On the Universal Rigidity of Tensegrity Frameworks

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(joint work with Viet-Hang Nguyen)

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Workshop on Discrete Geometry, Optimization and Symmetry
Fields Institute, Nov 2013
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Tensegrity Frameworks

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A tensegrity framework in $\mathbb{R}^r$, denoted by $(G, p)$, is a tensegrity graph where each node $i$ is mapped to a point $p^i$ in $\mathbb{R}^r$. 
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A tensegrity framework in $\mathbb{R}^r$, denoted by $(G, p)$, is a tensegrity graph where each node $i$ is mapped to a point $p^i$ in $\mathbb{R}^r$. If $\dim(\text{affine hull of } p^1, \ldots, p^n) = k$, we say that tensegrity $(G, p)$ is $k$-dimensional.
Tensegrity Frameworks

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A tensegrity graph is a simple undirected graph $G$ where each edge is labeled as either a **bar**, a **cable**, or a **strut**.

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If $\dim(\text{affine hull of } p^1, \ldots, p^n) = k$, we say that tensegrity $(G, p)$ is $k$-dimensional.

A tensegrity framework has two aspects: a **geometric one** $(p)$ and a **combinatorial one** $(G)$.
Applications

tensegrities have important applications in:

1. Molecular conformation theory.
2. Wireless sensor network localization problem.
3. Art.
Tensegrity as an Artwork

Kenneth Snelson needle tower sculpture in Washington D.C.
Tensegrity as an Artwork Cont’d

Kenneth Snelson Indexer II sculpture at the University of Michigan, Ann Arbor
Definition

Tensegrity \((G, q)\) in \(\mathbb{R}^s\) is said to be dominated by tensegrity \((G, p)\) in \(\mathbb{R}^r\) if

\[
\|q^i - q^j\| = \|p^i - p^j\| \quad \text{for all bar } \{i,j\}.
\]

\[
\|q^i - q^j\| \leq \|p^i - p^j\| \quad \text{for all cable } \{i,j\}.
\]

\[
\|q^i - q^j\| \geq \|p^i - p^j\| \quad \text{for all strut } \{i,j\}.
\]
Domination and Affine-Domination

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\| q_i - q_j \| \geq \| p_i - p_j \| \quad \text{for all strut } \{i, j\}.
\]

**Definition**

Tensegrity \((G, q)\) in \(\mathbb{R}^r\) is said to be **affinely-dominated by** tensegrity \((G, p)\) in \(\mathbb{R}^r\) if \((G, q)\) is dominated by \((G, p)\) and

\[
q_i = Ap_i + b \quad \text{for all } i = 1, \ldots, n
\]

for some \(r \times r\) matrix \(A\) and an \(r\)-vector \(b\).
Dimensional and Universal Rigidities

Definition

Tensegrity \((G, q)\) in \(\mathbb{R}^r\) is said to be congruent to tensegrity \((G, p)\) in \(\mathbb{R}^r\) if \(\|q^i - q^j\| = \|p^i - p^j\|\) for every \(i = 1, \ldots, n\).
Definition

Tensegrity \((G, q)\) in \(\mathbb{R}^r\) is said to be **congruent** to tensegrity \((G, p)\) in \(\mathbb{R}^r\) if \(||q^i - q^j|| = ||p^i - p^j||\) for every \(i = 1, \ldots, n\).

Definition

An \(r\)-dimensional tensegrity \((G, p)\) in \(\mathbb{R}^r\) is said to be **dimensionally rigid** if no \(s\)-dimensional tensegrity \((G, q)\), for any \(s \geq r + 1\), is dominated by \((G, p)\).
**Definition**

Tensegrity \((G, q)\) in \(\mathbb{R}^r\) is said to be **congruent** to tensegrity \((G, p)\) in \(\mathbb{R}^r\) if \(\|q^i - q^j\| = \|p^i - p^j\|\) for every \(i = 1, \ldots, n\).

**Definition**

An \(r\)-dimensional tensegrity \((G, p)\) in \(\mathbb{R}^r\) is said to be **dimensionally rigid** if no \(s\)-dimensional tensegrity \((G, q)\), for any \(s \geq r + 1\), is dominated by \((G, p)\).

**Definition**

An \(r\)-dimensional tensegrity \((G, p)\) in \(\mathbb{R}^r\) is said to be **universally rigid** if every \(s\)-dimensional tensegrity \((G, q)\), for any \(s\), that is dominated by \((G, p)\) is in fact congruent to \((G, p)\).
Example

Cable

Strut

1

3

4

2

Not universally rigid. It folds on the diagonal.
Example

Cable

Strut

universally rigid.

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Example

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Not universally rigid. It folds on the diagonal.
Characterization of Universal Rigidity

**Theorem**

An $r$-dimensional Tensegrity $(G, p)$ in $\mathbb{R}^r$ is **universally rigid** if and only if

1. $(G, p)$ is *dimensionally rigid*.
2. There does not exist an $r$-dimensional tensegrity $(G, q)$ in $\mathbb{R}^r$ affinely-dominated by, but not congruent to, $(G, p)$.

Condition 2 is known as the “no conic at infinity” condition.
In This Talk, I’ll:

1. Present the well-known sufficient condition for dimensional rigidity.
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1. Present the well-known sufficient condition for dimensional rigidity.
2. Present conditions under which the “no conic at infinity” holds.
A stress of a tensegrity \((G, p)\) is a real-valued function \(\omega\) on 
\(E(G) = B \cup C \cup S\) such that:

\[
\sum_{j : \{i, j\} \in E(G)} \omega_{ij} (p^i - p^j) = 0 \text{ for all } i = 1, \ldots, n.
\]
Stress Matrices

- A stress of a tensegrity $(G, p)$ is a real-valued function $\omega$ on $E(G) = B \cup C \cup S$ such that:

\[
\sum_{j: \{i, j\} \in E(G)} \omega_{ij}(p^i - p^j) = 0 \text{ for all } i = 1, \ldots, n.
\]

- A stress $\omega$ is proper if $\omega_{ij} \geq 0$ for every $\{i, j\} \in C$ and $\omega_{ij} \leq 0$ for every $\{i, j\} \in S$. 
Stress Matrices

- A **stress** of a tensegrity \((G, p)\) is a real-valued function \(\omega\) on \(E(G) = B \cup C \cup S\) such that:

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- A stress \(\omega\) is **proper** if \(\omega_{ij} \geq 0\) for every \(\{i, j\} \in C\) and \(\omega_{ij} \leq 0\) for every \(\{i, j\} \in S\).

- The **stress matrix** associated with stress \(\omega\) is the \(n \times n\) symmetric matrix \(\Omega\) where

\[
\Omega_{ij} = \begin{cases} 
-\omega_{ij} & \text{if } (i, j) \in E(G), \\
0 & \text{if } (i, j) \notin E(G), \\
\sum_{k : \{i, k\} \in E(G)} \omega_{ik} & \text{if } i = j.
\end{cases}
\]
Example

\begin{align*}
\omega_{12} &= 1, \quad \omega_{14} = 1, \\
\omega_{13} &= -1.
\end{align*}

$$\begin{pmatrix}
1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1
\end{pmatrix}$$

\(\Omega\) is proper positive semidefinite of rank 1.
Theorem (Connelly ’82)

An $r$-dimensional Tensegrity $(G, p)$ on $n$ nodes in $\mathbb{R}^r$ ($r \leq n - 2$) is dimensionally rigid if there exists a proper positive semidefinite stress matrix $\Omega$ of $(G, p)$ of rank $n - r - 1$. 
Example

- Bar
- Cable
- Strut

A dimensionally but not universally rigid tensegrity. The "No Conic at Infinity" Condition does not hold. In the sequel we concentrate on this condition.

A.Y. Alfakih (joint work with Viet-Hang Nguyen)
A dimensionally but not universally rigid tensegrity.
A dimensionally but not universally rigid tensegrity. The “No Conic at Infinity” Condition does not hold. In the sequel we concentrate on this condition.
A configuration $p = (p_1, \ldots, p^n)$ in $\mathbb{R}^r$ is generic if the coordinates of $p_1, \ldots, p^n$ are algebraically independent over the rationals, i.e., the coordinates of $p_1, \ldots, p^n$ do not satisfy any nonzero polynomial with rational coefficients.
Generic Configurations

**Definition**

A configuration \( p = (p^1, \ldots, p^n) \) in \( \mathbb{R}^r \) is **generic** if the coordinates of \( p^1, \ldots, p^n \) are algebraically independent over the rationals, i.e., the coordinates of \( p^1, \ldots, p^n \) do not satisfy any nonzero polynomial with rational coefficients.

**Lemma (Connelly ’05)**

Let \((G, p)\) be an \( r \)-dimensional tensegerity. If configuration \( p \) is **generic** and every node of \( G \) has degree at least \( r \), then the “no conic at infinity” condition holds. Consequently, **dimensional rigidity implies universal rigidity**.
A configuration $p = (p^1, \ldots, p^n)$ in $\mathbb{R}^r$ is in general position if every subset of $\{p^1, \ldots, p^n\}$ of cardinality $r + 1$ is affinely independent.
Configurations in General Position

**Definition**

A configuration \( p = (p^1, \ldots, p^n) \) in \( \mathbb{R}^r \) is **in general position** if every subset of \( \{p^1, \ldots, p^n\} \) of cardinality \( r + 1 \) is affinely independent.

**Definition**

A bar framework \((G, p)\) is a tensegrity framework where all the edges are bars, i.e., \( E(G) = B \) and \( C = S = \emptyset \).
Definitions

A configuration \( p = (p^1, \ldots, p^n) \) in \( \mathbb{R}^r \) is in general position if every subset of \( \{ p^1, \ldots, p^n \} \) of cardinality \( r + 1 \) is affinely independent.

A bar framework \((G, p)\) is a tensegrity framework where all the edges are bars, i.e., \( E(G) = B \) and \( C = S = \emptyset \).

Lemma (A. and Ye ’13)

Let \((G, p)\) be an \( r \)-dimensional bar framework. If \((G, p)\) admits a stress matrix \( \Omega \) of rank \( n - r - 1 \) and configuration \( p \) is in general position, then the “no conic at infinity” condition holds. Consequently, dimensional rigidity implies universal rigidity.
Let $C^*$ and $S^*$ be the sets of stressed cables and stressed struts respectively, i.e,

$C^* = \{ \{i, j\} \in C : \omega_{ij} \neq 0 \}$ and $S^* = \{ \{i, j\} \in S : \omega_{ij} \neq 0 \}$.
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$C^* = \{\{i,j\} \in C : \omega_{ij} \neq 0\}$ and $S^* = \{\{i,j\} \in S : \omega_{ij} \neq 0\}$.

**Theorem (A. and V-T Nguyen ’13)**

Let $(G, p)$ be an $r$-dimensional tensegrity in $\mathbb{R}^r$. If the following conditions hold:

1. there exists a proper stress matrix $\Omega$ of $(G, p)$ of rank $n - r - 1$.
2. for each node $i$, the set $\{p^i\} \cup \{p^j : \{i,j\} \in B \cup C^* \cup S^*\}$ affinely span $\mathbb{R}^r$.

Then the “no conic at infinity” condition holds. Consequently, dimensional rigidity implies universal rigidity.
Corollary (A. and V-T Nguyen ’13)

Let \((G, p)\) be an \(r\)-dimensional tensegrity in \(\mathbb{R}^r\). If the following conditions hold:

1. there exists a proper stress matrix \(\Omega\) of \((G, p)\) of rank \(n - r - 1\).
2. for each node \(i\), the set \(\{p^i\} \cup \{p^j : \{i, j\} \in B \cup C^* \cup S^*\}\) is in general position in \(\mathbb{R}^r\).

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Theorem (A. and V-T Nguyen ’13)

Let \((G, p)\) be an \(r\)-dimensional bar framework in \(\mathbb{R}^r\). If the following conditions hold:

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2. for each node \(i\), the set \(\{p^i\} \cup \{p^j : \{i, j\} \in E(G)\}\) affinely span \(\mathbb{R}^r\).

Then the “no conic at infinity” condition holds. Consequently, dimensional rigidity implies universal rigidity.
We use **Gram matrices** to represent configuration 
\( p = (p_1, \ldots, p^n) \).
The Idea Behind the Proof

1. We use **Gram matrices** to represent configuration $p = (p^1, \ldots, p^n)$.

2. Let $P^T = [p^1 \cdots p^n]$. $P$ is called the **configuration matrix**. Then the Gram matrix is $PP^T$. 

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2. Let $P^T = [p^1 \cdots p^n]$. $P$ is called the **configuration matrix**. Then the Gram matrix is $PP^T$.

3. Thus the universal rigidity problem becomes amenable to semi-definite programming.

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**Theorem (A. and V-T Nguyen ’13)**

Let $(G, p)$ be an $r$-dimensional tensegrity in $\mathbb{R}^r$ and let $\Omega$ be a proper positive semidefinite stress matrix of $(G, p)$. Then $\Omega$ is a proper stress matrix for all tensegrities $(G, p')$ dominated by $(G, p)$.
A Gale matrix of $r$-dimensional tensegrity $(G, p)$ in $\mathbb{R}^r$ is any $n \times (n - r - 1)$ matrix $Z$ such that the columns of $Z$ form a basis of the null space of:

$$
\begin{bmatrix}
p^1 & p^2 & \cdots & p^n \\
1 & 1 & \cdots & 1
\end{bmatrix} =
\begin{bmatrix}
P^T \\
e^T
\end{bmatrix}.
$$
Gale Matrices

- A **Gale matrix** of \( r \)-dimensional tensegrity \((G, p)\) in \(\mathbb{R}^r\) is any \(n \times (n - r - 1)\) matrix \(Z\) such that the columns of \(Z\) form a basis of the null space of:

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P^T \\
e^T
\end{bmatrix}.
\]

- In Polytope theory, the **rows** of \(Z\) (\(z^1, \ldots, z^n\) in \(\mathbb{R}^{n-r-1}\)) are called **Gale transforms** of \(p^1, \ldots, p^n\).
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\end{bmatrix}
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P^T \\
e^T
\end{bmatrix}.
$$

In Polytope theory, the rows of $Z$ ($z^1, \ldots, z^n$ in $\mathbb{R}^{n-r-1}$) are called Gale transforms of $p^1, \ldots, p^n$.

The Gale matrix $Z$ encodes the affine dependencies among the points $p^1, \ldots, p^n$. 
Theorem (A ‘07)

Let $\Omega$ and $Z$ be, respectively, a stress matrix and a Gale matrix of $(G, p)$. Then

$$\Omega = Z\Psi Z^T$$

for some symmetric matrix $\Psi$.

On the other hand, let $\Psi'$ be any symmetric matrix such that

$$z^i \Psi' z^j = 0$$

for all $\{i, j\} \notin E$,

where $z^i$ is the $i$th row of $Z$. Then $Z\Psi' Z^T$ is a stress matrix of $(G, p)$.
Example

Gale matrix is

\[ Z = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}. \]

and stress matrix \( \Omega = ZZ^T \).
Properties of Gale Transform

Lemma

Let \((G, p)\) be an \(r\)-dimensional tensegrity in \(\mathbb{R}^r\) and let \(z^1, \ldots, z^n\) be, respectively, Gale transforms of \(p^1, \ldots, p^n\). Let \(J \subseteq \{1, \ldots, n\}\) and assume that the set of vectors \(\{p^i : i \in J\}\) affinely span \(\mathbb{R}^r\). Then the set \(\{z^i : i \in \overline{J}\}\) is linearly independent, where \(\overline{J} = \{1, \ldots, n\} \setminus J\).
Let $F_{ij} = (e^i - e^j)(e^i - e^j)^T$, $e^i$ is the $i$th standard unit vector in $\mathbb{R}^n$. Recall that the configuration matrix $P^T = [p^1 \ldots p^n]$. 
Let $F_{ij} = (e^i - e^j)(e^i - e^j)^T$, $e^i$ is the $i$th standard unit vector in $\mathbb{R}^n$. Recall that the configuration matrix $P^T = [p^1 \ldots p^n]$.

**Lemma**

Let $(G, p)$ be an $r$-dimensional tensegrity in $\mathbb{R}^r$. Then the “no conic at infinity” holds iff there does not exist a nonzero symmetric matrix $\Phi$ such that:

- $\text{trace}(F_{ij}(P\Phi P^T)) = 0$ for all $\{i, j\} \in B$.
- $\text{trace}(F_{ij}(P\Phi P^T)) \leq 0$ for all $\{i, j\} \in C$.
- $\text{trace}(F_{ij}(P\Phi P^T)) \geq 0$ for all $\{i, j\} \in S$. 
Affine-Domination

$E_{ij}$ is the matrix with 1s in the $ij$th and $ji$th entries and 0’s elsewhere.
Affine-Domination

$E_{ij}$ is the matrix with 1s in the $ij$th and $ji$th entries and 0’s elsewhere.

**Lemma**

Let $(G, p)$ be an $r$-dimensional tensegrity in $\mathbb{R}^r$ and let $Z$ be a Gale matrix of $(G, p)$. Then the “no conic at infinity” holds iff there does not exist a nonzero $y = (y_{ij}) \in \mathbb{R}^{\bar{E} + |C| + |S|}$ and $\xi = (\xi_i) \in \mathbb{R}^{n-r-1}$ where $y_{ij} \geq 0$ for all $\{i, j\} \in C$ and $y_{ij} \leq 0$ for all $\{i, j\} \in S$ such that:

$$\mathcal{E}(y)Z = e\xi^T,$$

where $\mathcal{E}(y) = \sum_{\{i,j\} \in \bar{E} \cup C \cup S} y_{ij}E_{ij}$. 
Affine-Domination when a proper $\Omega$ is Known

The following are equivalent:

1. the ‘no conic at infinity” holds.

2. (Whiteley unpublished) $\exists$ symmetric $\Phi \neq 0$ such that:
   
   \[ \text{trace}(F_{ij}(P\Phi P^T)) = 0 \text{ for all } \{i, j\} \in B \cup C^* \cup S^*. \]
   
   \[ \text{trace}(F_{ij}(P\Phi P^T)) \leq 0 \text{ for all } \{i, j\} \in C^0. \]
   
   \[ \text{trace}(F_{ij}(P\Phi P^T)) \geq 0 \text{ for all } \{i, j\} \in S^0. \]

3. $\exists y = (y_{ij}) \neq 0 \in \mathbb{R}^{|\bar{E}|+|C^0|+|S^0|}$ and $\xi = (\xi_i) \in \mathbb{R}^{n-r-1}$ where $y_{ij} \geq 0 \ \forall \ \{i, j\} \in C^0$ and $y_{ij} \leq 0 \ \forall \ \{i, j\} \in S^0$ such that:
   
   \[ E^0(y)Z = e\xi^T, \]
   
   where $E^0(y) = \sum_{\{i,j\} \in \bar{E} \cup C^0 \cup S^0} y_{ij}E_{ij}$. 

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Lemma

Assume that $\Omega = Z\Psi Z^T$ is a proper stress matrix of $(G, p)$ of rank $n - r - 1$. Then the following are equivalent:

1. the “no conic at infinity” holds
2. $\forall y = (y_{ij}) \neq 0 \in \mathbb{R}^{|\tilde{E}|+|C^0|+|S^0|}$ and $\xi = (\xi_i) \in \mathbb{R}^{n-r-1}$ where $y_{ij} \geq 0$ for all $\{i, j\} \in C^0$ and $y_{ij} \leq 0$ for all $\{i, j\} \in S^0$ such that:

$$\mathcal{E}^0(y)Z = 0,$$

where $\mathcal{E}^0(y) = \sum_{\{i,j\} \in \tilde{E} \cup C^0 \cup S^0} y_{ij} E_{ij}$. 
Outline of the Proof of the main Theorem

It suffices to prove that under the theorem assumptions, the only solution of

\[ \mathcal{E}^0(y)Z = 0 \]  

is the trivial solution \( y = 0 \). Hence, the “no conic at infinity” condition holds.
Outline of the Proof of the main Theorem

It suffices to prove that under the theorem assumptions, the only solution of

$$E^0(y)Z = 0$$  \hspace{1cm} (1)

is the trivial solution $y = 0$. Hence, the “no conic at infinity” condition holds.

Equation (1) can be written as

$$\sum_{j=1}^{n} (E^0(y))_{ij} z^i = 0$$ for all $i = 1, \ldots, n$.

which reduces to

$$\sum_{j: \{i,j\} \in E \cup C^0 \cup S^0} (E^0(y))_{ij} z^i = 0.$$
Outline of the Proof of the main Theorem

It suffices to prove that under the theorem assumptions, the only solution of

$$E^0(y)Z = 0 \quad (1)$$

is the trivial solution $y = 0$. Hence, the “no conic at infinity” condition holds.

Equation (1) can be written as

$$\sum_{j=1}^{n} (E^0(y))_{ij} z^i = 0 \text{ for all } i = 1, \ldots, n.$$ 

which reduces to

$$\sum_{j : \{i, j\} \in \bar{E} \cup C^0 \cup S^0} (E^0(y))_{ij} z^i = 0.$$ 

Thus the result follows from the linear independence of

$$\{z^i : \{i, j\} \in \bar{E} \cup C^0 \cup S^0\}.$$
Thank You