Classical Mirror Constructions II

The Batyrev-Borisov Construction

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Outline

Reflexive Polytopes

Hypersurfaces in Toric Varieties

K3 Surfaces

Symmetric Subfamilies

References
The Batyrev-Borisov Strategy

We can describe mirror families of Calabi-Yau manifolds using combinatorial objects called reflexive polytopes.
Lattice Polygons

Let $N$ be a lattice isomorphic to $\mathbb{Z}^2$.

A **lattice polygon** is a polygon in the plane $N_{\mathbb{R}}$ which has vertices in the lattice.
Fano Polygons

We say a lattice polygon is Fano if it has only one lattice point, the origin, in its interior.
Describing a Fano Polygon

- List the vertices

\[
\begin{align*}
\{(0,1), (1,0), (-1,-1)\}
\end{align*}
\]
Describing a Fano Polygon

- List the vertices

\{(0, 1), (1, 0), (-1, -1)\}
Describing a Fano Polygon

- List the vertices
  
  \{(0, 1), (1, 0), (−1, −1)\}

- List the equations of the edges
Describing a Fano Polygon

- List the vertices
  \{ (0, 1), (1, 0), (-1, -1) \}

- List the equations of the edges
  
  \[-x - y = -1\]
  
  \[2x - y = -1\]
  
  \[-x + 2y = -1\]
A Dual Lattice

The dual lattice $M$ of $N$ is given by $\text{Hom}(N, \mathbb{Z})$; it is also isomorphic to $\mathbb{Z}^2$. We write the pairing of $v \in N$ and $w \in M$ as $\langle v, w \rangle$. After choosing a basis, we may also use dot product notation:

$$ (n_1, n_2) \cdot (m_1, m_2) = n_1 m_1 + n_2 m_2 $$

The pairing extends to a real-valued pairing on elements of $N_\mathbb{R}$ and $M_\mathbb{R}$. 
Polar Polygons

Edge equations define new polygons

Let $\Delta$ be a lattice polygon in $\mathbb{N}_\mathbb{R}$ which contains $(0,0)$. The polar polygon $\Delta^\circ$ is the polygon in $\mathbb{M}_\mathbb{R}$ given by:

$$\{(m_1, m_2) : (n_1, n_2) \cdot (m_1, m_2) \geq -1 \text{ for all } (n_1, n_2) \in \Delta\}$$
Polar Polygons

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\[
\begin{align*}
(x, y) \cdot (-1, -1) &= -1 \\
(x, y) \cdot (2, -1) &= -1 \\
(x, y) \cdot (-1, 2) &= -1
\end{align*}
\]
Polar Polygons

Edge equations define new polygons

Let $\Delta$ be a lattice polygon in $N_R$ which contains $(0, 0)$. The polar polygon $\Delta^\circ$ is the polygon in $M_R$ given by:

$$\{(m_1, m_2) : (n_1, n_2) \cdot (m_1, m_2) \geq -1 \text{ for all } (n_1, n_2) \in \Delta\}$$

$$(x, y) \cdot (-1, -1) = -1$$
$$(x, y) \cdot (2, -1) = -1$$
$$(x, y) \cdot (-1, 2) = -1$$

Figure: Our triangle's polar polygon
Mirror Pairs

If $\Delta$ is a Fano polygon, then:
- $\Delta^\circ$ is a lattice polygon
- In fact, $\Delta^\circ$ is another Fano polygon
- $(\Delta^\circ)^\circ = \Delta$.

We say that . . .
- $\Delta$ is a reflexive polygon.
- $\Delta$ and $\Delta^\circ$ are a mirror pair.
A Polygon Duality

Mirror pair of triangles

Figure: 3 boundary lattice points

Figure: 9 boundary lattice points

\[3 + 9 = 12\]
Classifying Fano Polygons

- We can classify Fano polygons up to a change of coordinates that acts bijectively on lattice points
- There are 16 isomorphism classes of Fano polygons
Mirror Pairs of Polygons

Figure: F. Rohsiepe, “Elliptic Toric K3 Surfaces and Gauge Algebras”
**Definition**

Let \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q \} \) be a set of points in \( \mathbb{R}^k \). The **polytope** with vertices \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q \} \) is the convex hull of these points.
Polar Polytopes

Let \( N \cong \mathbb{Z}^n \) be a lattice. A lattice polytope is a polytope in \( N \mathbb{R} \) with vertices in \( N \).
As before, we have a dual lattice \( M \) and a pairing \( \langle v, w \rangle \).

**Definition**
Let \( \Delta \) be a lattice polytope in \( N \mathbb{R} \) which contains \((0, \ldots, 0)\). The polar polytope \( \Delta^\circ \) is the polytope in \( M \mathbb{R} \) given by:

\[
\{(m_1, \ldots, m_k) : \langle(n_1, \ldots, n_k), (m_1, \ldots, m_k)\rangle \geq -1 \text{ for all } (n_1, \ldots, n_k) \in \Delta\}.
\]
Reflexive Polytopes

Definition
A lattice polytope $\Delta$ is reflexive if $\Delta^\circ$ is also a lattice polytope.

- If $\Delta$ is reflexive, $(\Delta^\circ)^\circ = \Delta$.
- $\Delta$ and $\Delta^\circ$ are a mirror pair.
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Fano vs. Reflexive

- Every reflexive polytope is Fano
- In dimensions $n \geq 3$, not every Fano polytope is reflexive
Classifying Reflexive Polytopes

Up to a change of coordinates that preserves the lattice, there are ...

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<tr>
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Mirror Polytopes Yield Mirror Spaces

\[
\begin{align*}
\text{polytope} & \leftrightarrow \text{polar polytope} \\
\downarrow & \downarrow \\
\text{toric variety} & \leftrightarrow \text{polar toric variety} \\
\downarrow & \downarrow \\
\text{hypersurface family} & \leftrightarrow \text{mirror hypersurface family}
\end{align*}
\]
Cones

A cone in $N$ is a subset of the real vector space $N_\mathbb{R} = N \otimes \mathbb{R}$ generated by nonnegative $\mathbb{R}$-linear combinations of a set of vectors $\{v_1, \ldots, v_m\} \subset N$. We assume that cones are strongly convex, that is, they contain no line through the origin.

Figure: Cox, Little, and Schenk
Fans

A fan $\Sigma$ consists of a finite collection of cones such that:

- Each face of a cone in the fan is also in the fan
- Any pair of cones in the fan intersects in a common face.

Figure: Cox, Little, and Schenk
We say a fan $\Sigma$ is simplicial if the generators of each cone in $\Sigma$ are linearly independent over $\mathbb{R}$. 
Fans from polytopes

We may define a fan using a polytope in several ways:

1. Take the fan $R$ over the faces of $\Diamond \subset N$.

2. Refine $R$ by using other lattice points in $\Diamond$ as generators of one-dimensional cones.

3. Take the normal fan $S$ to $\Diamond^\circ \subset M$. 
Toric varieties as quotients

- Let $\Sigma$ be a fan in $\mathbb{R}^n$.
- Let $\{\nu_1, \ldots, \nu_q\}$ be generators for the one-dimensional cones of $\Sigma$.
- $\Sigma$ defines an $n$-dimensional toric variety $V_\Sigma$.
- $V_\Sigma$ is the quotient of a subset $\mathbb{C}^q - Z(\Sigma)$ of $\mathbb{C}^q$ by a subgroup of $(\mathbb{C}^*)^q$.
- Each one-dimensional cone corresponds to a coordinate $z_i$ on $V_\Sigma$. 
Construction details: $Z(\Sigma)$

- Let $\mathcal{S}$ denote any subset of $\Sigma(1)$ that does not span a cone of $\Sigma$.
- Let $\mathcal{V}(\mathcal{S}) \subseteq \mathbb{C}^q$ be the linear subspace defined by setting $z_j = 0$ if the corresponding cone is in $\mathcal{S}$.
- $Z(\Sigma) = \bigcup_{\mathcal{S}} \mathcal{V}(\mathcal{S})$. 
Construction details: \( \ker(\phi) \)

- \((\mathbb{C}^\ast)^q\) acts on \(\mathbb{C}^q - Z(\Sigma)\) by coordinatewise multiplication.
- Write \(v_j = (v_{j1}, \ldots, v_{jn})\)
- Let \(\phi : (\mathbb{C}^\ast)^q \rightarrow (\mathbb{C}^\ast)^n\) be given by

\[
\phi(t_1, \ldots, t_q) \mapsto \left( \prod_{j=1}^{q} t_j^{v_{j1}}, \ldots, \prod_{j=1}^{q} t_j^{v_{jn}} \right)
\]

The toric variety \(V_\Sigma\) associated with the fan \(\Sigma\) is given by

\[
V_\Sigma = (\mathbb{C}^q - Z(\Sigma))/\text{Ker}(\phi).
\]
A Small Example

Let $R$ be the fan obtained by taking cones over the faces of $\diamond$. $Z(\Sigma)$ consists of points of the form $(0,0)$.

\[ V_R = (\mathbb{C}^2 - Z(\Sigma))/\sim \]

\[(z_1, z_2) \sim (\lambda z_1, \lambda z_2)\]

where $\lambda \in \mathbb{C}^*$. Thus, $V_R = \mathbb{P}^1$. 

Figure: 1D Polytope $\diamond$
Another Example

Let $R$ be the fan obtained by taking cones over the faces of $\diamondsuit$. $Z(\Sigma)$ consists of points of the form $(0, 0, z_3, z_4)$ or $(z_1, z_2, 0, 0)$.

Figure: Polygon $\diamondsuit$

$$V_R = (\mathbb{C}^4 - Z(\Sigma))/\sim$$

$$(z_1, z_2, z_3, z_4) \sim (\lambda_1 z_1, \lambda_1 z_2, z_3, z_4)$$

$$(z_1, z_2, z_3, z_4) \sim (z_1, z_2, \lambda_2 z_3, \lambda_2 z_4)$$

where $\lambda_1, \lambda_2 \in \mathbb{C}^*$. Thus, $V_R = \mathbb{P}^1 \times \mathbb{P}^1$. 
Anticanonical Hypersurfaces

For each lattice point $m$ in $\diamondsuit^\circ$, choose a parameter $\alpha_m$. Use this information to define a polynomial:

$$p_\alpha = \sum_{m \in M \cap \diamondsuit^\circ} \alpha_m \prod_{j=1}^{q} z_j^{\langle v_j, m \rangle + 1}$$
Calabi-Yau Varieties

- If we use the fan $R$ over the faces of $\diamond$ (or, equivalently, the normal fan to $\diamond^\circ$), $p_\alpha$ defines a Calabi-Yau variety.
- If we take a maximal simplicial refinement of $R$ (using all the lattice points of $\diamond$), and $k \leq 4$, then $p$ defines a smooth Calabi-Yau manifold $V_\alpha$.
- Reversing the roles of $\diamond$ and $\diamond^\circ$ yields paired families of hypersurfaces.
- In particular, we can use pairs of 4-dimensional reflexive polytopes to define paired families of Calabi-Yau threefolds.
Toric Divisors

Each nonzero lattice point $v_j$ in $\diamond$ defines a toric divisor, $z_j = 0$. We can intersect these divisors with $V_\alpha$ to yield elements of $H^{1,1}(V_\alpha)$.

- Not all of the toric divisors are independent.
- For general $\alpha$, a divisor corresponding to the interior lattice point of a facet will not intersect $V_\alpha$.
- The intersection of a toric divisor with $V_\alpha$ may “split” into several components.
Counting Kähler Moduli

For $k \geq 4$,

$$h^{1,1}(V_\alpha) = \ell(\diamond) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta})$$

- $\ell()$ = number of lattice points
- $\ell^*(())$ = number of lattice points in the relative interior of a polytope or face
- The $\Gamma$ are codimension 1 faces of $\diamond$
- The $\Theta$ are codimension 2 faces of $\diamond$
- $\hat{\Theta}$ is the face of $\diamond$ dual to $\Theta$
Counting Complex Moduli

We know each lattice point in $\diamondsuit^\circ$ corresponds to a monomial in $p_\alpha$. For $k \geq 4$,

$$h^{d-1,1}(V_\alpha) = \ell(\diamondsuit^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ) \ell^*(\hat{\Theta}^\circ)$$

- $\ell() = \text{number of lattice points}$
- $\ell^*(()) = \text{number of lattice points in the relative interior of a polytope or face}$
- The $\Gamma^\circ$ are codimension 1 faces of $\diamondsuit^\circ$
- The $\Theta^\circ$ are codimension 2 faces of $\diamondsuit^\circ$
- $\hat{\Theta}^\circ$ is the face of $\diamondsuit$ dual to $\Theta^\circ$
Comparing $V$ and $V^\circ$

For $k \geq 4$,

\[ h_{1,1}^1(V_\alpha) = \ell(\Diamond) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta}) \]

\[ h_{d-1,1}^{d-1}(V_\alpha) = \ell(\Diamond^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ) \]
Comparing $V$ and $V^\circ$

For $k \geq 4$,

\[ h^{1,1}(V_\alpha) = \ell(\diamond) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta}) \]

\[ h^{d-1,1}(V_\alpha) = \ell(\diamond^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ) \]

\[ h^{1,1}(V_\alpha^\circ) = \ell(\diamond^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ) \]

\[ h^{d-1,1}(V_\alpha^\circ) = \ell(\diamond) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta}) \]
Mirror Symmetry from Mirror Polytopes

We have mirror families of Calabi-Yau varieties $V_\alpha$ and $V_\beta$ of dimension $d = k - 1$.

\[
h^{1,1}(V_\alpha) = h^{d-1,1}(V_\alpha)
\]

\[
h^{d-1,1}(V_\alpha) = h^{1,1}(V_\alpha)
\]
An Example

Four-dimensional analogue:

- ▶ ◊ has vertices \((1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1),\) and \((-1,-1,-1,-1)\).
- ▶ ◇ has vertices \((-1,-1,-1,-1), (4,-1,-1,-1), (-1,4,-1,-1), (-1,-1,4,-1),\) and \((-1,-1,-1,4)\).
An Example

Four-dimensional analogue:

- $\diamond$ has vertices $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, and $(-1, -1, -1, -1)$.
- $\diamond^\circ$ has vertices $(-1, -1, -1, -1)$, $(4, -1, -1, -1)$, $(-1, 4, -1, -1)$, $(-1, -1, 4, -1)$, and $(-1, -1, -1, 4)$.

$$h^{1,1}(V_{\alpha}) = \ell(\diamond) - n - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta) \ell^*(\hat{\Theta})$$

$$= 6 - 4 - 1 - 0 - 0 = 1.$$
Example (Continued)

- ◊ has vertices $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, and $(-1, -1, -1, -1)$.
- ◊° has vertices $(-1, -1, -1, -1)$, $(4, -1, -1, -1)$, $(-1, 4, -1, -1)$, $(-1, -1, 4, -1)$, and $(-1, -1, -1, 4)$.

\[
h^{1,1}(V_\alpha) = 1
\]

\[
h^{3-1,1}(V_\alpha) = \ell(◊°) - n - 1 - \sum_{\Gamma°} \ell^*(\Gamma°) + \sum_{\Theta°} \ell^*(\Theta°)\ell^*(\widehat{\Theta°})
\]

\[
= 126 - 4 - 1 - 20 - 0 = 101.
\]
The Hodge Diamond

Calabi-Yau Threefolds

\[
V_\alpha
\begin{array}{cccc}
1 \\
0 & 0 \\
0 & 1 & 0 \\
1 & 101 & 101 & 1 \\
0 & 1 & 0 \\
0 & 0 \\
1 \\
\end{array}
\]

\[
V_\alpha^0
\begin{array}{cccc}
1 \\
0 & 0 \\
0 & 101 & 0 \\
1 & 1 & 1 & 1 \\
0 & 101 & 0 \\
0 & 0 \\
1 \\
\end{array}
\]
Extrapolations

By looking more carefully at the structure of a reflexive polytope, one can study . . .

- Fibrations of Calabi-Yau varieties
- Degenerations of Calabi-Yau varieties
- Calabi-Yau complete intersections
Dolgachev’s K3 Mirror Prescription

- Let $X$ be a K3 surface.

$$H^2(X, \mathbb{Z}) \cong U \oplus U \oplus U \oplus E_8 \oplus E_8$$

- If $X_\alpha$ is a family of K3 surfaces polarized by a lattice $L$, then the mirror family $X_\alpha^\circ$ should be polarized by a lattice $\hat{L}$ such that

$$L^\perp = \hat{L} \oplus nU$$

- In particular, $\text{rank}(L) + \text{rank}(\hat{L}) = 20$. 
Following Falk Rohsiepe, we observe . . .

- We can intersect toric divisors with \( X_\alpha \) to create a sublattice of \( \text{Pic}(X_\alpha) \)
- We can compute the lattice pairings using purely combinatorial information about lattice points
Examining the Data

Set

\[ \rho(\diamond) = \ell(\diamond^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ) \ell^*(\hat{\Theta}^\circ). \]

<table>
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<th>\diamond^\circ</th>
<th>\rho(\diamond)</th>
<th>\rho(\diamond^\circ)</th>
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<td>19</td>
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<tr>
<td>1</td>
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<td>4286</td>
<td>2</td>
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<td>2</td>
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<tr>
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A Toric Correction Term

Set

$$\delta(\diamond) = \sum_{\Theta^\circ} \ell^* (\Theta^\circ) \ell^* (\hat{\Theta}^\circ).$$

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Let $\Diamond$ and $\Diamond^\circ$ be a mirror pair of 3-dimensional reflexive polytopes, and let $X_\alpha$ and $X_\alpha^\circ$ be the corresponding families of K3 surfaces.

Write $i : X_\alpha \to W$ be the inclusion in the ambient toric variety, and let $D_j$ be the toric divisors.

Let $L$ be the sublattice of $\text{Pic}(X_\alpha)$ generated by $i^*(D_j)$

Let $\hat{L}$ be the sublattice of $\text{Pic}(X_\alpha^\circ)$ generated by all of the components of the intersections $D_j \cap X_\alpha^\circ$

$$L^\perp = \hat{L} \oplus U$$
Some Picard rank 19 families

- Hosono, Lian, Oguiso, Yau:
  \[ x + 1/x + y + 1/y + z + 1/z - \Psi = 0 \]

- Verrill:
  \[
  (1 + x + xy + xyz)(1 + z + zy + zyx) = (\lambda + 4)(xyz)
  \]

- Narumiya-Shiga:
  \[
  Y_0 + Y_1 + Y_2 + Y_3 - 4tY_4
  \]
  \[
  Y_0 Y_1 Y_2 Y_3 - Y_4^4
  \]
Toric realizations of the rank 19 families

The polar polytopes $\Diamond$ for [HLOY04], [V96], and [NS01].

$$f(t) = \left( \sum_{x \in \text{vertices}(\Diamond)} \prod_{k=1}^{q} z_{k}^{\langle v_k, x \rangle + 1} \right) + t \prod_{k=1}^{q} z_{k}. $$
What do these polytopes have in common?

- The only lattice points of these polytopes are the vertices and the origin.
- The group of orientation-preserving symmetries of the polytope acts transitively on the vertices.
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- The only lattice points of these polytopes are the vertices and the origin.
What do these polytopes have in common?

- The only lattice points of these polytopes are the vertices and the origin.
- The group $G$ of orientation-preserving symmetries of the polytope acts transitively on the vertices.
Another symmetric polytope

Figure: The skew cube

\[ f(t) = \left( \sum_{x \in \text{vertices}(\diamondsuit)} \prod_{k=1}^{q} z_k^{\langle v_k, x \rangle + 1} \right) + t \prod_{k=1}^{q} z_k. \]
We may view a rotation as acting either on $\vartriangle$ (inducing automorphisms on $X_t$) or on $\diamondsuit$ (permuting the monomials of $f(t)$).
Symplectic Group Actions

Let $G$ be a finite group of automorphisms of a K3 surface. For $g \in G$, 

$$g^*(\omega) = \rho \omega$$

where $\rho$ is a root of unity.

Definition
We say $G$ acts \textit{symplectically} if

$$g^*(\omega) = \omega$$

for all $g \in G$. 
A subgroup of the Picard group

Definition

\[ S_G = \left( (H^2(X, \mathbb{Z})^G)^\perp \right) \]

Theorem ([N80a])

\( S_G \) is a primitive, negative definite sublattice of \( \text{Pic}(X) \).
The rank of $S_G$

Lemma

- If $X$ admits a symplectic action by the permutation group $G = S_4$, then $\text{Pic}(X)$ admits a primitive sublattice $S_G$ which has rank 17.

- If $X$ admits a symplectic action by the alternating group $G = A_4$, then $\text{Pic}(X)$ admits a primitive sublattice $S_G$ which has rank 16.
Why is the Picard rank 19?

We can use the orbits of $G$ on $\diamond$ to identify divisors in $(H^2(X_t, \mathbb{Z}))^G$. 

Figure: $\diamond$
Why is the Picard rank 19?

We can use the orbits of $G$ on $\diamond$ to identify divisors in $(H^2(X_t, \mathbb{Z}))^G$.

- For the families of [HLOY04] and [V96], and the family defined by the skew cube, we conclude that $17 + 2 = 19$.
- For the family of [NS01], we conclude that $16 + 3 = 19$. 
Collaborators

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- Ursula Whitcher (UWEC)
K3 surfaces from elliptic curves

Let $E_1$ and $E_2$ be elliptic curves, and let $A = E_1 \times E_2$.

- The **Kummer surface** $Km(A)$ is the minimal resolution of $A/\{\pm 1\}$.
- The **Shioda-Inose surface** $SI(A)$ is the minimal resolution of $Km(A)/\beta$, where $\beta$ is an appropriately chosen involution.
A period is the integral of a differential form with respect to a specified homology class.

Periods of holomorphic forms encode the complex structure of varieties.

The Picard-Fuchs differential equation of a family of varieties is a differential equation that describes the way the value of a period changes as we move through the family.

Solutions to Picard-Fuchs equations for holomorphic forms on Calabi-Yau varieties define the mirror map.
Picard-Fuchs equations for rank 19 families

Let $M$ be a free abelian group of rank 19, and suppose $M \hookrightarrow \text{Pic}(X_t)$.

- The Picard-Fuchs equation is a rank 3 ordinary differential equation.
- The coefficients of the Picard-Fuchs equation are rational functions.
- The equation is Fuchsian (the singularities of the rational functions are controlled).
Symmetric Squares

Let $L(y)$ be a homogeneous linear differential equation with coefficients in $\mathbb{C}(t)$.

There exists a homogeneous linear differential equation $M(y) = 0$ with coefficients in $\mathbb{C}(t)$, such that . . .

The solution space of $M(y)$ is the $\mathbb{C}$-span of

$$\{\nu_1\nu_2 \mid L(\nu_1) = 0 \text{ and } L(\nu_2) = 0\}.$$ 

**Definition**

$M(y)$ is the symmetric square of $L$. 

Symmetric Square Formula

The symmetric square of the differential equation

\[ a_2 \frac{\partial^2 A}{\partial t^2} + a_1 \frac{\partial A}{\partial t} + a_0 A = 0 \]

is

\[ a_2 \frac{\partial^3 A}{\partial t^3} + 3a_1 a_2 \frac{\partial^2 A}{\partial t^2} + (4a_0 a_2 + 2a_1^2 + a_2 a'_1 - a_1 a'_2) \frac{\partial A}{\partial t} + (4a_0 a_1 + 2a'_0 a_2 - 2a_0 a'_2) A = 0 \]

where primes denote derivatives with respect to \( t \).
Picard-Fuchs equations and symmetric squares

Theorem

[D00, Theorem 5] The Picard-Fuchs equation of a family of rank-19 lattice-polarized K3 surfaces can be written as the symmetric square of a second-order homogeneous linear Fuchsian differential equation.
Quasismooth and regular hypersurfaces

Let \( \Sigma \) be a simplicial fan, and let \( X \) be a hypersurface in \( V_\Sigma \). Suppose that \( X \) is described by a polynomial \( f \) in homogeneous coordinates.

**Definition**
If the derivatives \( \frac{\partial f}{\partial z_i}, i = 1 \ldots q \) do not vanish simultaneously on \( X \), we say \( X \) is quasismooth.
Quasismooth and regular hypersurfaces

Let $\Sigma$ be a simplicial fan, and let $X$ be a hypersurface in $V_{\Sigma}$. Suppose that $X$ is described by a polynomial $f$ in homogeneous coordinates.

**Definition**
If the derivatives $\frac{\partial f}{\partial z_i}$, $i = 1 \ldots q$ do not vanish simultaneously on $X$, we say $X$ is **quasismooth**.

**Definition**
If the products $z_i \frac{\partial f}{\partial z_i}$, $i = 1 \ldots q$ do not vanish simultaneously on $X$, we say $X$ is **regular** and $f$ is **nondegenerate**.
The Skew Octahedron

- Let ♦ be the reflexive octahedron shown above.
- ♦ contains 19 lattice points.
- Let $R$ be the fan obtained by taking cones over the faces of ♦. Then $R$ defines a toric variety $V_R \cong (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$.
- Consider the family of K3 surfaces $X_t$ defined by $f(t) = \left( \sum_{x \in \text{vertices}(♦)} \prod_{k=1}^{q} z_k^{\langle v_k, x \rangle + 1} \right) + t \prod_{k=1}^{q} z_k$.
- $X_t$ are generally quasismooth but not regular.
The Picard-Fuchs equation

**Theorem ([KLMSW10])**

Let \( A = \int \text{Res} \left( \frac{\Omega_0}{f} \right) \). Then \( A \) is the period of a holomorphic form on \( X_t \), and \( A \) satisfies the Picard-Fuchs equation

\[
\frac{\partial^3 A}{\partial t^3} + \frac{6(t^2 - 32)}{t(t^2 - 64)} \frac{\partial^2 A}{\partial t^2} + \frac{7t^2 - 64}{t^2(t^2 - 64)} \frac{\partial A}{\partial t} + \frac{1}{t(t^2 - 64)} A = 0.
\]

As expected, the differential equation is third-order and Fuchsian.
Symmetric square root

The symmetric square root of our Picard-Fuchs equation is:

\[
\frac{\partial^2 A}{\partial t^2} + \frac{(2t^2 - 64)}{t(t^2 - 64)} \frac{\partial A}{\partial t} + \frac{1}{4(t^2 - 64)} A = 0.
\]
Semiample hypersurfaces

- Let $R$ be a fan over the faces of a reflexive polytope
- Let $\Sigma$ be a refinement of $R$
- We have a proper birational morphism $\pi : V_\Sigma \rightarrow V_R$
- Let $Y$ be an ample divisor in $V_R$, and suppose $X = \pi^*(Y)$

Then $X$ is **semiample**:

**Definition**
We say that a Cartier divisor $D$ is **semiample** if $D$ is generated by global sections and the intersection number $D^n > 0$. 
The residue map

We will use a residue map to describe the cohomology of a K3 hypersurface $X$:

$$\text{Res} : H^3(V_\Sigma - X) \rightarrow H^2(X).$$

Anvar Mavlyutov showed that $\text{Res}$ is well-defined for quasismooth, semiample hypersurfaces in simplicial toric varieties.
Two ideals

**Definition**
The Jacobian ideal $J(f)$ is the ideal of $\mathbb{C}[z_1, \ldots, z_q]$ generated by the partial derivatives $\frac{\partial f}{\partial z_i}, i = 1 \ldots q$.

**Definition**
[BC94] The ideal $J_1(f)$ is the ideal quotient
\[
\langle z_1 \frac{\partial f}{\partial z_1}, \ldots, z_q \frac{\partial f}{\partial z_q} \rangle : z_1 \cdots z_q.
\]
The induced residue map

Let $\Omega_0$ be a holomorphic 3-form on $V_{\Sigma}$. We may represent elements of $H^3(V_{\Sigma} - X)$ by forms $\frac{P\Omega_0}{f^k}$, where $P$ is a polynomial in $C[z_1, \ldots, z_q]$.

Mavlyutov described two induced residue maps on semiample hypersurfaces:

- $\text{Res}_J : C[z_1, \ldots, z_q]/J \to H^2(X)$ is well-defined for quasismooth hypersurfaces
- $\text{Res}_{J_1} : C[z_1, \ldots, z_q]/J_1 \to H^2(X)$ is well-defined for regular hypersurfaces.
Whither injectivity?

Res$_J$ is injective for smooth hypersurfaces in $\mathbb{P}^3$, but this does not hold in general.

**Theorem**

[M00] If $X$ is a regular, semiample hypersurface, then the residue map $\text{Res}_{J_1}$ is injective.
The Griffiths-Dwork technique

Plan

We want to compute the Picard-Fuchs equation for a one-parameter family of K3 hypersurfaces $X_t$.

- Look for $\mathbb{C}(t)$-linear relationships between derivatives of periods of the holomorphic form
- Use $\text{Res}_J$ to convert to a polynomial algebra problem in $\mathbb{C}(t)[z_1, \ldots, z_q]/J(f)$
The Griffiths-Dwork technique

Procedure

1.

\[
\frac{d}{dt} \int \text{Res} \left( \frac{P\Omega}{f^k(t)} \right) = \int \text{Res} \left( \frac{d}{dt} \left( \frac{P\Omega}{f^k(t)} \right) \right) = -k \int \text{Res} \left( \frac{f'(t)P\Omega}{f^{k+1}(t)} \right)
\]
The Griffiths-Dwork technique

Procedure

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\[
\frac{d}{dt} \int \text{Res} \left( \frac{P\Omega}{f^k(t)} \right) = \int \text{Res} \left( \frac{d}{dt} \left( \frac{P\Omega}{f^k(t)} \right) \right) = -k \int \text{Res} \left( \frac{f'(t)P\Omega}{f^{k+1}(t)} \right)
\]

2. Since \( H^*(X_t, \mathbb{C}) \) is a finite-dimensional vector space, only finitely many of the classes \( \text{Res} \left( \frac{d^j}{dt^j} \left( \frac{\Omega}{f^k(t)} \right) \right) \) can be linearly independent
The Griffiths-Dwork technique

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3. Use the **reduction of pole order** formula to compare classes of the form \( \text{Res} \left( \frac{P\Omega}{f^{k+1}(t)} \right) \) to classes of the form \( \text{Res} \left( \frac{Q\Omega}{f^k(t)} \right) \).
The Griffiths-Dwork technique

Implementation

Reduction of pole order

\[
\frac{\Omega_0}{f^{k+1}} \sum_i P_i \frac{\partial f}{\partial x_i} = \frac{1}{k} \frac{\Omega_0}{f^k} \sum_i \frac{\partial P_i}{\partial x_i} + \text{exact terms}
\]

We use Groebner basis techniques to rewrite polynomials in terms of \( J(f) \).
The Griffiths-Dwork technique
Advantages and disadvantages

**Advantages**
We can work with arbitrary polynomial parametrizations of hypersurfaces.

**Disadvantages**
We need powerful computer algebra systems to work with $J(f)$ and $\mathbb{C}(t)[z_1, \ldots, z_q]/J(f)$. 
Modular Groups and Modular Curves

- Consider a modular group $\Gamma \subset PSL_2(\mathbb{R})$.
- $\Gamma$ acts on the upper half-plane $\mathbb{H}$ by linear fractional transformations:
  \[ z \mapsto \frac{az + b}{cz + d} \]
- $\mathbb{H}/\Gamma$ is a Riemann surface called a modular curve.
- The function field of a genus 0 modular curve is generated by a transcendental function called a hauptmodul.
Some modular groups

Congruence subgroups

\[ \Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z}) \ \middle| \ c \cong 0 \ (\text{mod} \ n) \right\} \]

Atkin-Lehner map

\[ w_h = \begin{pmatrix} 0 & \frac{-1}{\sqrt{h}} \\ \sqrt{h} & 0 \end{pmatrix} \in PSL_2(\mathbb{R}) \]

\( \Gamma_0(n) + h \) is generated by \( \Gamma_0(n) \) and \( w_h \).
Mirror Moonshine

Mirror Moonshine for a one-parameter family of K3 surfaces arises when there exists a genus 0 modular group $\Gamma$ such that . . .

- The Picard-Fuchs equation gives the base of the family the structure of a modular curve $\overline{\mathbb{H}}/\Gamma$, or a finite cover of the modular curve.
- The hauptmodul for $\Gamma$ can be expressed as a rational function of the mirror map.
- The holomorphic solution to the Picard-Fuchs equation is a $\Gamma$-modular form of weight 2.
Mirror Moonshine from geometry

<table>
<thead>
<tr>
<th>Example</th>
<th>[HLOY04]</th>
<th>[V96]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shioda-Inose structure</td>
<td>SI($E_1 \times E_2$)</td>
<td>SI($E_1 \times E_2$)</td>
</tr>
<tr>
<td></td>
<td>$E_1, E_2$ are 6-isogenous</td>
<td>$E_1, E_2$ are 3-isogenous</td>
</tr>
<tr>
<td>Pic($X)^\perp$</td>
<td>$H \oplus \langle 12 \rangle$</td>
<td>$H \oplus \langle 6 \rangle$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$\Gamma_0(6) + 6$</td>
<td>$\Gamma_0(6) + 3 \subset \Gamma_0(3) + 3$</td>
</tr>
</tbody>
</table>
Geometry of the skew octahedron family

- $X_t$ is a family of Kummer surfaces
- Each surface can be realized as $Km(E_t \times E_t)$
- The generic transcendental lattice is $2H \oplus \langle 4 \rangle$
The modular group

We use our symmetric square root and the table of [LW06] to show that:

$$\Gamma = \Gamma_0(4|2) = \left\{ \begin{pmatrix} a & b/2 \\ 4c & d \end{pmatrix} \in PSL_2(\mathbb{R}) \mid a, b, c, d \in \mathbb{Z} \right\}$$

$\Gamma_0(4|2)$ is conjugate in $PSL_2(\mathbb{R})$ to $\Gamma_0(2) \subset PSL_2(\mathbb{Z}) = \Gamma_0(1) + 1$. 

Doran, C. Picard-Fuchs uniformization and modularity of the mirror map. *Communications in Mathematical Physics* 212 (2000), no. 3, 625–647.


http://people.brandeis.edu/~lian/Schiff.pdf


SAGE Mathematics Software, Version 3.4,  
http://www.sagemath.org/