Lecture II: Stochastic volatility modeling in energy markets

Fred Espen Benth

Centre of Mathematics for Applications (CMA)
University of Oslo, Norway

Fields Institute, 19-23 August 2013
Overview

1. Motivate and introduce a class of stochastic volatility models
2. Empirical example from UK gas prices
3. Comparison with the Heston model
4. Forward pricing
5. Discussion of generalizations to cross-commodity modelling
Stochastic volatility model
Motivation

- Annualized volatility of NYMEX sweet crude oil spot
  - Running five-day moving volatility
  - Plot from Hikspoors and Jaimungal 2008
- Stochastic volatility with fast mean-reversion
• Signs of stochastic volatility in financial time series
  • Heavy-tailed returns
  • Dependent returns
  • Non-negative autocorrelation function for squared returns

• Energy markets
  • Mean-reversion of (log-)spot prices
  • seasonality
  • Spikes
  • ... so, how to create reasonable stochastic volatility models?
The stochastic volatility model

- Simple one-factor Schwartz model
  - but with stochastic volatility

\[ S(t) = \Lambda(t) \exp(X(t)), \quad dX(t) = -\alpha X(t) \, dt + \sigma(t) \, dB(t) \]

- \( \sigma(t) \) is a stochastic volatility (SV) process
  - Positive
  - Fast mean-reversion

- \( \Lambda(t) \) deterministic seasonality function (positive)
• Motivated by Barndorff-Nielsen and Shephard (2001): 
  \( n \)-factor volatility model

\[
\sigma^2(t) = \sum_{j=1}^{n} \omega_j Y_j(t)
\]

where

\[
dY_j(t) = -\lambda_j Y_j(t) \, dt + dL_j(t)
\]

• \( \lambda_j \) is the speed of mean-reversion for factor \( j \)
• \( L_j \) are Lévy processes with only positive jumps
  • subordinators being driftless
  • \( Y_j \) are all positive!
• The positive weights \( \omega_j \) sum to one
• Simulation of a 2-factor volatility model
• Path of $\sigma^2(t)$
Stationarity of the log-spot prices

- After de-seasonalizing, the log-prices become stationary

\[ X(t) = \ln S(t) - \ln \Lambda(t) \sim \text{stationary}, \quad t \to \infty \]

- The limiting distribution is a variance-mixture
  - Conditional normal distributed with zero mean

\[ \ln S(t) - \ln \Lambda(t) | Z = z \sim N(0, z) \]

- \( Z \) is characterized by \( \sigma^2(t) \) and the spot-reversion \( \alpha \)
• Explicit expression the cumulant (log-characteristic function) of the stationary distribution of $X(t)$:

$$\psi_X(\theta) = \sum_{j=1}^{n} \int_{0}^{\infty} \psi_j \left( \frac{1}{2} i \theta^2 \omega_j \gamma(u; 2\alpha, \lambda_j) \right) du$$

• $\psi_j$ cumulant of $L_j$

• The function $\gamma(u; a, b)$ defined as

$$\gamma(u; a, b) = \frac{1}{a - b} \left( e^{-bu} - e^{-au} \right)$$

• $\gamma$ is positive, $\gamma(0) = \lim_{u \to \infty} \gamma(u) = 0$, and has one maximum.
• Each term in the limiting cumulant of $X(t)$ can be written as the cumulant of centered normal distribution with variance

$$\tilde{\psi}_X(\theta) = \int_0^\infty \psi_j (\theta \omega_j \gamma(u; 2\alpha, \lambda_j)) \, du$$

• One can show that $\tilde{\psi}_X(\theta)$ is the cumulant of the stationary distribution of

$$\int_0^t \gamma(t - u; 2\alpha, \lambda_j) \, dL_j(u)$$
• Recall the constant volatility model $\sigma^2(t) = \sigma^2$
  • The Schwartz model

• Explicit stationary distribution

$$\ln S(t) - \ln \Lambda(t) \sim \mathcal{N} \left( 0, \frac{\sigma^2}{2\alpha} \right)$$

• SV model gives heavy-tailed stationary distribution
  • Special cases: Gamma distribution, inverse Gaussian distribution....
Probabilistic properties

- ACF of $X(t)$ is given as

$$
corr(X(t), X(t + \tau)) = \exp(-\alpha \tau)
$$

- No influence of the volatility on the ACF of log-prices
  - Energy prices have multiscale reversion
  - Above model is too simple, multi-factor models required
• Consider reversion-adjusted returns over \([t, t + \Delta]\)

\[ R_\alpha(t, \Delta) := X(t) - e^{-\alpha \Delta} X(t-1) = \int_t^{t+\Delta} \sigma(s) e^{-\alpha(t+\Delta-s)} dB(s) \]

• Approximately,

\[ R_\alpha(t, \Delta) \approx \sqrt{\frac{1 - e^{-2\alpha \Delta}}{2\alpha}} \sigma(t) \Delta B(t) \]
• $R_\alpha(t, \Delta)$ is a variance-mixture model

$$R_\alpha(t, \Delta)|\sigma^2(t) \sim \mathcal{N}(0, \frac{1 - e^{-2\alpha \Delta}}{2\alpha} \sigma^2(t))$$

• Thus, knowing the stationary distribution of $\sigma^2(t)$, we can create distributions for $R_\alpha(t, \Delta)$
  • Based on empirical observations of $R_\alpha(t, \Delta)$, we can create desirable distributions from the variance mixture

• The reversion-adjusted returns are uncorrelated
• ...but squared reversion-adjusted returns are correlated

\[
corr(R_\alpha^2(t + \tau, \Delta), R_\alpha^2(t, \Delta)) = \sum_{j=1}^{n} \hat{\omega}_j e^{-\lambda_j \tau}
\]

• \(\hat{\omega}_j\) positive constants summing to one, given by the second moments of \(L_j\)
• ACF for squared reversion-adjusted returns given as a sum of exponentials
  • Decaying with the speeds \(\lambda_j\) of mean-reversions
• This can be used in estimation
Empirical example: UK gas prices
• NBP UK gas spot data from 06/02/2001 till 27/04/2004
  • 806 records, weekends and holidays excluded
• Seasonality modelled by a sine-function for log-spot prices
Estimate $\alpha$ by regressing $\ln \tilde{S}(t + 1)$ against $\ln \tilde{S}(t)$

$$\tilde{\alpha} = 0.127$$

Regression has $R^2 = 78$

Half-life: expected time until a shock is halfed in size

$$\text{half life} = \frac{\ln 2}{\tilde{\alpha}}$$

Half-life corresponding to 5.5 days
• Plot of residuals: histogram, ACF and ACF of squared residuals
  • Fitted speed of mean-reversion of volatility: $\hat{\lambda} = 1.1$. 
The normal inverse Gaussian distribution

- The residuals are not reasonably modelled by the normal distribution
  - Peaky in the center, heavy tailed
- Motivated from finance, use the normal inverse Gaussian distribution (NIG)
- Four-parameter family of distributions
  - $a$: tail heaviness
  - $\delta$: scale (or volatility)
  - $\beta$: skewness
  - $\mu$: location
• Density of the NIG

\[
f(x; a, \beta, \delta, \mu) = c \exp(\beta(x - \mu)) \frac{K_1 \left( a\sqrt{\delta^2 + (x - \mu)^2} \right)}{\sqrt{\delta^2 + (x - \mu)^2}}
\]

where \( K_1 \) is the modified Bessel function of the third kind with index one

\[
K_1(x) = \frac{1}{2} \int_0^\infty \exp \left( -\frac{1}{2} x(z + z^{-1}) \right) \, dz
\]

• Explicit (log-)moment generating function

\[
\phi(u) := \ln \mathbb{E}[e^{uL}] = u\mu + \delta \left( \sqrt{a^2 - \beta^2} - \sqrt{a^2 - (\beta + u)^2} \right)
\]
• Fitted symmetric centered NIG using maximum likelihood

\[ \hat{a} = 4.83, \quad \hat{\delta} = 0.071 \]
• Question: Does there exist SV driver $L$ such that residuals become NIG distributed?

• Answer is YES!

• There exists $L$ such that stationary distribution of $\sigma^2(t)$ is Inverse Gaussian distributed
  • Let $Z$ be normally distributed
  • The positive part of $1/Z$ is then Inverse Gaussian

• Conclusion:
  • Choose $L$ such that $\sigma^2(t)$ is Inverse Gaussian with specified parameters from the NIG estimation
  • Choose $\alpha$, $\lambda$ as estimated
  • Choose the seasonal function as estimated
  • Full specification of the SV volatility spot price dynamics
The Heston Model: a comparison
• Heston’s stochastic volatility: $\sigma^2(t) = Y(t)$,

$$dY(t) = \eta(\zeta - Y(t)) dt + \delta \sqrt{Y(t)} \, d\tilde{B}(t)$$

• $\tilde{B}$ independent Brownian motion of $B(t)$
  • In general Heston, $\tilde{B}$ correlated with $B$
  • Allows for leverage
• $Y$ recognized as the Cox-Ingersoll-Ross dynamics
  • Ensures positive $Y$
• The cumulant of stationary $Y$ is known (Cox, Ingersoll and Ross, 1981)

$$
\psi_Y(\theta) = \zeta c \ln \left( \frac{c}{c - i\theta} \right), \quad c = \frac{2\eta}{\delta^2}
$$

• Cumulant of a $\Gamma(c, \zeta c)$-distribution
• Can obtain the same stationary distribution from our SV-model
• Choose a one-factor model $\sigma^2(t) = Y(t)$

$$dY(t) = -\lambda Y(t)\,dt + dL(t)$$

• $L(t)$ a compound Poisson process with exponentially distributed jumps with expected size $1/c$

• Choose $\lambda$ and the jump frequency $\rho$ such that $\rho/\lambda = \zeta c$

• Stationary distribution of $Y$ is $\Gamma(c, \zeta c)$. 
• Question: what is the stationary distribution of $X(t)$ under the Heston model?

• Expression for the cumulant at time $t$

$$\psi_{X}(t, \theta) = i\theta X(0)e^{-\alpha t} + \ln \mathbb{E} \left[ \exp \left( -\frac{1}{2} \theta^2 \int_0^t Y(s)e^{-2\alpha(t-s)} \, ds \right) \right]$$

• An expression for the last expectation is unknown to us
  • The cumulant can be expressed as an affine solution
  • Coefficients solutions of Riccati equations, which are not analytically solvable
  • ...at least not to me....

• In our SV model the same expression can be easily computed
Application to forward pricing
• Forward price at time $t$ an delivery at time $T$

$$F(t, T) = \mathbb{E}_Q [S(T) | \mathcal{F}_t]$$

• $Q$ an equivalent probability, $\mathcal{F}_t$ the information filtration
• Incomplete market
  • No buy-and-hold strategy possible in the spot
  • Thus, no restriction to have $S$ as $Q$-martingale after discounting
• Choose $Q$ by a Girsanov transform

$$dW(t) = dB(t) - \frac{\theta(t)}{\sigma(t)} \, dt$$

• $\theta(t)$ bounded measurable function
  • Usually simply a constant
  • Known as the *market price of risk*

• Novikov’s condition holds since

$$\sigma^2(t) \geq \sum_{j=1}^{n} \omega_j Y_j(0)e^{-\lambda_j t}$$
The $Q$ dynamics of $X(t)$, the deseasonalized log-spot price

$$dX(t) = (\theta(t) - \alpha X(t)) \, dt + \sigma(t) \, dW(t)$$

For simplicity it is supposed that there is no market price of volatility risk

- No measure change of the $L_j$'s

Esscher transform could be applied

- Exponential tilting of the Lévy measure, preserving the Lévy property
- Will make big jumps more or less pronounced
- Scale the jump frequency
• Analytical forward price available (suppose one-factor SV for simplicity)

\[
F(t, T) = \Lambda(T)H_\theta(t, T)\exp\left(\frac{1}{2}\gamma(T - t; 2\alpha, \lambda)\sigma^2(t)\right) \\
\times \left(\frac{S(t)}{\Lambda(t)}\right)^{\exp(-\alpha(T-t))}
\]

• Recall the scaling function

\[
\gamma(u; 2\alpha, \lambda) = \frac{1}{2\alpha - \lambda} \left( e^{-\lambda u} - e^{-2\alpha u} \right)
\]
• $H_\theta$ is a risk-adjustment function

$$\ln H_\theta(t, T) = \int_t^T \theta(u) e^{-\alpha(T-s)} ds + \int_0^{T-t} \psi\left(-i\frac{1}{2}\gamma(u; 2\alpha, \lambda)\right) du$$

• Here, $\psi$ being cumulant of $L$
• Note: Forward price may jump, although spot price is continuous
  • The volatility is explicitly present in the forward dynamics
• Recall $\gamma(0; 2\alpha, \lambda) = \lim_{u \to \infty} \gamma(u; 2\alpha, \lambda) = 0$
  - In the short and long end of the forward curve, the SV-term will not contribute

• Scale function has a maximum in
  $u^* = (\ln 2\alpha - \ln \lambda)/(2\alpha - \lambda)$
  - Increasing for $u < u^*$, and decreasing thereafter
  - Gives a hump along the forward curve
  - Hump size is scaled by volatility level $Y(t)$

• Many factors in the SV model gives possibly several humps

• Observe that the term $(S(t)/\Lambda(t))^{\exp(-\alpha(T-t))}$ gives
  - backwardation when $S(t) > \Lambda(t)$
  - Contango otherwise
• Shapes from the “deseasonalized spot”-term in $F(t, T)$ (top) and SV term (bottom)
• The hump is produced by the scale function $\gamma$
• Parameters chosen as estimated for the UK spot prices
Forward price dynamics

\[
\frac{dF(t, T)}{F(t-, T)} = \sigma(t)e^{-\alpha(T-t)} \, dW(t)
\]

\[
+ \sum_{j=1}^{n} \int_{0}^{\infty} \{ e^{\omega_{j} \gamma(T-t;2\alpha,\lambda_{j}) z/2} - 1 \} \, \tilde{N}_{j}(dz, dt)
\]

- \(\tilde{N}\) compensated Poisson random measure of \(L_{j}\)
- Samuelson effect in \(dW\)-term. The jump term goes to zero as \(t \to T\)
Comparison with the Heston model

- Forward price dynamics

\[ F(t, T) = \Lambda(T) G_\theta(t, T) \exp\left(\xi(T - t) Y(t)\right) \left(\frac{S(t)}{\Lambda(t)}\right)^{\exp(-\alpha(T-t))} \]

where

\[ \ln G_\theta(t, T) = \int_t^T \theta(u)e^{-\alpha(T-u)} du + \eta \zeta \int_0^{T-t} \xi(u) du \]
• $\xi(u)$ solves a Riccati equation

$$\xi'(u) = \delta \left( \xi(u) - \frac{\eta}{2\delta} \right)^2 - \frac{\eta^2}{4\delta} + \frac{1}{2} e^{-2\alpha u}$$

• Initial condition $\xi(0) = 0$
• It holds $\lim_{u \to \infty} \xi(u) = 0$ and $\xi$ has one maximum for $u = u^* > 0$
  • Shape much like $\gamma(u; 2\alpha, \lambda)$
Extensions of the SV model
Spikes and inverse leverage

- **Spikes**: sudden large price increase, which is rapidly killed off
  - sometimes also negative spikes occur
- **Inverse leverage**: volatility increases with increasing prices
  - Effect argued for by Geman, among others
  - Is it an effect of the spikes?
• Spot price model

\[ S(t) = \Lambda(t) \exp \left( X(t) + \sum_{i=1}^{m} Z_i(t) \right) \]

where

\[ dZ_i(t) = (a_i - b_i Z_i(t)) \, dt + d\tilde{L}_i(t) \]

• Spikes imply that \( b_i \) are fast mean-reversions
• Typically, \( \tilde{L}_i \) are time-inhomogeneous jump processes, with only positive jumps
  • Negative spikes: must choose \( \tilde{L}_i \) having negative jumps
• Inverse leverage: Let $\tilde{L}_i = L_i$ for one or more of the jump processes

• A spike in the spot price will drive up the vol as well
  • Or opposite, an increase in volatility leads to an increase (spike) in the spot

• Spot model analytically tractable
  • Stationary, with analytical cumulant
  • Probabilistic properties available
  • Forward prices analytical in terms of cumulants of the noises
Cross-commodity modelling

- Suppose that $X(t)$ and $Z_i(t)$ are vector-valued Ornstein-Uhlenbeck processes.
- The volatility structure follows Stelzer’s multivariate extension of the BNS SV model.

\[ dX(t) = AX(t)\, dt + \Sigma(t)^{1/2} \, dW(t) \]

- $A$ is a matrix with eigenvalues having negative real parts
  - ...to ensure stationarity
- $\Sigma(t)$ is a matrix-valued process, $W$ is a vector-Brownian motion.
• The volatility process:

\[ \Sigma(t) = \sum_{j=1}^{n} \omega_j Y_j(t) \]

where

\[ dY_j(t) = \left( C_j Y_j(t) + Y_j(t) C_j^T \right) dt + dL_j(t) \]

• \( C_j \) are matrices with eigenvalues having negative real part
  • ...again to ensure stationarity
• \( L_j \) are matrix-valued subordinators
• The structure ensures that \( \Sigma(t) \) becomes positive definite
• Modelling approach allows for
  • Marginal modelling as above
  • Analyticity in forward pricing, say
  • Flexibility in linking different commodities

• However...not easy to estimate on data
Conclusions

- Proposed an SV model for power/energy markets
- Discussed probabilistic properties, and compared with the Heston model
- Forward pricing, and hump-shaped forward curves
- Extensions to cross-commodity and multi-factor models
- Empirical example from UK gas spot prices
Coordinates

- fredb@math.uio.no
- folk.uio.no/fredb
- www.cma.uio.no


