Introduction to Mirror Moonshine

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This is an introduction to "Mirror Moonshine" for Calabi–Yau manifolds. By this we mean loosely that mirror maps of Calabi–Yau manifolds have some modular properties.

We will illustrate in some examples that monodromy groups are modular groups (e.g., finite index genus zero congruence subgroups of $SL(2,\mathbb{Z})$) and that mirror maps are related to modular functions.

However, we also exhibit Calabi–Yau manifolds whose monodromy groups cannot have modular properties (i.e., thin). Cf. The talk by Hugh Thomas.

Picard–Fuchs differential equations

Let M_t be a 1-parameter family of Calabi–Yau manifolds of dimension n, parametrized by $t \in \mathbb{P}^1(\mathbb{C})$.

Let ω_t be the holomorphic top n-form on M_t (up to scalar). The periods $\int_{\gamma_t} \omega_t$ (with γ_t n-cycles on M_t) satisfy certain differential equations, called the *Picard–Fuchs differential equations* of M_t.

Monodromy groups

Let

$$L : r_n(t)y^{(n)} + r_{n-1}(t)y^{(n-1)} + \cdots + r_1(t)y' + r_0(t),\ r_i \in \mathbb{C}(t) \quad \forall i$$

be a differential operator with regular singularities. (We say that t_0 is a regular singular point if $r_{n-i}(t)$ has a pole of order at most i at t_0.) Assuming that $t = 0$ is a regular singularity, and write

$$r_{n-i}(t) = t^{-i}\tilde{r}_{n-i}(t),\ i = 1, \cdots, n,$$

where the functions $\tilde{r}_{n-i}(t)$ are analytic at $t = 0$. The roots of the *indicial equation*

$$s(s-1)\cdots(s-n+1) + \tilde{r}_{n-1}(0)s(s-1)\cdots(s-n+2) + \cdots$$
$$+\tilde{r}_1(0)s + \tilde{r}_0(0) = 0$$

determine the *exponents* of L at $t = 0$. We say that $t = 0$ is a point of *maximal unipotent monodromy (MUM)* if the exponents at $t = 0$

are all zero. (However, points of MUM may not always exist.)

Let S be the solution space of L at t_0. Then the analytic continuation along a closed curve γ circling around t_0 gives rise to an automorphism of S, called *monodromy*. If a basis $\{u_1, u_2, \cdots, u_n\}$ of S is chosen, then we have a matrix representation $A = (a_{i,j})_{1 \leq i,j \leq n}$ of the monodromy. The group of all such matrices is called the *monodromy group* relative to the basis $\{u_i\}$ of the differential equation L. If we choose a different bases, we will get another monodromy group. The two monodromy groups are related by conjugation. Thus the monodromy group is defined by up to conjugation.

Definition of the mirror map

Assume that $t = 0$ is a point of maximal unipotent monodromy. At $t = 0$, the Picard–Fuchs differential equation has a unique holomorphic solution

$$y_0(t) = 1 + \sum_{n \geq 1} a_n t^n$$

and a logarithmic solution

$$y_1(t) = y_0(t) \log(t) + g_1(t)$$

where $g_1(t)$ is holomorphic near $t = 0$ with $g_1(0) = 0$. Set

$$z = \frac{y_1(t)}{2\pi i y_0(t)}.$$

Then

$$q := q(z) = \exp(2\pi i z) = t\exp(\frac{g_1(t)}{y_0(t)})$$

gives an invertible analytic map from a disc $|t| < K_0$ to some disc $|q| < K_1$. The inverse which expresses t as a function of $q = q(z)$, is denoted by $t(q)$. Then $t(q)$ is holomorphic at $t = 0$, and it is defined to be the *mirror map* of the Calabi–Yau family.

We want to study

- Modular properties of monodromy groups, i.e, are they finite index subgroups of some known groups like $PSL(2,\mathbb{Z})$, or $Sp(n,\mathbb{Z})$?
- Modular properties of mirror maps, are they related to modular functions, or some other automorphic functions for the above monodromy groups?

Examples: $n = 2$

Let $E_t : y^2 = x^3 + a(t)x + b(t)$, $t \in \mathbb{P}^1(\mathbb{C})$ be a family of elliptic curves. Then for any 1-cycle γ we have

$$\frac{d}{dt}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} -(2a^2a' + 9bb') & 3(3a'b - 2ab') \\ a(3a'b - 2ab') & 2a^2a' + 9bb' \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},$$

where

$$\Delta = 2(4a^3 + 27b^2)$$

and

$$\eta_1 = \int_\gamma \frac{dx}{y}, \eta_2 = \int_\gamma \frac{xdx}{y}$$

and prime denotes the differentiation with respect to t.

For instance, if E_t is given by the Legendre family

$$E_t : y^2 = x(x-1)(x-t)$$

Then η_1 and η_2 are solutions of the second-order differential equation

$$(*) \quad \left(\theta^2 + \frac{1-2t}{t(1-t)}\theta - \frac{1}{4t(1-t)}\right) u = 0, \text{ with } \theta = t\frac{d}{dt}.$$

Lemma: *There is a basis for the solution space of (*) such that the projective monodromy group is* $\Gamma(2) \subset PSL(2,\mathbb{Z})$, *which is a genus zero principal congruence subgroup of index* 6.

The holomorphic solution is given by

$${}_2F_1(t) = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1, t)$$

and the other solution is

$$ {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1, 1-t). $$

The mirror map is the modular function for $\Gamma(2)$, *namely,* $(\frac{\theta_2(z)}{\theta_3(z)})^4$ *where* θ_2 *and* θ_3 *are classical Jacobi theta functions.*

Remark: Doran gave a necessary and sufficient condition for the mirror map of an elliptic family with non-constant j-invariant to be a Hauptmodul of a subgroup of $PSL(2, \mathbb{Z})$ of finite index.

K3 surfaces

Let S be a K3 surface. Then

$$H^2(S,\mathbb{Z}) = \Lambda := U_2^3 \perp (-E_8)^2$$

where U_2 is the rank 2 hyperbolic lattice and E_8 is the unique positive definite unimodular lattice of rank 8. The Picard group $Pic(S) = NS(S)$ generated by algebraic cycles on S is the sublattice of $\Lambda \cap H^{1,1}(S,\mathbb{R})$ of rank $\rho(S)$, so $\rho(S)$ is at most 20. The orthogonal complement of $Pic(S)$ in $H^2(S,\mathbb{Z})$ is $T(S)$ the lattice of transcendental cycles on S.

1-parameter families of K3 surfaces

Let S_t be a 1-parameter family of K3 surfaces parametrized by $t \in B$ where $B := \mathbb{P}^1 \setminus \{\, t \,|\, S_t \text{ singular} \,\}$. There is, up to scalar multiplication, a unique holomorphic 2-form $\omega_t \in H^2(S_t, \mathbb{C})$. Now fix $t_0 \in B$ and let $\pi_1(B, t_0)$ be the fundamental group. Then there is the monodromy representation

$$\pi_1(B, t_0) \rightarrow \mathrm{Aut}(\mathbb{P}(H_2(S_t, \mathbb{Z}))).$$

Its image, denoted by G is the *monodromy group* of S_t. Let $\{\gamma_i, \,|\, i = 1, \cdots, 22\,\}$ be a $\mathbb{Z}$–basis for $H_2(S_t, \mathbb{Z})$. Then the period map is the maps:

$$B \rightarrow \mathbb{P}^{21}/G : t \rightarrow [\int_{\gamma_1} \omega_t : \cdots \int_{\gamma_{22}} \omega_t]$$

where each function $\int_{\gamma_i} \omega_t$ is a period. The Picard–Fuchs differential equation is a differential equation satisfied by the periods with the monodromy group G.

Lemma *Let S_t be a 1-parameter family of K3 surfaces parametrized by $t \in B$. Suppose that $\rho(S_t) = r$ for generic t and put $k = 22 - r$. Then the period of S_t satisfies a Picard–Fuchs differential equation of order k.*

Proof: We know that $H^2_{DR}(S_t)$ has dimension 22. If $\rho(S_t) = r$, then $H^2_{DR}(S_t)/Pic(S_t)$ has dimension $22 - r$. Hence there is a linear relation satisfied by the classes:

$$[\omega_t], [\partial\omega_t/\partial t], \cdots, [\partial^k\omega_t/\partial^k t]$$

in $(H^2_{DR}(S_t)/Pic(S_t) \otimes \mathbb{C}$. Hence there are $g_0, g_1, \cdots, g_k \in \mathbb{C}$ such that

$$G := g_0\omega_0 + g_1\partial\omega_t/\partial t + \cdots + g_k\partial^k\omega_t/\partial^k t \in Pic(S_t),$$

which means that $\int_\gamma G = 0$ for any $\gamma \in H_2(S_t, \mathbb{Z})$. Since integrating around a cycle $\gamma \in H_2(S_t, \mathbb{Z})$ commutes with differentiation with respect to t in the sense that

$$\int_\gamma g_i \frac{\partial^i \omega_t}{\partial^i t} = g_i \frac{d^i}{dt^i} \int_\gamma \omega_i$$

the linear relation becomes a differential equation for $\int_\gamma \omega_t$ upon changing the order of integration and differentiation.

Examples : $n = 3$

We consider 1-parameter families of K3 surfaces with generic Picard number 19. Hence their Picard–Fuchs differential equations are of order 3.

Remark: Looking at the so-called Shioda–Inose structures on these K3 surfaces, Doran has shown that Picard–Fuchs differential equations are in fact symmetric squares of order 2 differential equations.

What are the monodromy groups?

Proposition: *Let S_t be a 1-parameter family of K3 surfaces with Picard number 19 for generic $t \in B$. Let $T = T(S_t)$ be the group of transcendental cycles on S_t, and let disc(T) be the intersection matrix of T. Put*

$$SO(T) := \{M \in PSL(3, \mathbb{R}) \mid M^T \mathrm{disc}(T) M = \mathrm{disc}(T)\}$$

and

$$SO(T, \mathbb{Z}) := SO(T) \cap PSL(3, \mathbb{Z}).$$

Then the monodromy group of S_t is isomorphic to a subgroup of $SO(T, \mathbb{Z})$.

Example:*Let* $\mathcal{M}_n := U_2 \perp (-E_8)^2 \perp < -2n >$ *for some integer* n, *and let* S_t *be a pencil of* $\mathcal{M}_n$*-polarized K3 surfaces. Then* $T(S_t) = U_2 \perp < 2n >$ *and the intersection matrix of* T *is given by*

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2n \end{pmatrix}.$$

The monodromy group of S_t *is isomorphic to a congruence subgroup of* $PSL(2, \mathbb{R})$ *of level* n.

Mirror maps and moonshine

Now what can we say about mirror maps for 1-parameter families of $\mathcal{M}_n$-polarized K3 surfaces?

First we ought to determine the Picard–Fuchs differential equation of S_t of generic Picard number 19 by computing the period. The Picard–Fuchs differential equation has order 3, and it has a unique holomorphic solution

$$\omega_0(t) = \sum_{n \geq 0} c_n t^n$$

with $\omega_0(0) = 1$, and also has a unique solution of the form:

$$\omega_1(t) = \omega_0(t) \log(t) + \sum_{n \geq 1} d_n t^n,$$

where c_n, d_n are polynomials in the constants appearing in the Picard–Fuchs differential equation. Here we assumed that $t = 0$ is the point of maximal unipotent monodromy. Put $z = \dfrac{\omega_1(t)}{2\pi i \omega_0(t)}$.

Then $q = e^{2\pi i z} = t\, e^{\sum d_n t^n / \sum c_n t^n}$ gives an invertible analytic map from a disc $|t| < K_0$ to some disc $|q| < K_1$.

The inverse $t(q)$ is the mirror map of S_t, which is holomorphic in q. To relate $t(q)$ to Monstrous Moonshine, we consider $\frac{1}{t(q)}$.

Theorem: *(1) Let*

$$S_t : x_0^4 + x_1^4 + x_2^4 + x_3^4 - t^{-1} x_0 x_1 x_2 x_3 = 0.$$

Then $\rho(S_t) = 19$ *for generic* t *and the mirror map* $t(q)$ *is given by*

$$t(q) = q - 104q^2 + 6444q^3 - 3111744q^4 + \cdots$$

and

$$\frac{1}{t(q)} - 96 = \frac{1}{q} + 8 + 4372q + 96256q^2 + 1240002q^3 + \cdots$$

is the Hauptmodul T_{2A} *for* $\Gamma_0(2)+$.

(2) Let

$$S_t : x_0^6 + x_1^6 + x_2^6 + x_3^2 + t^{-1/6} x_0 x_1 x_2 x_3 = 0.$$

Then $\rho(S_t) = 19$ *for generic* t, *and the mirror map* $t(q)$ *is given by*

$$t(q) = q - 744q + 356652q^2 - 140361152q^3 + \cdots$$

and

$$\frac{1}{t(q)} = j(q)$$

is the Hauptmodul T_{1A} *for* $\Gamma = PSL(2, \mathbb{Z})$.

(3) Let

$$S_t : (x_1 + x_2 + x_3 + x_4)(x_1^{-1} + x_2^{-1} + x_3^{-1} + x_4^{-1}) = t + t^{-1}$$

be a 1-parameter family of K3 surfaces associated to the root lattice A_3. *Then* $\rho(S_t) = 19$ *for generic* t, *and the reciprocal of the mirror map is the Hauptmodul* T_{6C} *for* $\Gamma_0(6) + 3$.

(4)Let

$$S_t : 1 - (1 - xy)z - txyz(1 - x)(1 - y)(1 - z) = 0$$

be a 1-parameter family of K3 surfaces arising from the Apéry sequences for $\zeta(3)$. *Then* $\rho(S_t) = 19$ *for generic* t, *and the* *reciprocal of the mirror map is a Hauptmodul for* $\Gamma_0(12|2) + 6$, *i.e.*,

$$\frac{1}{t(q)} = \left(\frac{\eta(4\tau)\eta(6\tau)}{(\eta(2\tau)\eta(12\tau)} \right)^6 .$$

All these K3 surfaces are $\mathcal{M}_n$-polarized.

Examples : $n = 4$

Now we consider 1-parameter families of Calabi–Yau threefolds X_t. We will be focusing on 14 families of Calabi–Yau threefolds with $h^{2,1} = 1$, so that $B_3 = 4$ and Picard–Fuchs differential equations are of order 4. A most well-known example of such a family is the quintic threefold:

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - t^{-1} x_0 x_1 x_2 x_3 x_4 = 0.$$

Actually, its mirror partner has $h^{2,1} = 1$ and hence the Picard–Fuchs differential equation of order 4. The Picard–Fuchs differential operator of this family is

$$\theta^4 - 5^5 t(\theta + \frac{1}{5})(\theta + \frac{2}{5})(\theta + \frac{3}{5})(\theta + \frac{4}{5})$$

where $\theta = t\frac{d}{dt}$. It has $0, 5^{-5}$ and ∞ as regular singularities.

The 14 families have the Picard–Fuchs differential equation of the form (which are all of hypergeometric type ${}_4F_3$):

$$\theta^4 - Ct(\theta + A)(\theta + 1 - A)(\theta + B)(\theta + 1 - B)$$

where $A, B, C \in \mathbb{Q}$.

Basic geometric invariants of Calabi–Yau threefolds are degree $:= H^3$, the second Chern numbers $c_2 \cdot H$ and the Euler number $c_3 = \chi_{top}$.

Theorem: *Let*

$$L : \theta^4 - Ct(\theta + A)(\theta + 1 - A)(\theta + B)(\theta + 1 - B)$$

be one of the 14 equations of hypergeometric type, and let $H^3, c_2 \cdot H$ and c_3 be geometric invariants of the associated Calabi–Yau threefold (given in the table). Let $\{y_0, y_1, y_2, y_3\}$ be the Frobenius basis at the point $t = 0$ of maximal unipotent monodromy. Then with respect to the ordered basis $\{y_3/(2\pi i)^3, y_2/(2\pi i)^2, y_1/2\pi i, y_0\}$, the monodromy matrices around $z = 0$ and $z = 1/C$ are

$$\begin{pmatrix} 1 & 1 & 1/2 & 1/6 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1+a & 0 & ab/d & a^2/d \\ -b & 1 & -b^2/d & -ab/d \\ 0 & 0 & 1 & 0 \\ -d & 0 & -b & 1-a \end{pmatrix}$$

where

$$a = \frac{c_3}{(2\pi i)^3}\zeta(3),\ b = \frac{c_2 \cdot H}{24},\ d = H^3.$$

#	A	B	C	Description	H^3	$c_2 \cdot H$	c_3	Ref
1	1/5	2/5	3125	$X(5) \subset \mathbb{P}^4$	5	50	−200	[9]
2	1/10	3/10	$8 \cdot 10^5$	$X(10) \subset \mathbb{P}^4(1,1,1,2,5)$	1	34	−288	[20]
3	1/2	1/2	256	$X(2,2,2,2) \subset \mathbb{P}^7$	16	64	−128	[19]
4	1/3	1/3	729	$X(3,3) \subset \mathbb{P}^5$	9	54	−144	[19]
5	1/3	1/2	432	$X(2,2,3) \subset \mathbb{P}^6$	12	60	−144	[19]
6	1/4	1/2	1024	$X(2,4) \subset \mathbb{P}^5$	8	56	−176	[19]
7	1/8	3/8	65536	$X(8) \subset \mathbb{P}^4(1,1,1,1,4)$	2	44	−296	[20]
8	1/6	1/3	11664	$X(6) \subset \mathbb{P}^4(1,1,1,1,2)$	3	42	−204	[20]
9	1/12	5/12	12^6	$X(2,12) \subset \mathbb{P}^5(1,1,1,1,4,6)$	1	46	−484	[11]
10	1/4	1/4	4096	$X(4,4) \subset \mathbb{P}^5(1,1,1,1,2,2)$	4	40	−144	[17]
11	1/4	1/3	1728	$X(4,6) \subset \mathbb{P}^5(1,1,1,2,2,3)$	6	48	−156	[17]
12	1/6	1/4	27648	$X(3,4) \subset \mathbb{P}^5(1,1,1,1,1,2)$	2	32	−156	[17]
13	1/6	1/6	$2^8 \cdot 3^6$	$X(6,6) \subset \mathbb{P}^5(1,1,2,2,3,3)$	1	22	−120	[17]
14	1/6	1/2	6912	$X(2,6) \subset \mathbb{P}^5(1,1,1,1,1,3)$	4	52	−256	[17]

By conjugating the above matrices by

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & d & d/2 & -b \\ -d & 0 & -b & -a \end{pmatrix}$$

the matrices can be brought into the symplectic group $Sp(4, \mathbb{Z})$.

Theorem: *The monodromy group is generated by the two matrices*

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ d & d & 1 & 0 \\ 0 & -k & -1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for $z = 0$ and $z = 1/C$, respectively, where $k = 2b + d/6$.

They are contained in the congruence subgroups $\Gamma(d, gcd(d,k))$ *of* $Sp(4,\mathbb{Z})$ *of finite index, where*

$$\Gamma(d_1, d_2) = \left\{ \gamma \in Sp(4,\mathbb{Z}) \,|\, \gamma \equiv \begin{pmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & * & * & * \end{pmatrix} \mod d_1 \right\}$$

$$\cap \left\{ \gamma \in Sp(4,\mathbb{Z}) \,|\, \gamma \equiv \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \mod d_2 \right\}.$$

Remark: Even though the congruence subgroup $\Gamma(d_1, d_2)$ has finite index in $Sp(4, \mathbb{Z})$ (by Erdenberger), we are not able to show that the monodromy group is of finite index in $\Gamma(d, gcd(d, k))$.

A physics consequence of finite indexness of the monodromy group is that there are only finitely many string vacua ($\sim 10^{500}$), or equivalently the finiteness of the Hodge numbers (~ 500).

Theorem (Chris Brav and Hugh Thomas): *Among the monodromy group of the* 14 *Calabi–Yau threefolds of hypergeometric type,* 7 *are thin.*

By Sarnak et al, among the remaining 7 cases, 3 are arithmetic, and but the remaining 4 are unknown.

Consequence: For the 7 Calabi-Yau threefolds with thin monodromy groups, the monodromy group cannot be a finite index subgroup of $Sp(4, \mathbb{Z})$. So it is not modular.

• thin
A = Arithmetic
? = unknown

	#	A	B	C	Description	H^3	$c_2 \cdot H$	c_3	Ref
•	1	1/5	2/5	3125	$X(5) \subset \mathbb{P}^4$	5	50	−200	[9]
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Kontsevich observed that the mirror map for the quintic Calabi–Yau family has bounded denominator.

Glossaries

- The Zariski closure of Γ is the smallest matrix group that contains Γ. In our case, the Zariski closure of the monodromy group is $Sp(4, \mathbb{Z})$.
- If Γ is of infinite index in its Zariski closure, Γ is called *thin*.
- A typical example of an arithmetic group is a subgroup of $GL_n(\mathbb{Z})$.