Classification of Regular and Chiral Polytopes by Topology

Egon Schulte
Northeastern University, Boston

November 2013, Toronto
Classical Regular Polytopes — Review

*Convex polytope*: convex hull of finitely many points in $\mathbb{E}^n$

Key observation: topologically spherical, both globally and locally!

*Regularity*: flag transitivity of the symmetry group (other equivalent definitions).

- $n=2$: polygons $\{p\}$ (Schläfli-symbol)
- $n=3$: Platonic solids $\{p,q\}$

\[
\{3, 5\}
\]
**DIMENSION $n \geq 4$**

<table>
<thead>
<tr>
<th>name</th>
<th>symbol</th>
<th>#facets</th>
<th>group</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>simplex</td>
<td>${3,3,3}$</td>
<td>5</td>
<td>$S_5$</td>
<td>120</td>
</tr>
<tr>
<td>cross-polytope</td>
<td>${3,3,4}$</td>
<td>16</td>
<td>$B_4$</td>
<td>384</td>
</tr>
<tr>
<td>cube</td>
<td>${4,3,3}$</td>
<td>8</td>
<td>$B_4$</td>
<td>384</td>
</tr>
<tr>
<td>24-cell</td>
<td>${3,4,3}$</td>
<td>24</td>
<td>$F_4$</td>
<td>1152</td>
</tr>
<tr>
<td>600-cell</td>
<td>${3,3,5}$</td>
<td>600</td>
<td>$H_4$</td>
<td>14400</td>
</tr>
<tr>
<td>120-cell</td>
<td>${5,3,3}$</td>
<td>120</td>
<td>$H_4$</td>
<td>14400</td>
</tr>
<tr>
<td>simplex</td>
<td>${3,\ldots,3}$</td>
<td>$n+1$</td>
<td>$S_{n+1}$</td>
<td>$(n + 1)!$</td>
</tr>
<tr>
<td>cross-polytope</td>
<td>${3,\ldots,3,4}$</td>
<td>$2^n$</td>
<td>$B_{n+1}$</td>
<td>$2^n n!$</td>
</tr>
<tr>
<td>cube</td>
<td>${4,3,\ldots,3}$</td>
<td>$2n$</td>
<td>$B_{n+1}$</td>
<td>$2^n n!$</td>
</tr>
</tbody>
</table>
24-cell \{3, 4, 3\}
(with thickened edges)

4D cube \{4, 3, 3\}
Symmetry group of \( \{p, q, r\} \) is the Coxeter group with string diagram

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
p & q & r
\end{array}
\]

Presentation

\[
\begin{align*}
\rho_0^2 &= \rho_1^2 = \rho_2^2 = \rho_3^2 = 1 \\
(\rho_0 \rho_1)^p &= (\rho_1 \rho_2)^q = (\rho_2 \rho_3)^r = 1 \\
(\rho_0 \rho_2)^2 &= (\rho_1 \rho_3)^2 = (\rho_0 \rho_3)^2 = 1
\end{align*}
\]

Generators are reflections in the walls of a fundamental chamber.
Presentation for 3-cube

\[ \rho_0^2 = \rho_1^2 = \rho_2^2 = 1 \]
\[ (\rho_0 \rho_1)^4 = (\rho_1 \rho_2)^3 = (\rho_0 \rho_2)^2 = 1 \]
• **Regular star-polyhedra — Kepler-Poinsot polyhedra** (Kepler 1619, Poinsot 1809). Cauchy (1813).

• **Ten regular star-polytopes in dimension 4. None in dimension > 4.**
<table>
<thead>
<tr>
<th>Dim.</th>
<th>Symbol</th>
<th>$f_0$</th>
<th>$f_{n-1}$</th>
<th>Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 3$</td>
<td>${3, \frac{5}{2}}$</td>
<td>12</td>
<td>20</td>
<td>$H_3$</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>${\frac{5}{2}, 3}$</td>
<td>20</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>$n = 3$</td>
<td>${5, \frac{5}{2}}$</td>
<td>12</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>$n = 3$</td>
<td>${\frac{5}{2}, 5}$</td>
<td>12</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td>${3, 3, \frac{5}{2}}$</td>
<td>120</td>
<td>600</td>
<td>$H_4$</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>${\frac{5}{2}, 3, 3}$</td>
<td>600</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td>${3, 5, \frac{5}{2}}$</td>
<td>120</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td>${\frac{5}{2}, 5, 3}$</td>
<td>120</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td>${3, \frac{5}{2}, 5}$</td>
<td>120</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td>${5, \frac{5}{2}, 3}$</td>
<td>120</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td>${5, 3, \frac{5}{2}}$</td>
<td>120</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td>${\frac{5}{2}, 3, 5}$</td>
<td>120</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td>${5, \frac{5}{2}, 5}$</td>
<td>120</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td>${\frac{5}{2}, 5, \frac{5}{2}}$</td>
<td>120</td>
<td>120</td>
<td></td>
</tr>
</tbody>
</table>

Regular Star-Polytopes in $\mathbb{E}^n \ (n \geq 3)$
Regular Honeycombs

Euclidean space

- $n=2$: with triangles, hexagons, squares
  - $\{3,6\}$, $\{6,3\}$, $\{4,4\}$
- $n\geq 2$: with cubes, $\{4,3,\ldots,3,4\}$
- $n=4$: with 24-cells, $\{3,4,3,3\}$
  - with cross-polytopes, $\{3,3,4,3\}$

Hyperbolic space

- $n=2$: each symbol $\{p,q\}$ with $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$
- $n=3$: $\# = 15$  $\{3,5,3\}$, $\{4,3,5\}$, $\{5,3,5\}$,$\{6,3,3\}$, ... 
- $n=4$: $\# = 7$  $\{5,3,3,4\}$, $\{5,3,3,5\}$,$\{3,4,3,4\}$, ... 
- $n=5$: $\# = 5$  $\{3,3,4,3,3\}$, $\{3,3,3,4,3\}$, ... 
- $n\geq 6$: none
Abstract Polytopes $P$ of rank $n$

$P$ ranked partially ordered set

- $i$-faces elements of rank $i$ ($= -1, 0, 1, \ldots, n$)
- $i=0$ vertices
- $i=1$ edges
- $i=n-1$ facets

- Faces $F_{-1}$, $F_n$ (of ranks $-1$, $n$)
- Each flag of $P$ contains exactly $n+2$ faces
- $P$ is connected
- Intervals of rank 1 are diamonds:

$P$ is regular iff $\Gamma(P)$ flag transitive.
P is \textit{chiral} iff $\Gamma(P)$ has two orbits on the flags such that adjacent flags always are in different orbits.

\textbf{Nothing new in ranks 0, 1, 2} (points, segments, polygons)!

\textbf{Rank 3:} maps (2-cell tessellations) on closed surfaces.

Rich history: Klein, Dyck, Brahana, Coxeter, Jones & Singer-man, Wilson, Conder ...........
Well-known: torus maps \( \{4, 4\}_{(b,c)}, \{3, 6\}_{(b,c)}, \{6, 3\}_{(b,c)} \).

Classification of regular and chirals maps by genus (Conder)
— orientable surfaces of genus 2 to 300
— non-orientable surfaces of genus 2 to 600

**Rank \( n \geq 4 \):** How about polytopes of rank 4 (or higher)?

**Local picture for a 4-polytope of type \( \{4, 4, 3\} \)**

- **Facets:** torus maps \( \{4, 4\}_{(s,0)} \) \((s \times s \text{ chessboard})\)
- **Vertex-figures:** cubes \( \{4, 3\} \)

  - 2 tori meeting at each 2-face
  - 3 tori surround each edge
  - 6 tori surround each vertex

**Problems:** local — global; universal polytopes; finiteness.
regular polytopes $\iff$ C-groups

C-group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$

\[
\begin{cases}
\rho_i^2 = (\rho_i \rho_j)^2 = 1 \ (|i - j| \geq 2) \\
(\rho_0 \rho_1)^{p_1} = (\rho_1 \rho_2)^{p_2} = \ldots = (\rho_{n-2} \rho_{n-1})^{p_{n-1}} = 1
\end{cases}
\]

& in general additional relations!

• Intersection property $\langle \rho_i | i \in I \rangle \cap \langle \rho_i | i \in J \rangle = \langle \rho_i | i \in I \cap J \rangle$

Polytope associated with $\Gamma$

$j$-faces — right cosets of $\Gamma_j := \langle \rho_i | i \neq j \rangle$

partial order: $\Gamma_j \varphi \leq \Gamma_k \psi$ iff $j \leq k$ and $\Gamma_j \varphi \cap \Gamma_k \psi \neq \emptyset$.

Quotient of the Coxeter group $\bullet \overline{p_1} \bullet \overline{p_2} \bullet \ldots \bullet \overline{p_{n-1}} \bullet$
**Topological classification** (of universal polytopes)

**Classical case** spherical or locally spherical

\[ \Updownarrow \]

quotient of a regular tessellation in \( S^{n-1} \), \( E^{n-1} \) or \( H^{n-1} \)

**Grünbaum’s Problem (mid 70’s):** Classify toroidal and locally toroidal regular polytopes.

Step 1: Tessellations on the \((n-1)\)-torus (globally toroidal)

Step 2: Locally toroidally polytopes only in ranks \( n = 4, 5, 6 \).

A lot of progress! Enumeration complete for \( n = 5 \); almost complete for \( n = 4 \); conjectures for \( n = 6 \).

McMullen & S.; also Weiss, Monson
**Toroids**

Torus maps \(\{4, 4\}_{(b,c)}, \{3, 6\}_{(b,c)}, \{6, 3\}_{(b,c)}\). How about higher-dimensional tori?

**Tessellations** \(\mathcal{T}\) **in euclidean space**

- \(n = 2\): with triangles, hexagons, squares, \(\{3, 6\}, \{6, 3\}, \{4, 4\}\)
- \(n \geq 2\): with cubes, \(\{4, 3, \ldots, 3, 4\}\)
- \(n = 4\): with 24-cells, \(\{3, 4, 3, 3\}\)
  
  with cross-polytopes, \(\{3, 3, 4, 3\}\)

**Regular toroids of rank** \(n + 1\) (McMullen & S.)

Quotients \(\mathcal{T}/\Lambda\) of regular tessellations \(\mathcal{T}\) in \(\mathbb{E}^n\) by suitable lattices \(\Lambda\).
A toroid with 27 cubical facets on the 3-torus (rank 4)

Type $\{4, 3, 4\}_{(3,0,0)}$

$$(\rho_0 \rho_1 \rho_2 \rho_3 \rho_2 \rho_1)^3 = 1$$
**Cubical Toroids** \( \{4, 3^{n-2}, 4\}_s \) on \( n \)-Torus

<table>
<thead>
<tr>
<th>s</th>
<th>vertices</th>
<th>facets</th>
<th>order</th>
<th>lattice</th>
</tr>
</thead>
<tbody>
<tr>
<td>((s,0,\ldots,0))</td>
<td>(s^n)</td>
<td>(s^n)</td>
<td>((2s)^n \cdot n!)</td>
<td>(s\mathbb{Z}^n)</td>
</tr>
<tr>
<td>((s,s,0,\ldots,0))</td>
<td>(2s^n)</td>
<td>(2s^n)</td>
<td>(2^{n+1}s^n \cdot n!)</td>
<td>(sD_n)</td>
</tr>
<tr>
<td>((s,\ldots,s))</td>
<td>(2^{n-1}s^n)</td>
<td>(2^{n-1}s^n)</td>
<td>(2^{2n-1}s^n \cdot n!)</td>
<td>(2sD^*_n)</td>
</tr>
</tbody>
</table>

Standard relations for \(\bullet \overbrace{4\quad\bullet\quad3\quad\bullet\ldots\bullet\quad3\quad\bullet\quad4\quad\bullet}^{s}\) and the single extra relation

\[
(\rho_0\rho_1\cdots\rho_n\rho_{n-1}\cdots\rho_k)^{ks} = 1 \quad (k = 1, 2 \text{ or } n, \text{ resp.})
\]
Exceptional Toroids \(\{3, 3, 4, 3\}_s\) on 4-Torus (up to duality)

<table>
<thead>
<tr>
<th>(s)</th>
<th>vertices</th>
<th>facets</th>
<th>order</th>
<th>lattice</th>
</tr>
</thead>
<tbody>
<tr>
<td>((s, 0, 0, 0))</td>
<td>(s^4)</td>
<td>(3s^4)</td>
<td>(1152s^4)</td>
<td>(sD_4) (self-reciprocal (D_4))</td>
</tr>
<tr>
<td>((s, s, 0, 0))</td>
<td>(4s^4)</td>
<td>(12s^4)</td>
<td>(4608s^4)</td>
<td>(sD_4)</td>
</tr>
</tbody>
</table>

Standard relations for \(\bullet \overbrace{3 \bullet 3 \bullet 4 \bullet 3} \bullet\) and the single extra relation

\[
\begin{align*}
(r_0 \sigma \tau \sigma)^s &= 1 & \text{if } s &= (s, 0, 0, 0), \\
(r_0 \sigma \tau)^{2s} &= 1 & \text{if } s &= (s, s, 0, 0),
\end{align*}
\]

where \(\sigma = r_1 r_2 r_3 r_2 r_1\) and \(\tau = r_4 r_3 r_2 r_3 r_4\).
Locally Toroidal Regular Polytopes

- universal polytopes = \{facets, vertex-figures\}

Rank n=4

\{
\{4, 4\}_s, \{4, 3\}\},
\{4, 4\}_s, \{4, 4\}_t\},
\{6, 3\}_s, \{3, r\}\} \ (r = 3, 4, 5),
\{6, 3\}_s, \{3, 6\}_t\},
\{3, 6\}_s, \{6, 3\}_t\},

where \(s = (s, 0)\) or \((s, s)\) and \(t = (t, 0)\) or \((t, t)\).
Locally toroidal 4-polytopes \( \{\{4, 4\}_{(s,0)}, \{4, 3\}\} \)

Coxeter group \( W_s \)

\[
\begin{align*}
\Gamma_s &:= \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle \cong W_s \times C_2 \\
\text{is the correct group!}
\end{align*}
\]

The universal polytope is finite iff \( s = 2 \) or \( s = 3 \).

The polytope for \( s = 3 \) (with group \( S_6 \times C_2 \)) can be realized by a tessellation on \( S^3 \) consisting of 20 tori (Grünbaum and Coxeter & Shephard).
More on Rank 4

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>$v$</td>
<td>$f$</td>
<td>$g$</td>
<td>Group</td>
</tr>
<tr>
<td>$(2,0)$</td>
<td>4</td>
<td>6</td>
<td>192</td>
<td>$D_4 \rtimes S_4$</td>
</tr>
<tr>
<td>$(3,0)$</td>
<td>30</td>
<td>20</td>
<td>1440</td>
<td>$S_6 \times C_2$</td>
</tr>
<tr>
<td>$(2,2)$</td>
<td>16</td>
<td>12</td>
<td>768</td>
<td>$C_2 \wr D_6$</td>
</tr>
</tbody>
</table>

The finite polytopes $\{\{4,4\}_s, \{4,3\}\}$, $s = (s,0), (s,s)$. 
<table>
<thead>
<tr>
<th>s</th>
<th>t</th>
<th>v</th>
<th>f</th>
<th>g</th>
<th>Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 0)</td>
<td>(t, t), t ≥ 2</td>
<td>4</td>
<td>2t²</td>
<td>64t²</td>
<td>((D_t \times D_t \times C_2 \times C_2)) \times (C_2 \times C_2)</td>
</tr>
<tr>
<td>(2, 0)</td>
<td>(2m, 0), m ≥ 1</td>
<td>4</td>
<td>4m²</td>
<td>128m²</td>
<td>((C_2 \times C_2) \times [4, 4]<em>{(2,0)}) if m = 1; ((D_m \times D_m) \times [4, 4]</em>{(2,0)}) if m ≥ 2</td>
</tr>
<tr>
<td>(3, 0)</td>
<td>(3, 0)</td>
<td>20</td>
<td>20</td>
<td>1440</td>
<td>(S_6 \times C_2)</td>
</tr>
<tr>
<td>(3, 0)</td>
<td>(4, 0)</td>
<td>288</td>
<td>512</td>
<td>36864</td>
<td>(C_2 \wr [4, 4]_{(3,0)})</td>
</tr>
<tr>
<td>(3, 0)</td>
<td>(2, 2)</td>
<td>36</td>
<td>32</td>
<td>2304</td>
<td>((S_4 \times S_4) \times (C_2 \times C_2))</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>(2, 2)</td>
<td>16</td>
<td>16</td>
<td>1024</td>
<td>(C_2^4 \times [4, 4]_{(2,2)})</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>(3, 3)</td>
<td>64</td>
<td>144</td>
<td>9216</td>
<td>(C_2^6 \times [4, 4]_{(3,3)})</td>
</tr>
<tr>
<td>(3, 0)</td>
<td>(5, 0)</td>
<td>19584</td>
<td>54400</td>
<td>3916800</td>
<td>(Sp_4(4) \times C_2 \times C_2)</td>
</tr>
</tbody>
</table>

The finite polytopes \(\{4, 4\}_s, \{4, 4\}_t\)
(except \(\{4, 4\}_{(s,0)}, \{4, 4\}_{(t,0)}\), with s, t odd and distinct)
Conjecture

The universal polytopes \( \{4,4\}_{(s,0)}, \{4,4\}_{(t,0)} \), with \( s, t \) odd and distinct, are finite iff the regular tessellation \( \{s,t\} \) is spherical (that is, iff \( (s,t) = (3,5), (5,3) \).)

Case \( (s,t) = (3,5) \): \( Sp_4(4) \times C_2 \times C_2 \).
Still more on Rank 4

<table>
<thead>
<tr>
<th>$r$</th>
<th>$s$</th>
<th>$v$</th>
<th>$f$</th>
<th>$g$</th>
<th>Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>(2,0)</td>
<td>10</td>
<td>5</td>
<td>240</td>
<td>$S_5 \times C_2$</td>
</tr>
<tr>
<td></td>
<td>(3,0)</td>
<td>54</td>
<td>12</td>
<td>1296</td>
<td>$[1 1 2]^3 \times C_2$</td>
</tr>
<tr>
<td></td>
<td>(4,0)</td>
<td>640</td>
<td>80</td>
<td>15360</td>
<td>$[1 1 2]^4 \times C_2$</td>
</tr>
<tr>
<td></td>
<td>(2,2)</td>
<td>120</td>
<td>20</td>
<td>2880</td>
<td>$S_5 \times S_4$</td>
</tr>
<tr>
<td>4</td>
<td>(1,1)</td>
<td>12</td>
<td>8</td>
<td>288</td>
<td>$S_3 \rtimes [3,4]$</td>
</tr>
<tr>
<td></td>
<td>(2,0)</td>
<td>16</td>
<td>16</td>
<td>768</td>
<td>$[3,3,4] \times C_2$</td>
</tr>
<tr>
<td>5</td>
<td>(2,0)</td>
<td>240</td>
<td>600</td>
<td>28800</td>
<td>$[3,3,5] \times C_2$</td>
</tr>
</tbody>
</table>

The finite polytopes $\{\{6,3\}_s, \{3, r\}\}$

$(s = (s,0), (s,s) \text{ and } r = 3, 4, 5)$. 
Thm The universal regular 4-polytope \( \{\{6, 3\}_s, \{3, 6\}_{(t, 0)}\} \) exists for all \( s, t \geq 2 \). In particular, it is finite if and only if \( (s, t) = (2, k) \) or \( (k, 2) \), with \( k = 2, 3, 4 \). In this case, its group is \( [1 1 2]^k \rtimes (C_2 \times C_2) \), of order \( 480, 108 \cdot 4!, 256 \cdot 5! \) if \( k = 2, 3, 4 \), respectively.

Thm The universal regular 4-polytope \( \{\{6, 3\}_{(s, s)}, \{3, 6\}_t\} \), with \( t = (t, 0) \) or \( (t, t) \), exists for all \( s, t \geq 2 \). In particular, it is finite if and only if \( s = 2 \) and \( t = (2, 0) \); in this case, its group is \( S_5 \times S_4 \times C_2 \).

Somewhat open: \( \{\{3, 6\}_s, \{6, 3\}_t\} \)
Locally toroidal regular polytopes (cont.)

**Rank** $n = 5$

<table>
<thead>
<tr>
<th>$s$</th>
<th>vertices</th>
<th>facets</th>
<th>group</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2,0,0)$</td>
<td>24</td>
<td>8</td>
<td>$C_2^3 \rtimes F_4$</td>
<td>9216</td>
</tr>
<tr>
<td>$(2,2,0)$</td>
<td>48</td>
<td>32</td>
<td>$C_2^5 \rtimes F_4$</td>
<td>36864</td>
</tr>
<tr>
<td>$(2,2,2)$</td>
<td>1536</td>
<td>2048</td>
<td>$(C_2^6 \rtimes C_2^5) \rtimes F_4$</td>
<td>2359296</td>
</tr>
</tbody>
</table>

**Finite polytopes** $\{\{3,4,3\}, \{4,3,4\}_s\}$

*(with $s = (s,0,0), (s,s,0), (s,s,s)$)*

\[
\begin{array}{cccc}
\bullet & 3 & \bullet & 4 & \bullet & 3 & F_4
\end{array}
\]
Locally toroidal regular polytopes (cont.)

Rank $n = 6$ (first type)

<table>
<thead>
<tr>
<th>$s$</th>
<th>vertices</th>
<th>facets</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2,0,0,0)$</td>
<td>20</td>
<td>960</td>
<td>368640</td>
</tr>
<tr>
<td>$(2,2,0,0)$</td>
<td>160</td>
<td>30720</td>
<td>11796480</td>
</tr>
<tr>
<td>$(3,0,0,0)$</td>
<td>780</td>
<td>189540</td>
<td>72783360</td>
</tr>
</tbody>
</table>

Conjectured finite polytopes of type
\{\{3,3,3,4\}, \{3,3,4,3\}_s\}
Rank $n = 6$ (second type)

<table>
<thead>
<tr>
<th>$s$</th>
<th>$t$</th>
<th>vertices</th>
<th>facets</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2,0,0,0)$</td>
<td>$(t,0,0,0)$</td>
<td>32</td>
<td>$2t^4$</td>
<td>$36864t^4$</td>
</tr>
<tr>
<td></td>
<td>$(t$ even)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(2,0,0,0)$</td>
<td>$(t,t,0,0)$</td>
<td>32</td>
<td>$8t^4$</td>
<td>$147476t^4$</td>
</tr>
<tr>
<td></td>
<td>$(t$ even)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(2,2,0,0)$</td>
<td>$(2,2,0,0)$</td>
<td>2048</td>
<td>2048</td>
<td>150994944</td>
</tr>
<tr>
<td>$(3,0,0,0)$</td>
<td>$(3,0,0,0)$</td>
<td>2340</td>
<td>2340</td>
<td>218350080</td>
</tr>
</tbody>
</table>

Conjectured finite polytopes of type

$\{\{3,3,4,3\}_s, \{3,4,3,3\}_t\}$
Rank $n = 6$ (third type)

<table>
<thead>
<tr>
<th>$s$</th>
<th>$t$</th>
<th>vertices</th>
<th>facets</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(s,0,0,0,0)$</td>
<td>$(2,0,0,0)$</td>
<td>$3s^4$</td>
<td>16</td>
<td>$18432s^4$</td>
</tr>
<tr>
<td>($s$ even)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(s,s,0,0)$</td>
<td>$(2,0,0,0)$</td>
<td>$12s^4$</td>
<td>16</td>
<td>$73728s^4$</td>
</tr>
<tr>
<td>$(s,0,0,0)$</td>
<td>$(2,2,0,0)$</td>
<td>$6s^4$</td>
<td>64</td>
<td>$73728s^4$</td>
</tr>
<tr>
<td>($s$ even)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(s,s,0,0)$</td>
<td>$(2,2,0,0)$</td>
<td>$24s^4$</td>
<td>64</td>
<td>$294912s^4$</td>
</tr>
<tr>
<td>($s$ even)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(2,0,0,0)$</td>
<td>$(2,2,2,2)$</td>
<td>$384$</td>
<td>1024</td>
<td>$18874368$</td>
</tr>
<tr>
<td>$(2,0,0,0)$</td>
<td>$(4,0,0,0)$</td>
<td>$12288$</td>
<td>65536</td>
<td>$1207959552$</td>
</tr>
<tr>
<td>$(3,0,0,0)$</td>
<td>$(3,0,0,0)$</td>
<td>$2340$</td>
<td>780</td>
<td>$72783360$</td>
</tr>
</tbody>
</table>

Conjectured finite polytopes of type

\[
\{\{3,4,3,3\}_s, \{4,3,3,4\}_t\}
\]
Open Problem

Classify all locally toroidal chiral polytopes!

Rank 4: \( \{\{4, 4\}_{(b,c)}, \{4, 3\}\} \), \( \{\{4, 4\}_{(b,c)}, \{4, 4\}_{(e,f)}\} \), ..... 

Almost completely open!
Chirality

$\Gamma(P)$ has 2 flag-orbits, represented by adjacent flags!

- Rank 3: Lots of chiral torus maps! Occurrence very sporadic, at least for small genus $g$ (next for $g = 7$).

Generators $\sigma_1, \sigma_2$ for type $\{p, q\}$ in rank 3

$$\sigma_1^p = \sigma_2^q = (\sigma_1 \sigma_2)^2 = 1 \quad \& \quad \text{generally more relations.}$$
**Local definition:** P not regular, but for some base flag $\Phi := \{F_1, F_0, \ldots, F_n\}$ there exist $\sigma_1, \ldots, \sigma_{n-1} \in \Gamma(P)$ such that $\sigma_i$ fixes each face in $\Phi \setminus \{F_{i-1}, F_i\}$ and cyclically permutes consecutive $i$-faces in the section $F_{i+1}/F_{i-2}$.

Two enantiomorphic forms: Chiral polytopes occur in a “right-hand” and a “left-hand” version, distinguished by the choice of base flag.
Rank 4

Generators $\sigma_1, \sigma_2, \sigma_3$ for type $\{p, q, r\}$ in rank 4

Standard relations

$$\sigma_1^p = \sigma_2^q = \sigma_3^r = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1$$

Example: The universal $\{\{4, 4\}_{(b,c)}, \{4, 3\}\}$ has extra relation

$$(\sigma_1^{-1}\sigma_2)^b(\sigma_1\sigma_2^{-1})^c = 1$$

Intersection property

$$\langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle = \langle \epsilon \rangle = \langle \sigma_2 \rangle \cap \langle \sigma_3 \rangle, \quad \langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_2, \sigma_3 \rangle = \langle \sigma_2 \rangle$$
Polytopes associated with the groups

Regular polytopes: $\Gamma$ generated by $\rho_0, \ldots, \rho_{n-1}$

$j$-faces: right cosets of $\Gamma_j := \langle \rho_i \mid i \neq j \rangle$

Chiral polytopes: $\Gamma$ generated by $\sigma_1, \ldots, \sigma_{n-1}$

$j$-faces: right cosets of

$$\Gamma_j := \begin{cases} 
\langle \sigma_2, \ldots, \sigma_{n-1} \rangle & \text{if } j = 0, \\
\langle \{\sigma_i \mid i \neq j, j + 1\} \cup \{\sigma_j \sigma_{j+1}\} \rangle & \text{if } j = 1, \ldots, n - 2, \\
\langle \sigma_1, \ldots, \sigma_{n-2} \rangle & \text{if } j = n - 1.
\end{cases}$$

Partial order in both cases:

$$\Gamma_j \varphi \leq \Gamma_k \psi \text{ iff } j \leq k \text{ and } \Gamma_j \varphi \cap \Gamma_k \psi \neq \emptyset.$$
Rank 4

Plenty of locally toroidal chiral 4-polytopes. (Coxeter, Weiss & S., Monson, Nostrand; 1990’s and earlier.)

Key idea: Relevant hyperbolic Coxeter groups have nice representations as groups of Möbius transformations over $\mathbb{Z}[i]$, $\mathbb{Z}[\omega]$, .... Rotation subgroups have generators like $\sigma_1, \sigma_2, \sigma_3$.
Then construct polytopes by modular reduction of the corresponding groups of $2 \times 2$ matrices.

Example: Take rotation subgroup of $\bullet 4 \bullet 4 \bullet 3 \bullet$ and work over $\mathbb{Z}_m$, where $-1$ is a quadratic residue mod $m$. Gives chiral polytopes of type $\{\{4, 4\}_{(b,c)}, \{4, 3\}\}$ with $m = b^2 + c^2$, $(b, c) = 1$ and group $PSL_2(\mathbb{Z}_m)$ or $PSL_2(\mathbb{Z}_m) \rtimes C_2$. (Work modulo the ideal in $\mathbb{Z}[i]$ generated by $b + ic$.)
Higher ranks

- Lots of finite examples in "low ranks" by Conder, Hubard & Pisanski; Breda, Jones & S.; Conder & Devillers, ...  

- Finite examples for every rank $n \geq 3$ (Pellicer, 2009)!

- Extension problem: Chiral $n$-polytope $P$ as the facet of a chiral $(n + 1)$-polytope $Q$? Facets of $P$ regular!
  (a) Universal: $\Gamma(Q) = \Gamma(P) * \Gamma_+(F) \Gamma(F)$ (Weiss & S., 1994)
  (b) Finite $Q$, if $P$ is finite. (Cunningham & Pellicer, 2013)

- $n$-torus is the only compact euclidean space form with regular or chiral tessellations. Chirality only when $n = 2$! (Hartley, McMullen & S., 1999)
..... The End ..... 

Thank you
Abstract

The past three decades have seen a revival of interest in the study of polytopes and their symmetry. The most exciting new developments all center around the concept and theory of abstract polytopes. The lecture gives a survey of currently known topological classification results for regular and chiral polytopes, focusing in particular on the universal polytopes which are globally or locally toroidal. While there is a great deal known about toroidal regular polytopes, there is almost nothing known about the classification locally toroidal chiral polytopes.
Example: $P = \{\{6, 3\}_{(s,s)}, \{3, 3\}\}$

$\begin{array}{c}
6 \cdot 3 \cdot 3 \cdot 3
\end{array}$

extra relation: $\left(\rho_2(\rho_1\rho_0)^2\right)^{2s} = 1$

Polytopes of type $\{6,3,r\}$

1. Normal subgroup $W$ of $\Gamma(P)$ of finite index!

2. "Locally unitary" representation

$$\varphi : W \hookrightarrow GL_m(C)$$
which preserves a hermitian form $h$ on $C^m$.

3. Finiteness of $P$ is decided by $h$!

$$P = \{\{6, 3\}_s, \{3, 3\}\}$$
\[ W = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle \]

\[
\begin{align*}
\sigma_i^2 &= (\sigma_i \sigma_j)^3 = 1 \\
(\sigma_i \sigma_j \sigma_k \sigma_j)^s &= 1
\end{align*}
\]

GROUP: \( \Gamma(P) = W \ltimes S_4 \)

\[ \rho_0 = \sigma_1, \ \rho_1 = \tau_1, \ \rho_2 = \tau_2, \ \rho_3 = \tau_3 \]

Structure of \( W = W_s \) ?
\[ \varphi : \ W \hookrightarrow GL_4(C) \]

\[ \sigma_i \mapsto S_i \ (i = 1, 2, 3, 4), \]

where

\[ S_i(x) = x - 2 \ h(x, e_i) \ e_i \]
HERMITIAN FORM:

\[ e_1, \ldots, e_4 \text{ canonical basis of } C^4 \]

\[ h(x, y) := \sum_{i=1}^{4} x_i \bar{y}_i - \sum_{i \neq j} c_{ij} x_i \bar{y}_j , \]

\[ \langle S_i, S_j, S_k \rangle \cong [1 1 1]^s \text{ ("locally unitary").} \]

Choice of \( c_{ij} \):

\[ c_{12} = c_{34} = c_{31} = \frac{e^{2\pi i/s}}{2} \]

\[ c_{23} = c_{24} = c_{41} = \frac{e^{-2\pi i/s}}{2} \]
Situation: $W$ acts on $\mathbb{C}^4$ as a reflection group

**Theorem:** $W$ finite iff $h$ positive definite

Classification of unitary reflection groups: Shephard, Todd, Coxeter, Cohen

**Consequence:** \{\{6, 3\}_{(s,s)}, \{3, 3\}\} finite iff $h$ positive definite

$$\det(h) = \frac{1}{16}(-9 - 16 \cos \frac{2\pi}{s} - 2 \cos \frac{4\pi}{s})$$
\[ h \text{ is } \begin{cases} \text{positive definite for } s = 2 \\ \text{positive semi-definite for } s = 3 \\ \text{indefinite for } s \geq 4 \end{cases} \]

**Thm:** \( P := \{\{6, 3\}_{(s,s)}, \{3, 3\}\} \) exists for each \( s \geq 2 \), and \( P \) is finite iff \( s = 2 \).

**s=2:** \( \Gamma(P) = S_5 \times S_4, \)
\[ W = S_5 = \langle (1 \, 5), (2 \, 5), (3 \, 5), (4 \, 5) \rangle \]