Rankin-Selberg methods for String Amplitudes

Boris Pioline
CERN & LPTHE

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based on work with C. Angelantonj and I. Florakis,
arXiv:1110.5318,1203.0566,1304.4271, and work in progress
In closed string theory, an interesting class of amplitudes are given by a modular integral

\[ \mathcal{A} = \int_{\mathcal{F}} d\mu \Gamma_{d+k,d} \Phi(\tau) , \quad d\mu = \frac{d\tau_1 d\tau_2}{\tau_2^2} \]

- \( \mathcal{F} = \Gamma \backslash \mathcal{H} \) : fundamental domain of the modular group \( \Gamma = SL(2, \mathbb{Z}) \) on the Poincaré upper half plane \( \mathcal{H} \);
- \( \Gamma_{(d+k,d)} = \tau_2^{d/2} \sum q_2^{1/2} p_L^2 \bar{q}_2^{1/2} p_R^2 \) : a Siegel-Narain series for an even self-dual lattice of signature \( (d + k, d) \);
- \( \Phi(\tau) \) : an (almost, weakly) holomorphic modular form of weight \( w = -k/2 \), which I will call the elliptic genus.
Such modular integrals arise in one-loop computations of certain BPS-saturated amplitudes, such as $F^2, R^2, F^4, R^4$, after integrating over the location of the vertex operators.

More general one-loop amplitudes are given by similar integrals, but $\Phi(\tau)$ is no longer (almost) holomorphic, hence much harder to compute.

$A$ provides a function on the moduli space of lattices,

$$G_{d+k,d} = \frac{O(d+k,d)}{O(d+k) \times O(d)} \ni (g_{ij}, B_{ij}, Y^a_i),$$

which is invariant T-duality, i.e. under the automorphism group $O(\Gamma_{d+k,d})$: an example of Theta correspondence.
In the physics literature, the time-honored way to evaluate such integrals has been the unfolding trick or orbit method:

\[ \int_{\Gamma \setminus \mathcal{H}} \sum_{\gamma \in \Gamma} f|_0 \gamma = \int_{\Gamma \setminus \mathcal{H}} f \]

\[ f|_w \gamma(\tau) = (c\tau + d)^{-w} f \left( \frac{a\tau + b}{c\tau + d} \right) \]

E.g for \( d = 1 \), representing \( \Gamma_{1,1} = R \sum_{m,n} e^{-\pi R^2 |m-n\tau|^2 / \tau^2} \),

\[ \int_{\mathcal{F}} \Gamma_{1,1} = R \int_{\mathcal{F}} d\mu + R \int_{S} d\mu \sum_{m \neq 0} e^{-\pi R^2 m^2 / \tau^2} \]

\[ = \frac{\pi}{3} R + \frac{\pi}{3} R^{-1} \]
Unfolding trick, revisited

For higher dimensional lattices, the theta series \( \Gamma_{d+k,d} \) involves several different orbits of \( SL(2,\mathbb{Z}) \). The orbit decomposition breaks manifest invariance under the automorphism group \( O(\Gamma_{d+k,d}) \).

I will present an alternative method for computing such modular integrals, which keeps T-duality manifest at all stages. The method is inspired by the Rankin-Selberg method commonly used in number theory.

The result is typically expressed as a field theory amplitude with an infinite number of BPS states running through the loop.

The method is in principle applicable to higher genus amplitudes, though for the most part I will focus on genus one.
Consider the completed non-holomorphic Eisenstein series

$$E^*(\tau; s) = \zeta^*(2s) \sum_{\gamma \in \Gamma \setminus \Gamma} \tau_2^s |\gamma = \frac{1}{2} \zeta^*(2s) \sum_{(c,d)\equiv 1} \frac{\tau_2^s}{|c\tau + d|^{2s}}$$

where $\zeta^*(s) \equiv \pi^{-s/2} \Gamma(s/2) \zeta(s) = \zeta^*(1 - s)$.

$E^*(\tau; s)$ is convergent for $\Re(s) > 1$, and has a meromorphic continuation to all $s$, invariant under $s \mapsto 1 - s$, with simple poles at $s = 0, 1$ with constant residue:

$$E^*(\tau; s) = \frac{1}{2(s - 1)} + \frac{1}{2} \left( \gamma - \log(4\pi \tau_2 |\eta(\tau)|^4) \right) + O(s - 1),$$
For any cusp form $F(\tau)$, consider the Rankin-Selberg transform

$$\mathcal{R}^*(F, s) = \int_{\mathcal{F}} d\mu \ E^*(\tau; s) \ F(\tau)$$

By the unfolding trick, $\mathcal{R}^*(F, s)$ is proportional to the Mellin transform of the constant term $F_0(\tau_2) = \int_{-1/2}^{1/2} d\tau_1 \ F(\tau)$,

$$\mathcal{R}^*(F; s) = \zeta^*(2s) \int_{S} d\mu \ \tau_2^s \ F(\tau)$$

$$= \zeta^*(2s) \int_{0}^{\infty} d\tau_2 \ \tau_2^{s-2} \ F_0(\tau_2) ,$$
The RS transform is in fact proportional to the L-function
\[ L(s) = \sum_n a_n n^{-s} \] associated to \( F \).

It inherits the meromorphy and functional relations of \( E^* \), e.g.
\[ \mathcal{R}^*(F; s) = \mathcal{R}^*(F; 1 - s). \]

Since the residue of \( E^*(\tau; s) \) at \( s = 0, 1 \) is constant, the residue of \( \mathcal{R}^*(F; s) \) at \( s = 1 \) is proportional to the modular integral of \( F \),
\[
\text{Res}_{s=1} \mathcal{R}^*(F; s) = \frac{1}{2} \int_{\mathcal{F}} d\mu F
\]
This was extended by Zagier to the case where $F^{(0)}$ is of power-like growth $F^{(0)}(\tau) \sim \varphi(\tau_2)$ at the cusp: the renormalized integral

$$\text{R.N. } \int_{\mathcal{F}} d\mu F(\tau) = \lim_{T \to \infty} \left[ \int_{\mathcal{F}_T} d\mu F(\tau) - \hat{\varphi}(T) \right]$$

$$\varphi(\tau_2) = \sum_{\alpha} c_\alpha \tau_2^\alpha, \quad \hat{\varphi}(T) = \sum_{\alpha \neq 1} c_\alpha \frac{\tau_2^{\alpha-1}}{\alpha - 1} + \sum_{\alpha = 1} c_\alpha \log \tau_2$$

is related to the Mellin transform of the (regularized) constant term

$$\mathcal{R}^*(F; s) = \zeta^*(2s) \int_0^\infty d\tau_2 \tau_2^{s-2} \left( F^{(0)} - \varphi \right),$$

via

$$\text{R.N. } \int_{\mathcal{F}} d\mu F(\tau) = 2 \text{Res}_{s=1} \mathcal{R}^*(F; s) + \delta$$
$\delta$ is a scheme-dependent correction which depends only on the leading behavior $\varphi(\tau_2)$,

$$\delta = 2 \text{Res}_{s=1} \left[ \zeta^*(2s) h_\mathcal{T}(s) + \zeta^*(2s-1) h_\mathcal{T}(1-s) \right] - \hat{\varphi}(\mathcal{T}) ,$$

where $h_\mathcal{T}(s) = \int_0^\mathcal{T} d\tau_2 \varphi(\tau_2) \tau_2^{s-2}$.

The Rankin-Selberg transform $\mathcal{R}^*(F; s)$ is itself equal to the renormalized integral

$$\mathcal{R}^*(F; s) = \text{R.N.} \int_{\mathcal{F}} d\mu F(\tau) \mathcal{E}^*(s; \tau)$$

According to this prescription, $\text{R.N.} \int_{\mathcal{F}} d\mu \mathcal{E}^*(\tau; s) = 0$ !
The RSZ method applies immediately to integrals with $\Phi = 1$:

$$
\mathcal{R}^*(\Gamma_d, d; s) = \zeta^*(2s) \int_0^\infty d\tau_2 \tau_2^{s+d/2-2} \sum_{p^2_l-p^2_R=0} e^{-\pi \tau_2 (p^2_L+p^2_R)}
$$

$$
= \zeta^*(2s) \frac{\Gamma(s + \frac{d}{2} - 1)}{\pi^{s+\frac{d}{2}-1}} \mathcal{E}^d_V(g, B; s + \frac{d}{2} - 1)
$$

where $\mathcal{E}^d_V(g, B; s)$ is the constrained Epstein series

$$
\mathcal{E}^d_V(g, B; s) \equiv \sum_{(m_i, n^i) \in \mathbb{Z}^{2d} \setminus (0,0)} \mathcal{M}^{-2s}, \quad \mathcal{M}^2 = p^2_L + p^2_R
$$
This is identified as a sum over all BPS states of momentum $m_i$ and winding $n^i$, with mass

$$M^2 = (m_i + B_{ik} n^k) g^{ij} (m_j + B_{jl} n^l) + n^i g_{ij} n^j$$

subject to the BPS condition $m_i n^i = 0$. Invariance under $O(\Gamma_{d,d})$ is manifest.

The constrained Epstein Zeta series $\mathcal{E}_V^d(g, B; s)$ converges absolutely for $\Re(s) > d$. The RSZ method shows that it admits a meromorphic continuation in the $s$-plane satisfying

$$\mathcal{E}_V^d(s) = \pi^{-s} \Gamma(s) \zeta^*(2s - d + 2) \mathcal{E}_V^d(s) = \mathcal{E}_V^{d*}(d - 1 - s),$$

with a simple pole at $s = 0, \frac{d}{2} - 1, \frac{d}{2}, d - 1$ (double poles if $d = 2$).
The residue at \( s = \frac{d}{2} \) produces the modular integral of interest:

\[
\text{R.N. } \int_{\mathcal{F}} d\mu \Gamma_{d,d}(g, B) = \frac{\Gamma\left(\frac{d}{2} - 1\right)}{\pi^{\frac{d}{2} - 1}} \mathcal{E}^d_V (g, B; \frac{d}{2} - 1)
\]

rigorously proving an old conjecture of Obers and myself (1999).

For \( d = 2 \), the BPS constraint \( m_i n^i = 0 \) can be solved, leading to

\[
\mathcal{E}^2_V^* (T, U; s) = 2 E^*(T; s) E^*(U; s)
\]

hence to Dixon-Kaplunovsky-Louis famous result (1989)

\[
\int_{\mathcal{F}} \left( \Gamma_{2,2}(T, U) - \tau_2 \right) d\mu = -\log \left( T_2 U_2 |\eta(T) \eta(U)|^4 \right) + \text{cte}
\]
The differential equations

\[ 0 = \left[ \Delta_{SO(d,d)} - 2 \Delta_{SL(2)} + \frac{1}{4} d(d - 2) \right] \Gamma_{d,d}(g, B) \]
\[ 0 = \left[ \Delta_{SL(2)} - \frac{1}{2} s(s - 1) \right] E^*(\tau; s), \]

imply that \( E^d_\nu^*(s) \) is an eigenmode of the Laplace-Beltrami operator on the Grassmannian \( G_{d,d} \) with eigenvalue \( s(s - d + 1) \), and more generally, of all \( O(d, d) \) invariant differential operators.

\( E^d_\nu^*(g, B; s) \) is proportional to the Langlands-Eisenstein series of \( O(d, d) \) with infinitesimal character \( \rho - 2s\alpha_1 \).

The residue at \( s = \frac{d}{2} \) is the minimal theta series, attached to the minimal representation of \( SO(d, d) \) (functional dimension \( 2d - 3 \)).

\textit{Ginzburg Rallis Soudry; Kazhdan BP Waldron; Green Vanhove Miller}
For many cases of interest, the integrand is NOT of moderate growth at the cusp, rather it grows exponentially, due to the heterotic unphysical tachyon, \( \Phi(\tau) \sim 1/q^\kappa + O(1) \) with \( \kappa = 1 \).

In mathematical terms, \( \Phi(\tau) \in \mathbb{C}[\hat{E}_2, E_4, E_6, 1/\Delta] \) is an almost, weakly holomorphic modular form with weight \( w = -k/2 \leq 0 \).

The RSZ method fails, however the unfolding trick could still work provided \( \Phi(\tau) \) can be represented as a uniformly convergent Poincaré series with seed \( f(\tau) \) is invariant under \( \Gamma_\infty : \tau \rightarrow \tau + n \),

\[
\Phi(\tau) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} f(\tau)|_W \gamma
\]

Convergence requires \( f(\tau) \ll \tau_2^{1 - \frac{w}{2}} \) as \( \tau_2 \rightarrow 0 \). The choice \( f(\tau) = 1/q^\kappa \) works for \( w > 2 \) but fails for \( w \leq 2 \).
Various Poincaré series representations

- One option is to insert a non-holomorphic convergence factor à la Hecke-Kronecker, i.e. choose a seed $f(\tau) = \tau_2^{s - \frac{w}{2}} q^{-\kappa}$

$$E(s, \kappa, w) \equiv \frac{1}{2} \sum_{(c,d)=1} \frac{(c\tau + d)^{-w}}{|c\tau + d|^{2s-w}} \tau_2^{s - \frac{w}{2}} e^{-2\pi i \kappa \frac{a\tau + b}{c\tau + d}}$$

- This converges absolutely for $\text{Re}(s) > 1$, but analytic continuation to desired value $s = \frac{w}{2}$ is tricky, and in general non-holomorphic.

- Moreover, $E(s, \kappa, w)$ is not an eigenmode of the Laplacian, rather

$$[\Delta_w + \frac{1}{2} s(1 - s) + \frac{1}{8} w(w + 2)] E(s, \kappa, w) = 2\pi \kappa (s - \frac{w}{2}) E(s+1, \kappa, w)$$

Selberg; Goldfeld Sarnak; Pribitkin
We shall use another regularization which does not require analytic continuation: the Niebur-Poincaré series

\[ \mathcal{F}(s, \kappa, w) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathcal{M}_{s,w}(-\kappa \tau_2) e^{-2\pi i \kappa \tau_1} |w \gamma \]

where \( \mathcal{M}_{s,w}(y) \) is proportional to a Whittaker function, so that

\[ \left[ \Delta_w + \frac{1}{2} s(1 - s) + \frac{1}{8} w(w + 2) \right] \mathcal{F}(s, \kappa, w) = 0 \]

The seed \( f(\tau) = \mathcal{M}_{s,w}(-\kappa \tau_2) e^{-2\pi i \kappa \tau_1} \) is uniquely determined by

\[ f(\tau) \sim_{\tau_2 \to 0} \tau_2^{s - \frac{w}{2}} e^{-2\pi i \kappa \tau_1} \quad f(\tau) \sim_{\tau_2 \to \infty} \frac{\Gamma(2s)}{\Gamma(s + \frac{w}{2})} q^{-\kappa} \]

ensuring that \( \mathcal{F}(s, \kappa, w) \) converges absolutely for \( \text{Re}(s) > 1 \).
Niebur-Poincaré series II

- Under raising and lowering operators,

\[ D_w = \frac{i}{\pi} \left( \partial_\tau - \frac{iw}{2\tau_2} \right), \quad \bar{D}_w = -i\pi \tau_2^2 \partial_\tau, \]

the NP series transforms as

\[ D_w \cdot F(s, \kappa, w) = 2\kappa (s + \frac{w}{2}) F(s, \kappa, w + 2), \]
\[ \bar{D}_w \cdot F(s, \kappa, w) = \frac{1}{8\kappa} (s - \frac{w}{2}) F(s, \kappa, w - 2). \]

- Under Hecke operators,

\[ H_{\kappa'} \cdot F(s, \kappa, w) = \sum_{d | (\kappa, \kappa')} d^{1-w} F(s, \kappa \kappa' / d^2, w). \]

- For congruence subgroups of SL(2, \mathbb{Z}), one can similarly define NP series \( F_a(s, \kappa, w) \) for each cusp.
For $s = 1 - \frac{w}{2}$, the value relevant for weakly holomorphic modular forms, the seed simplifies to

$$f(\tau) = \Gamma(2 - w) \left( q^{-\kappa} - \bar{q}^\kappa \sum_{\ell=0}^{-w} \frac{(4\pi \kappa \tau_2)^\ell}{\ell!} \right)$$

For $w < 0$, the value $s = 1 - \frac{w}{2}$ lies in the convergence domain, but $F(1 - \frac{w}{2}, \kappa, w)$ is in general NOT holomorphic, but rather a weakly harmonic Maass form,

$$\Phi = \sum_{m=-\kappa}^{\infty} a_m q^m + \sum_{m=1}^{\infty} m^{w-1} \bar{b}_m \Gamma(1 - w, 4\pi m \tau_2) q^{-m}$$

For any such form, $\bar{D}\Phi = \tau_2^{2-w} \bar{\Psi}$ where $\Psi = \sum_{m \geq 1} b_m q^m$ is a holomorphic cusp form of weight $2 - w$, the shadow of the Mock modular form $\Phi^- = \sum_{m=-\kappa}^{\infty} a_m q^m$. 
If \(|w|\) is small enough, the negative frequency coefficients \(b_m\) vanish and \(\Phi\) is in fact a weakly holomorphic modular form:

\[
\Phi = \mathcal{F}(1 - \frac{w}{2}, 1, w) = j + 24 - \frac{3!}{2^3} E_4 E_6/\Delta - \frac{5!}{2^4} E_4^2/\Delta - \frac{7!}{2^5} E_6/\Delta - \frac{9!}{2^6} E_4/\Delta - \frac{11!}{2^7} \Phi_{-10} - \frac{13!}{2^8} \Phi_{-14}
\]

where \(\Phi_{-10}\) and \(\Phi_{-14}\) are Mock modular forms with shadow 2.8402\ldots \times \Delta and 1.3061\ldots \times E_4 \Delta.
Theorem (Bruinier) : any weakly holomorphic modular form of weight $w \leq 0$ with polar part $\Phi = \sum_{-\kappa \leq m < 0} a_m q^m + \mathcal{O}(1)$ is a linear combination of Niebur-Poincaré series

$$\Phi = \frac{1}{\Gamma(2 - w)} \sum_{-\kappa \leq m < 0} a_m \mathcal{F}(1 - \frac{w}{2}, m, w) + a'_0 \delta_{w,0}$$

(The same holds for congruence subgroups of $SL(2, \mathbb{Z})$, including contributions from all cusps)

Weakly almost holomorphic modular forms of weight $w \leq 0$ can similarly be represented as linear combinations of $\mathcal{F}(1 - \frac{w}{2} + n, m, w)$ with $-\kappa \leq m < 0, 0 \leq n \leq p$ where $p$ is the depth. This fails for positive weight, as such forms are not necessarily harmonic !
Niebur-Poincaré series VI

\[ s = \frac{w}{2} : \text{weakly hol. (ghost)} \]

\[ s = \frac{w}{2} : \text{weakly almost hol.} \]

\[ s = \frac{w}{2} : \text{anti-hol. (shadow)} \]

\[ s = 1 - \frac{w}{2} : \text{weakly harmonic} \]

\[ E_4^2 \Delta \]

\[ E_4 E_6 / \Delta \]

\[ E_4 \]

\[ E_6 \]

\[ j + 24 \]

\[ D \to 2 \]

\[ \bar{D} \to 1 \]

\[ D \to 3 \]

\[ \bar{D} \to s \]

\[ F(s, \kappa, w) \]

Figure: Phase diagram for the Niebur-Poincaré series for integer values of \((w^2, s)\) with \(s \geq 1\). For low negative values of \(w\), \(F(s, \kappa, w)\) reduces to an ordinary weak almost holomorphic Maass form, see Table ??.
Unfolding the modular integral

- By Bruinier’s thm, any modular integral is a linear combination of

\[ I_{d+k,d}(s, \kappa) = R.N. \int_{\mathcal{F}} d\mu \, \Gamma_{d+k,d}(G, B, Y) \mathcal{F}(s, \kappa, -\frac{k}{2}) \]

- Using the unfolding trick, one arrives at the BPS state sum

\[ I_{d+k,d}(s, \kappa) = (4\pi \kappa)^{1-\frac{d}{2}} \Gamma(s + \frac{2d+k}{4} - 1) \times \sum_{\text{BPS}} 2F_1 \left( s - \frac{k}{4}, s + \frac{2d+k}{4} - 1; 2s; \frac{4\kappa}{p_L^2} \right) \left( \frac{p_L^2}{4\kappa} \right)^{1-s-\frac{2d+k}{4}} \]

where \( \sum_{\text{BPS}} \equiv \sum_p \delta(p_L^2 - p_R^2 - 4\kappa) \). This converges absolutely for \( \text{Re}(s) > \frac{2d+k}{4} \) and can be analytically continued to \( \text{Re}(s) > 1 \) with a simple pole at \( s = \frac{2d+k}{4} \).
Unfolding the modular integral

For values $s = 1 - \frac{w}{2} + n$ relevant for almost holomorphic modular forms, the summand can be written using elementary functions, e.g.

$$\mathcal{I}_{2+k,2}(1 + \frac{k}{4}, \kappa) = -\Gamma(2 + \frac{k}{2}) \sum_{\text{BPS}} \log \left( \frac{p_R^2}{p_L^2} \right) + \sum_{\ell=1}^{k/2} \frac{1}{\ell} \left( \frac{p_L^2}{4\kappa} \right)^{-\ell}$$

The result is manifestly $O(\Gamma_{d+k,d})$ invariant, and requires no choice of chamber in Narain modular space. Singularities on $G_{d+k,d}$ arise when $p_L^2 = 0$ for some lattice vector.
Fourier-Jacobi expansion

For $d = 2, k = 0$, the Fourier expansion in $T_1$ (or $U_1$) is obtained by solving the BPS constraint. E.g. for $\kappa = 1$, all solutions to $m_1 n^1 + m_2 n^2 = 1$ are

\[
\begin{align*}
    m_1 &= b + dM, \quad n^1 = -c \\
    m_2 &= a + cM, \quad n^2 = d
\end{align*}
\]

, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus SL(2, \mathbb{Z})$, $M \in \mathbb{Z}$

After Poisson resumming over $M$, the sum over $\gamma$ neatly produces a Niebur-Poincaré series in $U$,

\[
\mathcal{I}(s, 1) = 2^{2s} \sqrt{4\pi} \Gamma(s - \frac{1}{2}) T_2^{1-s} \mathcal{E}(U; s) + 4 \sum_{N > 0} \sqrt{\frac{T_2}{N}} T_1^{s - \frac{1}{2}} (2\pi N T_2) \left[ e^{2\pi i N T_1} \mathcal{F}(s, N, 0; U) + \text{cc} \right]
\]

Moreover, recall $\mathcal{F}(s, N, 0) = H_N \cdot \mathcal{F}(s, 1, 0)$...
Fourier-Jacobi expansion II

For \( s = 1 \), relevant for weakly holomorphic modular forms, one recovers the usual Borcherds products,

\[
\mathcal{A} = 8\pi \text{Res}_{s=1} \left[ T_2^{1-s} \mathcal{E}(s; U) \right] + 2 \sum_{N > 0} \left[ \frac{q_T^N}{N} H_N^{(U)} \cdot [j(U) + 24] + \text{cc} \right]
\]

\[
= -24 \log \left[ T_2 U_2 |\eta(T)\eta(U)|^4 \right] - 2 \sum_{M,N} c(MN) [\log(1 - q_T^N q_U^M) + \text{c.c.}]
\]

\[
= -24 \log \left[ T_2 U_2 |\eta(T)\eta(U)|^4 \right] - \log |j(T) - j(U)|^4
\]

where we have used \( \mathcal{F}(1, 1, 0; U) = j(U) + 24 \), \( j(U) = \sum c(M) q^M \).

Borcherds; Harvey Moore
Fourier-Jacobi expansion III

For $s = 1 + n$, relevant for almost holomorphic modular forms of depth $p \geq n$, we can use

$$D^n_T q^n_T = 2 (-2N)^n \sqrt{NT_2} K_{n+\frac{1}{2}}^n (2\pi NT_2) e^{2\pi i NT_1}$$

$$D^n_T 1 = (2n)! (-2\pi T_2)^{-n}/n!$$

$$D^n_U F(n + 1, \kappa, -2n; U) = (2\kappa)^n n! F(n + 1, \kappa, 0; U)$$

$$D^n_U E(n + 1, -2n; U) = (2\pi)^n E(U; n + 1)/n!$$

to express $\mathcal{I}_{2,2}(n + 1, 1)$ as the iterated derivative of a generalized prepotential formally of weight $(-2n, -2n)$,

$$\mathcal{I}_{2,2}(n + 1, 1) = 4 \operatorname{Re} \left[ \frac{(-D_T D_U)^n}{n!} f_n(T, U) \right]$$
The resulting prepotential is holomorphic in $T$ but harmonic in $U$,

$$f_n(T, U) = 2(2\pi)^{2n+1} E(n + 1, -2n; U) + \sum_{N>0} \frac{2q_T^N}{(2N)^{2n+1}} F(n + 1, N, -2n; U)$$

One can turn $f_n$ into a holomorphic function $\tilde{f}_n(T, U)$ by replacing $E(n + 1, -2n; U)$ and $F(n + 1, N, -2n; U)$ by their analytic parts without affecting the real part of its iterated derivative.

The generalized holomorphic prepotential $\tilde{f}_n(T, U)$ now transforms as an Eichler integral of weight $(-2n, -2n)$ under $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \ltimes (T \leftrightarrow U)$. 

Gangl Zagier
Fourier-Jacobi expansion V

- The generalized Yukawa coupling $\partial^{2n+1}_T f_n$ is an ordinary modular form of weight $(2n+2, -2n)$, e.g. for $n = 1$

$$\partial^3_T f_1 \propto \sum_{N>0} q^N_T H^{(U)}_N \cdot \frac{E_4(U)E_6(U)}{\Delta(U)} = \frac{E_4(T)E_4(U)E_6(U)}{\Delta(U)[j(T) - j(U)]}$$

- The case $n = 1$ describes the standard prepotential appearing in string vacua with $\mathcal{N} = 2$ supersymmetry. Its modular anomaly was discussed by Antoniadis, Ferrara, Gava, Narain, Taylor in 1995, which is the first occurrence of Eichler integrals in string theory!

- The case $n = 2$ has appeared in the context of 1/4-BPS amplitudes in $Het/K_3$.

  _Lerche Stieberger 1998_
String amplitudes at genus \( h \leq 3 \) take the form

\[
\mathcal{A}_h = \int_{\mathcal{F}_h} d\mu_h \Gamma_{d+k,d,h}(G, B, Y; \Omega) \Phi(\Omega), \quad d\mu_h = \frac{d\Omega_1 d\Omega_2}{[\det \Omega_2]^{h+1}}
\]

\( \mathcal{F}_h \) is a fundamental domain of the action of \( \Gamma = \text{Sp}(2h, \mathbb{Z}) \) on Siegel’s upper half plane \( \{ \Omega = \Omega^t \in \mathbb{C}^{h \times h}, \Omega_2 > 0 \} \)

\( \Gamma_{d+k,d,h} \) a Siegel-Narain theta series of signature \( (d + k, d) \)

\[
\Gamma_{d+k,d,h} = [\det \Omega_2]^{d/2} \sum_{(\Gamma_{d+k,d})^h} e^{i\pi \text{Tr}(\Omega P_L P_L^t) - i\pi \text{Tr}(\bar{\Omega} P_R P_R^t)}
\]

\( \Phi(\Omega) \) a Siegel modular form of weight \( -k/2 \).

We would like to generalize the previous methods to the case where \( \Phi(\Omega) \) is an almost holomorphic modular form with poles inside \( \mathcal{F}_h \), such as \( 1/\chi_{10} \). As a first step, take \( k = 0, \Phi = 1 \).
The genus $h$ analog of $\mathcal{E}^*(s; \tau)$ is the non-holomorphic Siegel-Eisenstein series

$$
\mathcal{E}^*_h(s; \Omega) = \zeta^*(2s) \prod_{j=1}^{[h/2]} \zeta^*(4s - 2j) \sum_{\gamma \in \Gamma \setminus \Gamma_\infty} |\Omega_2|^s |\gamma|
$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} A & B \\ 0 & A^{-1} \end{pmatrix} \right\} \subset \Gamma$, $|\Omega_2| = |\det \text{Im}\Omega|$.

The sum converges absolutely for $\Re(s) > \frac{h+1}{2}$ and can be meromorphically continued to the full $s$ plane. The analytic continuation is invariant under $s \mapsto \frac{h+1}{2} - s$, and has a simple pole at $s = \frac{h+1}{2}$ with constant residue $r_h = \frac{1}{2} \prod_{j=1}^{[h/2]} \zeta^*(2j + 1)$.
For any cusp form $F(\Omega)$, the Rankin-Selberg transform can be computed by unfolding the integration domain against the sum,

$$\mathcal{R}_h^*(F; s) = \int_{\mathcal{F}_h} d\mu_h F(\Omega) \mathcal{E}_h^*(\Omega, s)$$

$$= \zeta^*(2s) \prod_{j=1}^{[h/2]} \zeta^*(4s - 2j) \int_{\text{GL}(h, \mathbb{Z}) \backslash \mathcal{P}_h} d\Omega_2 |\Omega_2|^{s-h-1} F_0(\Omega_2)$$

where $\mathcal{P}_h$ is the space of positive definite real matrices, and $F_0(\Omega_2) = \int_0^1 d\Omega_1 F(\Omega)$ is the constant term of $F$.

The residue at $s = \frac{h+1}{2}$ is proportional to the average of $F$,

$$\text{Res}_{s=\frac{h+1}{2}} \mathcal{R}_h^*(F; s) = r_h \int_{\mathcal{F}_h} F.$$
The Siegel-Narain theta series is not a cusp form, instead its zero-th Fourier mode is

$$I_{d,d,h}^{(0)}(g, B; \Omega) = |\Omega_2|^{d/2} \sum_{(m^\alpha, n^i\alpha) \in \mathbb{Z}^{2d \times h}, m^\alpha_i n^i\beta = 0} e^{-\pi \text{Tr}(M^2 \Omega_2)}$$

where

$$M^2;\alpha\beta = (m^\alpha_i + B_{ik} n^{k\alpha}) g^{ij} (m^\beta_j + B_{jl} n^{l\beta}) + n^{i\alpha} g_{ij} n^{j\beta}$$

Terms with $\text{Rk}(m^\alpha_i, n^{i\alpha}) < h$ do not decay rapidly at $\Omega_2 \to \infty$. For $d < h$, this is always the case.

The Siegel-Eisenstein series $E^*_h(\Omega, s)$ similarly has non-decaying constant term of the form $\sum_T e^{-\text{Tr}(T \Omega_2)}$ with $\text{Rk}(T) < h$. 
The regularized Rankin-Selberg transform is obtained by subtracting non-suppressed terms, and yields a field theory-type amplitude, with BPS states running in the loops,

\[
\mathcal{R}_h(\Gamma_{d,d,h}; s) = \int_{\text{GL}(h,\mathbb{Z})\backslash \mathcal{P}_h} \frac{d\Omega_2}{\Omega_2|^{h+1-s-d/2}} \sum_{\text{BPS}} e^{-\pi \text{Tr}(M^2 \Omega_2)}
\]

\[
= \Gamma_h(s - \frac{h+1-d}{2}) \sum_{\text{BPS}} \left[ \det M^2 \right]^{\frac{h+1-d}{2} - s}
\]

\[
\sum_{\text{BPS}} = \sum_{(m^\alpha_i, n^i\alpha) \in \mathbb{Z}^{2d \times h}, m^\alpha_i n^i \beta = 0, \det M^2 \neq 0}
\]

\[
\Gamma_h(s) = \frac{1}{4} \frac{h(h-1)}{2} \prod_{k=0}^{h-1} \Gamma(s - \frac{k}{2})
\]
This is recognized as the Langlands-Eisenstein series of $SO(d, d, \mathbb{Z})$ with infinitesimal character $\rho - 2(s - \frac{h+1-d}{2})\lambda_h$, associated to $\wedge^h V$ where $V$ is the defining representation,

$$\mathcal{R}_h(\Gamma_{d,d}; s) \propto \mathcal{E}_{\wedge^h V}^{SO(d,d)}(s - \frac{h+1-d}{2})$$

(h > d)

For $h = d$, $\wedge^h V = S^2 \oplus C^2$ where $S, C$ are spinor representations,

$$\mathcal{R}_h(\Gamma_{h,h,h}; s) \propto \mathcal{E}_S^{SO(h,h)}(2s - 1) + \mathcal{E}_C^{SO(h,h)}(2s - 1)$$

The modular integral of $\Gamma_{d,d,h}$ is proportional to the residue of $\mathcal{R}_h(\Gamma_{d,d,h}; s)$ at $s = \frac{h+1}{2}$, up to a scheme dependent term $\delta$. For $d < h$, the entire result comes from $\delta$. 
For $d = 1$, any $h$,

$$\mathcal{A}_h = \mathcal{V}_h(R^h + R^{-h}), \quad \mathcal{V}_h = \int_{\mathcal{F}_h} d\mu_h = 2 \prod_{j=1}^{h} \zeta^*(2j)$$

For $h = d = 2$, either by computing the BPS sum, or by unfolding the Siegel-Narain theta series, one finds

$$\mathcal{R}^*_2(\Gamma_{2,2}, s) = 2\zeta^*(2s)\zeta^*(2s - 1)\zeta^*(2s - 2) \times [\mathcal{E}_1^*(T; 2s - 1) + \mathcal{E}_1^*(U; 2s - 1)]$$

hence

$$\mathcal{A}_2 = 2\zeta^*(2) [\mathcal{E}_1^*(T; 2) + \mathcal{E}_1^*(U; 2)]$$

proving the conjecture by Obers and BP (1999).
For $h = d = 3$,

$$R_3^*(\Gamma_{3,3}; s) = \zeta^*(2s) \zeta^*(2s - 1) \zeta^*(2s - 2) \zeta^*(2s - 3)$$

$$\left[ \mathcal{E}^*,_{SO(3,3)}(2s - 1) + \mathcal{E}^*,_{SO(3,3)}(2s - 1) \right]$$

hence

$$A_3 = 2\zeta^*(2)\zeta^*(4) \left[ \mathcal{E}^*,_{SO(3,3)}(3) + \mathcal{E}^*,_{SO(3,3)}(3) \right]$$
Modular integrals can be efficiently computed using Rankin-Selberg type methods. The result is expressed as a field theory amplitude with BPS states running in the loop.

T-duality and singularities from enhanced gauge symmetry are manifest. Fourier-Jacobi expansions can be obtained in some cases by solving the BPS constraint.

The RSZ method also works at higher genus, at least for $h = 2, 3$. For computing modular integrals with $\Phi \neq 1$ it will be important to develop Poincaré series representations for Siegel modular forms with poles at Humbert divisors, such as $1/\Phi_{10}$.

Non-BPS amplitudes where $\Phi$ is not almost weakly holomorphic are challenging! So are amplitudes with $h \geq 4$!