Regular Covers and Monodromy Groups of Abstract Polytopes

Barry Monson (UNB)

(from projects with L.B., M.M., D.O., E.S. and G.W.)

Fields Institute, November, 2013

(supported in part by NSERC)
A $d$-polytope $\mathcal{P}$ is regular if $\text{Aut}(\mathcal{P})$ is transitive on flags. But most polytopes of rank $d \geq 3$ are not regular.

Eg. The truncated tetrahedron $Q$, although quite symmetrical, has facets of two types (and 3 flag orbits under action of $\text{Aut}(Q) \simeq S_4$).
Now lift to covers ...

- Likewise, a map $Q$ on a compact surface will not usually be regular.
- But it is ‘well-known’ that $Q$ is covered by a regular map $P$ (usually on some other surface).
- The regular cover $P$ is unique (to isomorphism) if it covers $Q$ minimally.
- The proof is straightforward and works for any abstract 3-polytope (e.g. if $Q$ is a face-to-face tessellation of the plane). In fact, $\text{Aut}(P) \cong \text{Mon}(Q)$, the monodromy group of $Q$.
- So it’s crucial that $\text{Mon}(Q)$ is a string $C$-group when rank $d = 3$. 

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Hartley and Williams (2009) determined the **minimal regular cover** $\mathcal{P}$ for each classical (convex) Archimedean solid $Q$ in $\mathbb{E}^3$.

Here the regular toroidal map $\mathcal{P} = \{6, 3\}_{(2,2)}$ covers the truncated tetrahedron $Q$. 

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In the theory of covering spaces $f : C \to B$, the *monodromy group* is a representation of the fundamental group of the base $B$ as a permutation group on a generic fibre $f^{-1}(x)$.

This is definitely not how we think of $\text{Mon}(Q)$ in polytope theory!

The covering on the previous slide is $2 : 1$, except at four ramification points. There is no place for our monodromy group there.

But perhaps we can say, with futility, that the people working on covering spaces these last 200 years have misused the word!
Let's take stock:

- every polytope of small rank $d \leq 2$ is (combinatorially=abstractly) regular, hence equals its own minimal regular cover.
- every (abstract) 3-polytope $Q$ has a unique minimal regular cover $P$, and $\text{Mon}(Q) \cong \text{Aut}(P)$.
- So it's clear (in rank $d = 3$) that the cover $P$ is finite if-f $Q$ is finite.
- On the other hand, any polytope in any rank $d \geq 2$ is covered by the universal regular $d$-polytope $U = \{\infty, \ldots, \infty\}$.
- So what about finite covers in higher ranks, i.e. $d \geq 4$?
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Recently, Egon Schulte and I found a fix. From this we are able to prove, for the first time,

Every finite $d$-polytope $Q$ is covered by a finite regular $d$-polytope $P$. Moreover, if $Q$ has all its $k$-faces isomorphic to one particular regular $k$-polytope $K$, then we may choose $P$ to also have such $k$-faces.
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Suppose $Q$ is the pyramid over a cuboctahedral base. Then from our theorem, $Q$ has a regular cover $\mathcal{P}$ of type $\{12, 12, 12\}$ and with
\[2^{53} \cdot 3^{14} \cdot 5 \approx 2.15 \times 10^{23}\] flags. (This isn’t likely a minimal cover!)
Idea of proof.

- an induction based on rank of regular initial sections in $Q$
- crucial case is when $d$-polytope $Q$ has all facets isomorphic to some regular $(d - 1)$-polytope $\mathcal{K}$
- in that case, extend $\mathcal{K}$ 'trivially' to a regular $d$-polytope $\tilde{\mathcal{K}}$ of type $\{\mathcal{K}, 2\}$. Thanks ...
- next 'mix' to get

$$G = \text{Mon}(Q) \triangleright \text{Aut}(\tilde{\mathcal{K}})$$

- then $G = \text{Aut}(P)$ for desired regular cover $P$ of $Q$ (quotient criterion).
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Leah B., Mark M., Deborah O., Gordon W. and I have studied the monodromy group $M_n$ of the ordinary pyramid $Q_n$ over an $n$-gon (coming up in *Discrete Mathematics*).

The extreme cases $M_2$ and $M_\infty$ are most interesting. In fact, $M_\infty$ is isomorphic to one of the 4783 space groups acting on Euclidean 4-space. The ‘4’ is because most 3-pyramids have 4 flag-orbits under automorphisms. Here is a

**Problem of Sorts**

What is special about a $k$-orbit $d$-polytope $Q$ for which $\text{Mon}(Q)$ has a normal subgroup $N \cong \mathbb{Z}_b^k$? Maybe maximal among abelian subgroups? Here $b$ should be ‘meaningful’. For example, if $Q$ were infinite, we might want $b = \infty$, or even $N$ of finite index in $\text{Mon}(Q)$.
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Many thanks to our organizers!
References


What are abstract polytopes?

An abstract $n$-polytope $Q$ is a poset having some of the key structural properties of the face lattice of a convex $n$-polytope, although $Q$

- need not be a lattice
- need not be finite
- need not have a familiar geometric realization

The abstract 3-polytopes include all convex polyhedra, face-to-face tessellations and many less familiar structures. But you can safely think of a finite 3-polytope as a map on a compact surface.
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- \( Q \) has a unique minimal face \( F_{n-1} \) and maximal face \( F_n \).
- Every maximal chain or flag has \( n+2 \) faces so \( Q \) has a strictly monotone rank function onto \( \{-1, 0, \ldots, n\} \).
- \( Q \) is strongly flag connected via adjacency in the flag graph; this rules out, for example, the disjoint union of two polyhedra.
- \( Q \) satisfies the 'diamond' condition: whenever \( F < G \) with rank(\( F \)) = \( j-1 \) and rank(\( G \)) = \( j+1 \) there exist exactly two \( j \)-faces \( H \) with \( F < H < G \).
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F \quad H \quad G \\
| \quad | \quad |
\downarrow \quad \downarrow \quad \downarrow \\
F' \\
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is encoded in the group $\Gamma = \Gamma(Q)$ of all order-preserving bijections (= automorphisms) of $Q$.

Each automorphism is det’d by its action on any one flag $\Phi$; for a polyhedron, a flag

$$\Phi = \text{incident [vertex, edge, facet] triple}$$

**Def.** $Q$ is *regular* if $\Gamma$ is transitive on flags.

**Examples:**
- any polygon ($n = 2$) is (abstractly, i.e. combinatorially) regular
- the usual tiling of $\mathbb{E}^3$ by unit cubes is an infinite regular 4-polytope
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The convex regular polyhedra (=Platonic solids) and the Kepler-Poinsot star-polyhedra \( \mathcal{P} \)

Local data for both polyhedron \( \mathcal{P} \) and its group \( \Gamma(\mathcal{P}) \) reside in the Schl"afli symbol or type \( \{ p, q \} \).

**Platonic solids:** \( \{ 3, 3 \} \) (tetrahedron), \( \{ 3, 4 \} \) (octahedron), \( \{ 4, 3 \} \) (cube), \( \{ 3, 5 \} \) (icosahedron), \( \{ 5, 3 \} \) (dodecahedron)

**Kepler** (ca. 1619) \( \{ \frac{5}{2}, 5 \} \) (small stellated dodecahedron), \( \{ \frac{5}{2}, 3 \} \) (great stellated dodecahedron)

**Poinsot** (ca. 1809) \( \{ 5, \frac{5}{2} \} \) (great dodecahedron), \( \{ 3, \frac{5}{2} \} \) (great isosahedron)
## The classical convex regular polytopes, their Schl"afli symbols and finite Coxeter groups with string diagrams

<table>
<thead>
<tr>
<th>name</th>
<th>symbol</th>
<th># facets</th>
<th>(Coxeter) group</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 4$:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>simplex</td>
<td>${3,3,3}$</td>
<td>5</td>
<td>$A_4 \simeq S_5$</td>
<td>5!</td>
</tr>
<tr>
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<td>${3,3,4}$</td>
<td>16</td>
<td>$B_4$</td>
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<td>$B_4$</td>
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<tr>
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<td>${3,4,3}$</td>
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</tr>
<tr>
<td>600-cell</td>
<td>${3,3,5}$</td>
<td>600</td>
<td>$H_4$</td>
<td>14400</td>
</tr>
<tr>
<td>120-cell</td>
<td>${5,3,3}$</td>
<td>120</td>
<td>$H_4$</td>
<td>14400</td>
</tr>
<tr>
<td>$n &gt; 4$:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>simplex</td>
<td>${3,3,\ldots,3}$</td>
<td>$n + 1$</td>
<td>$A_n \simeq S_{n+1}$</td>
<td>$(n + 1)!$</td>
</tr>
<tr>
<td>cross-polytope</td>
<td>${3,\ldots,3,4}$</td>
<td>$2^n$</td>
<td>$B_n$</td>
<td>$2^n \cdot n!$</td>
</tr>
<tr>
<td>cube</td>
<td>${4,3,\ldots,3}$</td>
<td>$2n$</td>
<td>$B_n$</td>
<td>$2^n \cdot n!$</td>
</tr>
</tbody>
</table>
Schulte (1982) showed that the abstract regular $n$-polytopes $\mathcal{P}$ correspond exactly to the \textit{string C-groups of rank $n$} (which we often study in their place).

**The Correspondence Theorem.**

**Part 1.** If $\mathcal{P}$ is a regular $n$-polytope, then $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a \textit{string C-group}.

**Part 2.** Conversely, if $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a string C-group, then we can reconstruct an $n$-polytope $\mathcal{P}(\Gamma)$ (in a natural way as a coset geometry on $\Gamma$).

Furthermore, $\Gamma(\mathcal{P}(\Gamma)) \simeq \Gamma$ and $\mathcal{P}(\Gamma(\mathcal{P})) \simeq \mathcal{P}$.
Recap: what is a string C-group?

**Means:** having fixed a base flag $\Phi$ in $\mathcal{P}$, for $0 \leq j \leq n - 1$ there is a unique automorphism $\rho_j \in \Gamma(\mathcal{P})$ mapping $\Phi$ to the $j$-adjacent flag $\Phi^j$. These involutions generate $\Gamma(\mathcal{P})$ and satisfy the relations implicit in some string (Coxeter) diagram, like

$$\bullet p_1 \bullet p_2 \bullet \ldots \bullet p_{n-1} \bullet,$$

and perhaps other relations, so long as this *intersection condition* continues to hold:

$$\langle \rho_k : k \in I \rangle \cap \langle \rho_k : k \in J \rangle = \langle \rho_k : k \in I \cap J \rangle$$

(for all $I, J \subseteq \{0, \ldots, n - 1\}$).

Notice that $\mathcal{P}$ then has *Schläfli type* $\{p_1, \ldots, p_{n-1}\}$.
What is the monodromy group?

Look for example at the usual faithful realization of the regular dodecahedron $\mathcal{D}$

The flags of $\mathcal{D}$ correspond exactly to the triangles in a barycentric subdivision of the boundary. Here is part of that ⇒
A base flag for $\mathcal{D}$, adjacent flags and generators

By transitivity, pick any base flag $= \Phi$ [white]
Then
0-adjacent flag $=: \Phi^0$ [pink]
1-adjacent flag $=: \Phi^1$ [cyan]
2-adjacent flag $=: \Phi^2$ [orange]
For $i = 0, 1, 2$, there is a unique automorphism

$$\rho_i : \Phi \mapsto \Phi^i.$$ 

Then $\Gamma(\mathcal{D}) = \langle \rho_0, \rho_1, \rho_2 \rangle$.
Can think reflections $\Rightarrow$
Now DESTROY the polytope!

Consider any $d$-polytope $Q$, not necessarily regular. For each flag $\Phi$ of $Q$ and $i = 0, \ldots, d - 1$, there is a unique $i$-adjacent flag $\Phi^i$.

The mapping $s_i : \Phi \mapsto \Phi^i$ defines an involutory bijection $s_i$ on the set $\mathcal{F}(Q)$ of all flags.

**Defn.** The monodromy group of $Q$ is $\text{Mon}(Q) := \langle s_0, \ldots, s_{d-1} \rangle$.

(For maps, Steve Wilson [1994] calls this the “connection group”.)

It is easy to check that $s_i^2 = 1$ and that $(s_is_j)^2 = 1$, for $|j - i| > 1$, so $\text{Mon}(Q)$ is an sggi = string group generated by involutions.

But for ranks $d \geq 4$, $\text{Mon}(Q)$ can fail the intersection condition needed to be a string C-group = aut. group of regular $d$-poly.

Barry Monson (UNB), (from projects with L.B., M.M., D.O., E) Regular Covers and Monodromy Groups of Abstract Polytopes
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It is easy to check that $s_i^2 = 1$ and that $(s_i s_j)^2 = 1$, for $|j - i| > 1$, so $\text{Mon}(Q)$ is an *string group generated by involutions*. But for ranks $d \geq 4$, $\text{Mon}(Q)$ can fail the intersection condition needed to be a string C-group = aut. group of regular $d$-poly.
Example 1 - more on the regular dodecahedron $\mathcal{D}$

Note how seemingly destructive such flag swaps are. (Think Rubik.) Even so, here we do have

$$\text{Mon}(\mathcal{D}) \simeq \Gamma(\mathcal{D}).$$

**Theorem** [ours in high rank] For any abstract regular $d$-polytope $\mathcal{P}$,

$$\text{Mon}(\mathcal{P}) \simeq \Gamma(\mathcal{P}).$$

See *Mixing and Monodromy of Abstract Polytopes*, Monson, Pellicer and Williams, coming soon.
Example 2. The 4-gonal pyramid $\mathcal{E}$ is not regular

You can see that $\Gamma(\mathcal{E})$ has order 8. Guess the order of its monodromy group . . .
Example 2, continued

Here is a bit of the barycentric subdivision (left) with a few flags (right). Start flipping!