Polytopes derived from cubic tessellations

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including joint work with

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A Euclidean tessellation is a collection of $n$-polytopes, called cells, which cover $E^n$ and tile it in face-to-face manner.

A Euclidean tessellation $\mathcal{U}$ is said to be regular if its group of symmetries (isometries preserving $\mathcal{U}$) is transitive on the flags of $\mathcal{U}$. The cells of a regular tessellation are convex, isomorphic regular polytopes.
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**REGULAR TESSELLATIONS** \( E^n \):

\[
\begin{align*}
\{4,3^{n-2},4\} & , \ n \geq 2 \\
\{3,6\} & , \ \{6,3\} \\
\{3,3,4,3\} & , \ \{3,4,3,3\}
\end{align*}
\]
• Abstract polytope
• Abstract polytope

• Equivelar abstract polytope $\iff$ Schläfli symbol
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• Classification of equivelar abstract polytopes of type $\{4,4\}$ and $\{4,3,4\}$
The group of symmetries $\Gamma(\mathcal{U})$ of the tessellation $\mathcal{U}$ is a Coxeter group. In this talk we will mostly be concerned with cubic tessellations in dimension 2 and 3 so that $\Gamma(\mathcal{U}) = [4,4]$ or $[4,3,4]$. 
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\[
\Gamma(U) \cong T \rtimes S
\]

where \( T \) is the translation subgroup and \( S \) is the stabilizer of origin (point group of \( U \)).
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$$\Gamma(\mathcal{U}) \cong T \rtimes S$$

where $T$ is the translation subgroup and $S$ is the stabilizer of origin (point group of $\mathcal{U}$).

When $G$ is a fixed-point free subgroup of $\Gamma(\mathcal{U})$ the quotient

$$\mathcal{T} = \mathcal{U} / G$$

is called a (cubic) twistoid.
Twistoid $\mathcal{T}$ is an abstract polytope whose faces are orbits of faces of $\mathcal{U}$ under the action of $G$. 
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Note: $\mathcal{U}/G \cong \mathcal{U}/G'$ $\iff$ $G$ and $G'$ are conjugate in $\Gamma(\mathcal{U})$. 
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Note: $\mathcal{U}/G \cong \mathcal{U}/G' \iff G$ and $G'$ are conjugate in $\Gamma(\mathcal{U})$.

$$\text{Sym}(\mathcal{T}) := \{ \phi \in \Gamma(\mathcal{U}) \mid \phi^{-1} \alpha \phi \in G \text{ for all } \alpha \in G \}$$

$$\text{Aut}(\mathcal{T}) := \text{Sym}(\mathcal{T})/G$$
Fixed-point free crystallographic groups in Euclidean plane:

Generated by:

two independent translations  
two parallel glide reflections (same translation vectors)
Equivelar Toroids of type \{4, 4\}:

Conjugacy classes of vertex stabilizers for \{4, 4\}
Class 1: regular \{4,4\} maps on torus

(Coxeter 1948)

\[
\{4,4\}_{(a,0)(0,a)}, \quad a > 0
\]

\[
\{4,4\}_{(a,a)(a,-a)}, \quad a > 0
\]
Class 2: chiral \{4,4\} maps on torus

(Coxeter 1948)
Class $2_1$: vertex, edge and face transitive $\{4,4\}$ maps on torus

($\acute{S}$irán, Tucker, Watkins, 2001)

\[
\{4,4\}_{(a,a)(b,-b)}, \quad a > b > 0
\]

\[
\{4,4\}_{(a,b)(b,a)}, \quad a > b > 0
\]
Class $2_{02}$: vertex and face transitive $\{4,4\}$ maps on torus

(Hubard 2007; Duarte 2007)

$$\{4,4\}_{(a,0)(0,b)}, \quad a > b > 0$$

$$\{4,4\}_{(a,b)(a,-b)}, \quad a > b > 0$$
Class 4: vertex and face transitive \(\{4,4\}\) maps on torus

(Brehm, Khünel 2008; Hubard, Orbanić, Pellicer, Asia 2007)

\[ a > b > 0, \quad c \geq a - b, \quad c \neq 2a \neq 4c \]

and if \(b | a, c\), then \(\frac{c}{b} \uparrow 1 \pm \frac{a^2}{b^2}\)

\[ \{4,4\}_{(a,b)(c,0)} \]
Equivelar maps of type \{4,4\} on Klein bottle

(Wilson 2006)
Fixed-point free crystallographic groups in Euclidean space:

Six generated by orientation preserving isometries (twists)

Four have orientation reversing generators (glide reflections)

Platycosms are the corresponding 3-manifolds.

Classification of twistoids on platycosms is mostly completed (Hubard, Mixer, Orbanić, Pellicer, Asia) and partially published in two papers.
Platycosm arising from the group generated by

a six-fold twist and a three-fold twist

with parallel axes and congruent translation component is the only platycosm admitting no twistoids.
3-torus is the platycosm arising from the group G generated by three independent translations:
3-torus is the platycosm arising from the group $G$ generated by three independent translations:

How can we place this fundamental region into a fixed cubical lattice $\{4,3,4\}$ so that $G$ is a subgroup of the lattice symmetries?
Twistoid on 3-torus is commonly referred to as 3-toroid.

Conjugacy classes of vertex stabilizers of equivalar 3-toroids of type \{4, 3, 4\}:
Class 1:

Theorem: Each regular rank 4 toroid belongs to one of the three families. (McMullen & Schulte, 2002)

\[ \{4,3,4\}^{(a,0,0)(0,a,0)(0,0,a)} \]

\[ \{4,3,4\}^{(2a,0,0)(0,2a,0)(a,a,a)} \]

\[ \{4,3,4\}^{(a,a,0)(a,-a,0)(0,a,a)} \]
A “closer” view of \( \{4,3,4\}^{(2a,0,0)(0,2a,0)(a,a,a)} \)
Class 2:

**Theorem:** There are no chiral toroids of rank > 3.  
(McMullen & Schulte, 2002)
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**Theorem:** There are no rank 4 toroids with two flag orbits (in Class 2).
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**Theorem:** There are no rank 4 toroids with two flag orbits (in Class 2).

Examples in Class 3:
<table>
<thead>
<tr>
<th>Projection of ( v )</th>
<th>Class of ( P_{\Pi} )</th>
<th>Generators of ( P )</th>
<th>Parameters</th>
<th>Class of ( P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( o )</td>
<td>1</td>
<td>((a,0,0),(0,a,0),(0,0,d))</td>
<td>(a,d &gt; 0, d \neq a)</td>
<td>3</td>
</tr>
<tr>
<td>( o )</td>
<td>1</td>
<td>((a, a, 0),(a, a, 0),(0,0, d))</td>
<td>(a,d &gt; 0)</td>
<td>( 6_A )</td>
</tr>
<tr>
<td>( (v_1 + v_2)/2 )</td>
<td>1</td>
<td>((a,0,0),(0,a,0),(a/2,a/2,d))</td>
<td>(a,d&gt;0, d \neq a/2)</td>
<td>( 6_B )</td>
</tr>
<tr>
<td>( (v_1 + v_2)/2 )</td>
<td>1</td>
<td>((a,0,0),(0,a,0),(a,-a,0),(a,0,d))</td>
<td>(a,d&gt;0,d \neq a)</td>
<td>( 6_C )</td>
</tr>
<tr>
<td>( o )</td>
<td>2</td>
<td>((a,0,0),(0,0,0),(0,0,d))</td>
<td>(a&gt;b&gt;d&gt;0)</td>
<td>( 6_A )</td>
</tr>
<tr>
<td>( o )</td>
<td>2</td>
<td>((a,0,0),(0,0,0),(0,0,d))</td>
<td>(a&gt;b&gt;2d&gt;0)</td>
<td>( 6_B )</td>
</tr>
<tr>
<td>( (v_1 + v_2)/2 )</td>
<td>2</td>
<td>((a,0,0),(0,b,0),(a/2,b/2,d))</td>
<td>(a&gt;b&gt;2d&gt;0)</td>
<td>( 6_C )</td>
</tr>
<tr>
<td>( v_1/2 )</td>
<td>1</td>
<td>((a,0,0),(0,a,0),(a/2,a/2,d))</td>
<td>(a,b&gt;0,d&gt;0)</td>
<td>( 12_A )</td>
</tr>
<tr>
<td>( v_1/2 )</td>
<td>2</td>
<td>((a,0,0),(0,-a,0),(a/2,-a/2,d))</td>
<td>(a,d&gt;0)</td>
<td>( 12_A )</td>
</tr>
<tr>
<td>( v_2/2 )</td>
<td>2</td>
<td>((a,0,0),(0,b,0),(b/2,-b/2,d))</td>
<td>(a&gt;b&gt;0,d&gt;0)</td>
<td>( 12_A )</td>
</tr>
<tr>
<td>( (v_1 + v_2)/2 )</td>
<td>2</td>
<td>((a,0,0),(0,-a,0),(a/2,-a/2,d))</td>
<td>(a,d&gt;0)</td>
<td>( 12_A )</td>
</tr>
<tr>
<td>( (v_1 + v_2)/2 )</td>
<td>4</td>
<td>((a,0,0),(0,c,0),(0,0,d))</td>
<td>(a&gt;b&gt;0,d&gt;0)</td>
<td>( 12_A )</td>
</tr>
<tr>
<td>Projection of $v$</td>
<td>Class of $\mathcal{P}_{\Pi_e}$</td>
<td>Generators of $\mathcal{P}$</td>
<td>Parameters</td>
<td>Class of $\mathcal{P}$</td>
</tr>
<tr>
<td>------------------</td>
<td>-------------------------------</td>
<td>-----------------------------</td>
<td>------------</td>
<td>----------------------</td>
</tr>
<tr>
<td>$o$</td>
<td>1</td>
<td>$(a, 0, -a), (0, a, -a), (c, c, c)$</td>
<td>$a, c &gt; 0$</td>
<td>4</td>
</tr>
<tr>
<td>$(v_1 + v_2)/3$</td>
<td>1</td>
<td>$(a, 0, -a), (0, a, -a), (\frac{a+c}{3}, \frac{a+c}{3}, \frac{-2a+c}{3})$</td>
<td>$a, c &gt; 0, 3</td>
<td>(a + c)$</td>
</tr>
<tr>
<td>$o$</td>
<td>1</td>
<td>$(a, a, -2a), (2a, -a, -a), (c, c, c)$</td>
<td>$a, c &gt; 0$</td>
<td>4</td>
</tr>
<tr>
<td>$(v_1 + v_2)/3$</td>
<td>1</td>
<td>$(a, a, -2a), (2a, -a, -a), (a + c, c, -a + c)$</td>
<td>$a &gt; b &gt; 0, c &gt; 0$</td>
<td>8</td>
</tr>
<tr>
<td>$o$</td>
<td>2</td>
<td>$(a, b, -a - b), (-b, a + b, -a), (c, c, c)$</td>
<td>$a &gt; b &gt; 0, c &gt; 0$</td>
<td>8</td>
</tr>
<tr>
<td>$(v_1 + v_2)/3$</td>
<td>2</td>
<td>$(a, b, -a - b), (-b, a + b, -a), (\frac{a-b+c}{3}, \frac{a+2b+c}{3}, \frac{-2a-b+c}{3})$</td>
<td>$a &gt; b &gt; 0, c &gt; 0$</td>
<td>8</td>
</tr>
</tbody>
</table>
**Didicosm** is the platycosm arising from the group \( G \) generated by

- two half-turn twists with parallel axes and congruent translation component, and
- a twist whose axis does not intersect and is perpendicular to the axes of the other two twists and has the translation component equal to a vector between the other two axes:
Identification of points of the boundary of the fundamental region:
How can we place this fundamental region into a fixed cubical lattice \{4,3,4\} so that \( G \) is a subgroup of the lattice symmetries?
Classification of cubic tessellations on didicosm according to their automorphism groups:

<table>
<thead>
<tr>
<th>Group</th>
<th>Size</th>
<th>Generators</th>
<th>Conditions</th>
<th>Toroidal Covers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>96</td>
<td>$\tau_1, \tau_2, \tau_3, \chi, \chi_1, \chi_2, \chi_3, \rho$</td>
<td>$a = 2b = c$ even, $p = q = 0$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>48</td>
<td>$\tau_1, \tau_2, \tau_3, \chi, \chi_1 \chi_2, \rho$</td>
<td>$a = 2b = c$ odd, $p = \frac{1}{2}, q = 0$</td>
<td>1</td>
</tr>
<tr>
<td>3_1</td>
<td>32</td>
<td>$\tau_1, \tau_2, \tau_3, \chi, \chi_1, \rho$</td>
<td>$a = 2b$, $c$ even, $(p = 0$ or $a \not\in \mathbb{Z})$</td>
<td>1, 3</td>
</tr>
<tr>
<td>3_2</td>
<td>32</td>
<td>$\tau_1, \tau_2, \tau_3, \chi, \chi_2, \rho$</td>
<td>$c = 2b$, $a$ even, $(q = 0$, $c$ even or $q = \frac{1}{2}$, $c$ odd$)$</td>
<td>1, 3</td>
</tr>
<tr>
<td>3_3</td>
<td>32</td>
<td>$\tau_1, \tau_2, \tau_3, \chi, \chi_3, \rho$</td>
<td>$a = c$, $2b$ even, $p = q$</td>
<td>1, 3</td>
</tr>
<tr>
<td>6</td>
<td>16</td>
<td>$\tau_1, \tau_2, \tau_3, \chi, \rho$</td>
<td>$a, 2b \in \mathbb{Z} \cup \sqrt{2}\mathbb{Z}$</td>
<td>1, 3, 6_A, 6_B</td>
</tr>
<tr>
<td>6_1</td>
<td>16</td>
<td>$\tau_1, \chi, \chi_1, \rho$</td>
<td>$a = 2b \not\in \mathbb{Z} \cup \sqrt{2}\mathbb{Z}$, $c$ even</td>
<td>3</td>
</tr>
<tr>
<td>12_2</td>
<td>8</td>
<td>$\tau_1, \tau_2, \rho$</td>
<td>$a \in \sqrt{2}\mathbb{Z}$, $2b \not\in \sqrt{2}\mathbb{Z}$</td>
<td>6_B</td>
</tr>
<tr>
<td>12_3</td>
<td>8</td>
<td>$\tau_1, \tau_3, \rho$</td>
<td>$2b \in \sqrt{2}\mathbb{Z}$, $a \not\in \sqrt{2}\mathbb{Z}$</td>
<td>6_B</td>
</tr>
<tr>
<td>12</td>
<td>8</td>
<td>$\tau_1, \chi, \rho$</td>
<td>$(a + 2b) \not\in \frac{\sqrt{2}}{2}\mathbb{Z} \setminus \sqrt{2}\mathbb{Z}$</td>
<td>3, 6_B</td>
</tr>
</tbody>
</table>