Universality in Geometric Graph Theory

Csaba D. Tóth
Cal State Northridge
University of Calgary
Tufts University
Outline

• Introduction: Geometric Graphs
• Counting Problems on \( n \) Points
  – Labeled Plane Graphs
  – Unlabeled Plane Graphs
• Universality
  – Configurations Compatible with Many Graphs
    Universal Point Sets, Universal Slope Sets, etc.
  – Graphs Compatible with Many Parameters
    Globally Rigid Graphs, Length Universal Graphs, etc.
• Open Problems
Geometric Graphs

A geometric graph is $G = (V, E)$, where $V$ is the set of points in the plane, and $E$ is the set of line segments between points in $V$.

Applications:

- Cartography (GIS, Navigation, etc.)
- Networks (VLSI Design, Optimization, etc.)
- Combinatorial Geometry (Incidences, Unit Distances, etc.)
- Rigidity (Robot arms, etc.)
Counting labeled plane graphs

Giménez and Noy (2009): The asymptotic number of (labeled) planar graphs on $n$ vertices is $g \cdot n^{-7/2} \gamma^n n!$, where $\gamma \approx 27.22688$ and $g \approx 4.26 \cdot 10^{-6}$.

Fáry (1957): Every planar graph has an embedding in the plane as a geometric graph.

Ajtai, Chvátal, Newborn, & Szemerédi (1982): On any $n$ points in $\mathbb{R}^2$, at most $c^n$ labeled planar graphs can be embedded, where $c < 10^{13}$. Hoffmann et al. (2010): $c < 207.85$. 
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Requiring straight-line edges is a real restriction.
How to bridge the gap between $n!$ and $\exp(n)$?

- Allow the edges to bend
- Allow graph isomorphisms
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Given a set $V$ of $n$ points in the plane, we can embed every labeled planar graph $G = (V, E)$ with *curved* edges, or with polyline edges.
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**Thm.** (2013): On any $n$-element point set in $\mathbb{R}^2$, at most $2^{O(kn)}$ labeled planar graphs can be embedded with $k$ bends per edge.

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Problem: Can this bound be improved to $2^{O(n \log k)}$?
Every $n$-vertex planar graph has a straight line embedding, but not all of them can be embedded on an arbitrary set of $n$ points.

- $C_4$ can be embedded on any 4 points in the plane.
- $K_4$ cannot be embedded on 4 points in convex position.
Counting unlabeled planar geometric graphs

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A point set $S \subset \mathbb{R}^2$ is \textbf{$n$-universal} if every $n$-vertex planar graph has an embedding such that the vertices map into $S$.

Cardinal, Hoffmann, & Kusters (2013):

- For $n = 1, \ldots, 10$, there is an $n$-element point set that can host all $n$-vertex planar graphs (by exhaustive search).
- For $n \geq 15$, there is no $n$-element point set that can accommodate all $n$-vertex planar graphs (by counting argument).
Universal point sets

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De Fraysseix, Pach, & Pollack (1990) and Schnyder (1990): An $(n - 1) \times (n - 1)$ section of the integer lattice is $n$-universal.

Methods:

- partial orders defined on the vertices
- three Schnyder trees (Schnyder wood)

One method is an incremental algorithm, the other embedding all vertices at once. They have turned out to be equivalent...
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$\frac{n^2}{2}$ points suffice if we do not insist on a rectangular lattice.
Universality in Geometric Graphs

1. A structure is **universal** if it is “compatible” with every geometric graph from a certain family (e.g., universal point sets, universal slopes, etc.)

2. An abstract graph is **universal** if it has a geometric realization for any possible choice of certain parameters (e.g., globally rigid graphs, length-universal graphs, area universal floorplans).
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Frati & Patrignani (2008): If a rectangular section of the integer lattice is $n$-universal, it must contain at least $\frac{n^2}{9}$ points.
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\textbf{Open Problem:} Find $n$-universal point sets of size $o(n^2)$. 

Universal point sets

Bannister et al. (2013) there is an $n$-universal point set of size $n^2/4 + \Theta(n)$ for all $n \in \mathbb{N}$. (not a lattice section)

Kurowski (2004): The size of an $n$-universal set is at least $1.235n - o(n)$.
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\[ G \]

\[ T(G) \]

1. Planar 3-trees require $1.235n - o(n)$ points.
Universal point sets in special classes

Gritzman et al. (1991): Every $n$-element point set in general position is $n$-universal for outerplanar graphs.

Angelini et al. (2011): There is an $n$-universal point set of size $O(n(\log n / \log \log n)^2)$ for simply nested planar graphs.

Bannister et al. (2013): There is an $n$-universal point set of size $O(n \log n)$ for simply nested planar graphs, and of size $O(n \text{polylog } n)$ for planar graphs of bounded pathwidth.
**Thm. (2013):** There is an $n$-universal point set of size $O(n^{3/2} \log n)$ for planar 3-trees.

Our $n$-universal point set for planar 3-trees is constructed from an $14n \times 14n$ section of the integer lattice in two steps:
1. sparsening,
2. stretching.
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### Sparsening:

Suppose $\sqrt{n} \in \mathbb{N}$.

Pick all points $(i, j)$ were $\sqrt{n}$ divides $i \cdot j$. 
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**Sparsening:**

Suppose $\sqrt{n} \in \mathbb{N}$.
Pick all points $(i,j)$ were $\sqrt{n}$ divides $i \cdot j$.

Add the forward and backward diagonals of $\sqrt{n} \times \sqrt{n}$ “squares.”
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Total: $O(n^{3/2} \log n)$ points.
Construction for planar 3-trees

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Objective: The slope of an edge between rows $i$ and $j$ is larger than the slope of any other edge among rows $1..j - 1$. 
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Transformation \((x, y) \rightarrow (x, (28n)y)\)

Objective: The slope of an edge between rows \( i \) and \( j \) is larger than the slope of any other edge among rows \( 1, 2, \ldots, j - 1 \).

When we pull back the stretched grid to the integer grid, the straight-line edges become \( \Gamma \)-shaped curves.
Embedding algorithm

Every $n$-vertex planar 3-tree can be embedded such that the vertices are remapped into our point set.
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In a top-down traversal of $T(G)$, we allocate a rectangular region to each subtree (triangle).
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In a top-down traversal of $T(G)$, we allocate a rectangular region to each subtree (triangle).

When a new vertex is inserted, the rectangle is subdivided into four rectangles: **left**, **right**, and **bottom** rectangles.
Embedding algorithm

If a “large” rectangle $R(\Delta)$ is allocated to a subgraph lying in a triangle $\Delta$, then we can complete the embedding with the algorithm of de Fraysseix, Pach, & Pollack (1990).

This is possible when $k$ points has to be embedded in a triangle $\Delta$, and the full rows or full columns in the rectangle $R(\Delta)$ form a $k \times k$ grid.
Universal Point Sets: Summary

**Problem:** Is our point set universal for all planar graphs?

For all planar graphs, the currently best bounds are $1.235n - o(n)$ (Kurowski) and $n^2/4$ (Bannister et al.).

**Open Problem:** Find $n$-universal point sets of size $o(n^2)$. 
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Generalization:
A point set $S$ is **universal for** a family of graphs $\mathcal{G}$ if every graph $G \in \mathcal{G}$ has a geometric realization with $cr(G)$ crossings such that all vertices are mapped into $S$.

**Open Problem:** Find $n$-universal point sets for all graphs.

...might be elusive:
—computing the crossing number, $cr(G)$, is NP-hard,
—no optimal embedding is known for the complete graph $K_n$. 
Universal Slope Sets

Keszegh et al. (2008):

- Every (abstract) graph with maximum degree 3 has a geometric realization with 5 distinct slopes.
- Every graph with vertices of both degree 2 and 3 has a geometric realization with 4 slopes,
- A set $S$ of 4 slopes is universal for all such graphs iff $S = \{ \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{a} - \overrightarrow{b}, \overrightarrow{a} + \overrightarrow{b} \}$.

Keszegh et al. (2010): There is a function $f : \mathbb{N} \to \mathbb{N}$ such that every planar graph $G$ with maximum degree $d$ admits a geometric embedding with at most $f(d)$ different slopes.
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Open Problem: Which slope sets are universal for all planar graphs of maximum degree $d$?
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1. A structure is universal if it is “compatible” with every geometric graph from a certain family (e.g., universal point sets, universal slopes, etc.)

2. An abstract graph is universal if it has a geometric realization for any possible choice of certain parameters (e.g., globally globally rigid graphs, length-universal graphs, area universal floorplans).
Globally Rigid Graphs

A geometric graph $G = (V, E)$ is (locally) **rigid** if every small motion of the vertices that preserves all edge lengths is an isometry.
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**Def.:** An (abstract) graph $G = (V, E)$ is **generically globally rigid** if every realization as a geometric graph with vertices in general position is **rigid**.
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**Def.:** An (abstract) graph $G = (V, E)$ is **generically globally rigid** if every realization as a geometric graph with vertices in general position is rigid.

**Jackson & Jordán (2005):** A graph $G$ is generically globally rigid iff
- either $G = K_n$, $n \leq 3$,
- or $G$ is 3-connected and redundantly rigid.
Length Universal Graphs: An Easy Exercise

A graph $G = (V, E)$ is **length universal** if it admits a geometric embedding for all length assignments $\ell : E \to \mathbb{R}^+$. 
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A graph $G = (V, E)$ is **length universal** if it admits a geometric embedding for all length assignments $\ell : E \rightarrow \mathbb{R}^+$. E.g., a star is realizable with arbitrary positive edge lengths.
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But the edges of a cycle must satisfy the triangle inequality. The edge lengths cannot be chosen arbitrarily.
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But the edges of a cycle must satisfy the triangle inequality. The edge lengths cannot be chosen arbitrarily.

Observation: A graph is length universal iff it is a forest.
Free Graphs

Let $G = (V, E)$ be a subgraph of a planar graph $H$. Graph $G$ is **free in** $H$ if for every function $\ell : E \rightarrow \mathbb{R}^+$, $H$ has a geometric embedding such that every $e \in E$ has length $\ell(e)$.
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But a star with $n \geq 5$ vertices cannot have arbitrary positive edge lengths in a triangulation $H$. 

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**Thm.:** A graph $G$ is free in every planar $H$, $G \subseteq H$, iff $G$ is

- a matching
- a forest with at most 3 edges, or
- two disjoint paths of length 2.
**Lemma:** If $G = (V, E)$ is a perfect matching in a triangulation $H$, then all positive lengths $\ell(e), e \in E$, can be realized in an embedding of $H$. 
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1. Contract all edges $e \in E$ of the matching;
2. Embed the resulting graph with “giant” edges;
3. Expand all $e \in E$ to the required lengths.
Lemma: If $G = (V, E)$ is a perfect matching in a triangulation $H$, then all positive lengths $\ell(e)$, $e \in E$, can be realized in an embedding of $H$.

Naïve idea:
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**Lemma:** If $G = (V, E)$ is a perfect matching in a triangulation $H$, then all positive lengths $\ell(e)$, $e \in E$, can be realized in an embedding of $H$.

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A recursion on the hierarchy of separating triangles and separating 4-cycles works, using an appropriate linear transformation in each step.
Extrinsically Free Graphs

Let $G = (V, E)$ be a subgraph of a planar graph $H$. Graph $G$ is extrinsically free in $H$ if whenever if $G$ has a geometric embedding with edge lengths $\ell(e), e \in E$, then $H$ also has a geometric embedding such that every $e \in E$ has length $\ell(e)$. 
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A triangle $G = C_3$, and every triangulation $G = T$ is extrinsically free, since $H = G$. 
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The 4-cycle \( C_4 \) is not extrisically free: if all four edges have unit length, then \( C_4 \) is a rhombus (i.e., convex), and cannot have an external diagonal.
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**Thm.:** A graph $G$ is extrinsically free in every planar $H$, $G \subseteq H$, iff $G$ is
- a matching
- a forest with at most 3 edges,
- two disjoint paths of length 2,
- a triangulation, or
- a triangle and one edge.
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No cycle $C_k, k \geq 4$, is extrinsically free:
“Triangulated” Carpenter’s Rule

Connelly et al. (2003): Every simple polygonal cycle (with fixed edge lengths) can be continuously unfolded into convex position (i.e., its configuration space is connected).
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The unfolding algorithm by Streinu maintains a triangulation of $C'$: The edges of the interior triangulation are preserved, and the edges of the exterior triangulation vanish.
“Triangulated” Carpenter’s Rule

Given a simple polygonal cycle $C$ and an arbitrary curvilinear triangulation $H$, does $H$ admit a straight-line embedding such that the cycle $C$ keeps its given edge lengths?
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Thm. (Abel et al., 2013): “Yes” if the edge lengths are nondegenerate, that is, if the cycle cannot be “flattened” into 1D in two different ways with the given edge lengths.
Thank you!