1}{2} \sigma'(x) \nabla v + F(x, v)$$

such that

$$v(t, x) \to 0 \quad \text{as } t \to 0 \text{ uniformly in } x.$$

This solution satisfies:

$$v(t, x) \sim \frac{\eta(x)}{t^{\frac{1}{\beta}}} \quad \text{as } t \to 0 \text{ uniformly in } x.$$
Remark

- The operator \( A = \frac{1}{2} \Delta - \frac{1}{2} \sigma'(x) \nabla \) generates an analytic (yet not strongly continuous) semigroup \( e^{tA} \) in \( C(\mathbb{R}) \) and a priori bounds give that any short-time solution extends to a global solution.

- For the short-time solution, we express the asymptotics in terms of an equation:

\[
v(t, x) = \frac{\eta(x)}{t^{\frac{1}{\beta}}} + \text{‘correction’}
\]
Existence of a short-time solution

Our ansatz is to additively separate the “leading singular term”:

\[ v(t, x) = \frac{\eta(x)}{t^{\frac{1}{\beta}}} + \frac{u(t, x)}{t^{\frac{1}{\beta}} + 1}, \quad u(t, x) \in O(t^2) \text{ as } t \to 0 \text{ uniformly in } x \]

Results in an evolution equation in \( C(\mathbb{R}) \) for the correction term:

\[ u'(t) = Au + f(t, u(t)), \quad u(0) \equiv 0, \]

with the singular nonlinearity of the form:

\[ f(t, u(t)) = \ldots \sum_{k=2}^{\infty} \ldots \left( \frac{u(t)}{t\eta} \right)^k \ldots. \]

Remark

We move the singularity from the terminal condition into the non-linearity in such a way that it causes no harm.
Existence of a short-time solution

The contraction argument giving a short-time solution by a fixed point of the operator

$$\Gamma(u)(t) = \int_0^t e^{(t-s)A}f(s, u(s)) \, ds$$

is then carried out in the space

$$E = \{ u \in C([0, \delta]; C(\mathbb{R})) : \| u \|_E < \infty \}$$

where

$$\| u \|_E = \sup_{t \in (0, \delta]} \| t^{-2} u(t) \|$$

Theorem (Existence of solutions)

The operator $\Gamma$ has a fixed point for all sufficiently small $t \in [0, T]$. 
Lemma

It is enough to consider only strategies that yield monotone portfolio processes. For such strategies

\[ \mathbb{E} \left[ \nu(T - t, S_t)|X_t^{\xi,\nu}|^p \right] \rightarrow 0 \quad \text{as} \quad t \rightarrow T. \]

Theorem (Value Function)

The value function for our control problem is

\[ V(T, S, X) = \nu(T, X)|X|^p. \]
Non-Markovian Models
Probability space

Consider a probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) with \{\mathcal{F}_t\}_{t \geq 0}\) being generated by three mutually independent processes:

- \textit{m}-dimensional Brownian motion \(W\);
- \textit{m}-dimensional Brownian motion \(B\);
- stationary Poisson point process \(J\) on \(\mathbb{L} \subset \mathbb{R}^l\) with
  - \textit{finite} characteristic measure : \(\mu(dz)\);
  - counting measure \(\pi(dt, dz)\) on \(\mathbb{R}_+ \times \mathbb{L}\); and
  - \(\{\tilde{\pi}([0, t] \times A)\}_{t \geq 0}\) a martingale where

\[\tilde{\pi}([0, t] \times A := \pi([0, t] \times A) - t\mu(A).\]

- The filtration generated by \(W\) is denoted \(\mathcal{F}\).
The control problem

- The controlled process is

\[ x_t = x - \int_0^t \xi_s \, ds - \int_0^t \int \rho_s(z) \pi(dz, ds) ; \quad x_T^{-} = 0 \]

the set of admissible strategies is the set of all pairs

\[(\xi, \rho) \in \mathcal{L}^2(0, T) \times \mathcal{L}^4(0, T ; L^2(\mathcal{L})) \] with \( x_T^{-} = 0 \) a.s.

- The uncontrolled factors follow the dynamics

\[ y_t = y + \int_0^t b_s(y_s, \omega) \, ds + \int_0^t \bar{\sigma}_s(y_s, \omega) \, dB_s + \int_0^t \sigma_s(y_s, \omega) \, dW_s \]

where the processes \( b(y, \cdot), \sigma(y; \cdot), \bar{\sigma}(y, \cdot) \) are \( \mathcal{F} \)-adapted.
The value function

Just as above, the objective is to minimize the cost functional

$$J_t(x_t, y_t; \xi, \rho) =: \mathbb{E} \left[ \int_0^T \left( \eta_s(y_s, \omega)|\xi_s|^2 + \lambda_s(y_s, \omega)|x_s|^2 \right) ds ight. + \left. \int_{[0, T] \times \mathcal{Y}} \left( \gamma_s(y_s, z, \omega)|\rho_s(z)|^2 \mu(dz) \right) ds \right]$$

The resulting value function is

$$V_t(x, y) =: \text{ess inf}_{\xi, \rho} J_t(x_t, y_t; \xi, \rho) \bigg|_{x_t=x, y_t=y}$$
We expect the value function \( V_t(x, y) \) to satisfy the BSPDE:

\[
- dV_t(x, y) = \left[ \text{tr} \left( \frac{1}{2} \left( \sigma_t \sigma_t^\mathcal{T} + \tilde{\sigma}_t \tilde{\sigma}_t^\mathcal{T} \right) \right) \partial_{yy} V_t(x, y) + \partial_y \Psi_t(x, y) \sigma_t^\mathcal{T}(y) \right. \\
\left. + b_t^\mathcal{T} \partial_y V_t(x, y) + \text{ess inf}_{\xi, \rho} \left\{ \eta_t |\xi|^2 + \lambda_t |x|^2 - \xi \partial_x V_t(x, y) \right. \right. \\
\left. \left. + \int_{\mathcal{Z}} \left( V_t(x - \rho, y) - V_t(x, y) + \gamma_t(y, z) |\rho|^2 \right) \mu(dz) \right\} \right] dt \\
- \Psi_t(x, y) dW_t, \quad (t, x, y) \in [0, T) \times \mathbb{R} \times \mathbb{R}^d; \\
V_T(x, y) = (+\infty) 1_{x \neq 0}, \quad (x, y) \in \mathbb{R} \times \mathbb{R}^d.
\]

A solution is a pair of adapted processes \((V, \Psi)\) s.t. (i) ... (ii) ....
Making the same ansatz as before:

\[ V_t(x, y) = u_t(y)x^2 \quad \text{and} \quad \Psi_t(x, y) = \psi_t(y)x^2, \]

we now obtain a BSPDE for the inflator. It is of the form:

\[
\begin{cases}
-du_t(y) = \left[ \text{tr} \left( a_t \partial_{yy} u_t(y) + \partial_y \psi_t(y) \sigma^T_t \right) + b_t^T \partial_y u_t(y) \\
- \int_{\mathcal{Z}} \frac{|u_t(y)|^2}{\gamma(t, y, z) + u_t(y)} \mu(dz) - \frac{|u_t(y)|^2}{\eta_t(y)} + \lambda_t(y) \right] dt \\
- \psi_t(y) dW_t, \quad (t, y) \in [0, T] \times \mathbb{R}^d; \\
u_T(y) = +\infty, \quad y \in \mathbb{R}^d.
\end{cases}
\]
Theorem (Verification Theorem)

Suppose \((u, \psi)\) is a solution to BSPDE \((E)\) such that ... and a.s.

\[
\frac{c_0}{T - t} \leq u_t(y) \leq \frac{c_1}{T - t}.
\]

Then

\[
V(t, y, x) := u_t(y)x^2, \quad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d,
\]

coincides with the value function for almost every \(y \in \mathbb{R}^d\), and the optimal (feedback) control is given by

\[
(\xi^*_t, \rho^*_t(z)) = \left( \frac{u_t(y_t)x_t}{\eta_t(y_t)}, \frac{u_t(y_t)x_t-}{\gamma_t(z, y_t) + u_t(y_t)} \right).
\]
Theorem (Existence of solutions)

Our BSPDE \( \mathcal{E} \) admits a unique solution \((u, \psi)\) such that ... and

\[
\frac{c_0}{T - t} \leq u_t(y) \leq \frac{c_1}{T - t}, \quad \mathbb{P} \otimes dt \otimes dy - a.e. \tag{1}
\]

Under suitable stronger conditions on \( \sigma \) we have that

\[
V(t, y, x) := u_t(y)x^2, \quad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d, \tag{2}
\]

coincides with the value function for every \( y \in \mathbb{R}^d \).
Remark

The proof is based on the penalization method; consider BSPDEs

\[
- \, du_{t}^{N}(y) = \left[ \text{tr} \left( a_{t} \partial_{yy}^{2} u_{t}^{N}(y) + \partial_{y} \psi_{t}^{N}(y) \sigma_{t}^{\mathcal{F}} \right) + b_{t}^{\mathcal{F}} \partial_{y} u_{t}^{N}(y) \\
- \int_{\mathcal{F}} \frac{|u_{t}^{N}(y)|^2}{\gamma(t, y, z) + u_{t}^{N}(y)} \mu(dz) - \frac{|u_{t}^{N}(y)|^2}{\eta_{t}^{N}(y)} + \lambda_{t}^{N}(y) \right] dt \\
- \psi_{t}^{N}(y) dW_{t}, \quad (t, y) \in [0, T] \times \mathbb{R}^{d}; \\
u_{T}^{N}(y) = N, \quad y \in \mathbb{R}^{d}.
\]

and establish their convergence. Converge has to be fast enough. This is the hard part which our method in the Markovian case bypassed.
Conclusion

- We studied control problems with singular terminal conditions arising in models of optimal portfolio liquidation.
- In the Markovian framework we showed that the HJB PDE has a strong solution, and ...
- ... obtained detailed information about the degree of the singularity at the terminal time.
- In the non-Markovian framework we solved a BSPDE with singular terminal condition by means of penalization, and ...
- ... also obtained detailed information about the degree of the singularity at the terminal time.
- Open problem: permanent market impact
- Major open problem: different powers for active and passive orders (possible for non-price dependent impact functions).
Thanks